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A new family of complex, compact, non-symplectic manifolds.

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—Dedicado a la memoria de Ricardo Mañé.

Abstract. In this paper we study a family of complex, compact, non-symplectic manifolds arising from linear complex dynamical systems. For every integer $n > 3$, and an ordered partition of n into an odd number k of positive integers we construct such a manifold together with an $(n-2)$-dimensional space of complex structures. We show that, under mild additional hypotheses, these deformation spaces are universal. Some of these manifolds are holomorphically equivalent to some known examples and we stablish the identification with them. But we also obtain new manifolds admitting a complex structure, and we describe the differentiable type of some of them.

1. Introduction

Many examples of compact, complex manifolds are given by the smooth projective algebraic varieties in $\mathbb{C}P^n$. It has been known for some time that many other examples are not of this kind.

The first examples given were the *Hopf manifolds* ([**H**]) constructed as follows: Given a real number $r > 1$, we can take in $\mathbb{C}^n \backslash 0$ the action of the infinite cyclic group given by $m \cdot z = r^m z$. The quotient manifold is diffeomorphic to $S^{2n-1} \times S^1$. Since the action is holomorphic and totally discontinuous, this manifold inherits a natural complex structure. To see it is not symplectic, and therefore cannot be algebraic, we can use the following well known facts:

- *On every symplectic manifold M of real dimension $2n$ there exists an element $x \in H^2(M)$ such that $x^n \neq 0$ in $H^{2n}(M)$.*

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- *Every projective, algebraic manifold admits a Kähler structure and, in particular, it admits a symplectic structure.*

(For a proof of the second fact see [**We**], p.182).

It should be mentioned also that there are examples of symplectic, complex, but non-Kähler manifolds due to Kodaira and of Kähler, non algebraic ones (see [**We**]).

Coming back to the Hopf manifolds, since $H^2(S^{2n-1} \times S^1) = 0$ for $n > 1$, they cannot be symplectic.

More generally, Borcea ([**B**]), and Haefliger ([**Hae**]) have considered the generalized Hopf manifolds obtained by taking the quotient of $\mathbb{C}^n \backslash 0$ by any action of the infinite cyclyc group which is holomorphic and totally discontinuous. For all these, the quotient is again topologically $S^{2n-1} \times S^1$. Haefliger obtained a complete description of these manifolds and their small deformations, showing in particular that they come either from linear actions, like

$$m \cdot (z_1, \ldots, z_n) = (\alpha_1^m z_1, \ldots, \alpha_n^m z_n)$$

(where α_i are complex numbers with $|\alpha_i| > 1$), or from certain non linear deformations of them when there are resonances among the α_i.

Calabi and Eckmann ([**C-E**]) generalized Hopf's construction to give complex structures on $S^{2k-1} \times S^{2l-1}$, all of which are again non-symplectic for the same reason. In a recent paper, Loeb and Nicolau ([**L-N**]), using constructions coming from complex dynamical systems, extended Haefliger's results to the Calabi-Eckmann situation, thus obtaining a very complete description of a large class of complex structures on $S^{2k-1} \times S^{2l-1}$ together with their deformations.

Complex structures have also been constructed on manifolds of the form $\Sigma^{2n-1} \times S^1$, where Σ^{2n-1} is a homotopy sphere bounding a π-manifold (or a homology 3-sphere when $n = 2$), by Brieskorn and Van de Ven ([**B-VdV**]). Some of these, even in the case where Σ^{2n-1} is the standard sphere, are different from the ones studied by Borcea and Haefliger, their universal covering space being holomorphically different from $\mathbb{C}^n \backslash 0$ (see [**Mo**]).

Finally, complex structures have been constructed on certain connected sums of simple manifolds: Kato constructed complex manifolds whose differentiable structure was found in joint work with Yamada ([**K-Y**]). In particular they proved there is a complex structure on the connected sum

$$(\mathbb{C}P^1 \times S^4)\#(S^3 \times S^3)\#(S^3 \times S^3).$$

Friedman, Lu and Tian constructed complex structures on the connected sum of k copies of $S^3 \times S^3$ for any $k > 1$. In fact, Friedman showed this for $k > 103$ ([**F**]), while Lu and Tian ([**L-T**]) proved it for $k \leq 106$ by a different construction. These manifolds have the additional feature of having trivial canonical bundles. The case $k = 0$, which can be considered as the question of the existence of a complex structure on S^6, is one of the main open problems in this theory.

For results concerning the non-existence of complex structures on certain connected sums see [**A**].

2. Construction of the manifolds.

Given an n-tuple $\Lambda = (\lambda_1, \ldots, \lambda_n)$ of complex numbers satisfying the property of weak hyperbolicity (cf. [**Ch**], [**LdM1**]):

(WH) For no pair of indices i, j does the segment $[\lambda_i, \lambda_j]$ contain 0,

we shall construct a complex manifold $N = N(\Lambda)$, as follows:

Consider the system of linear, complex, differential equations in $\mathbb{C}^n$:

$$\dot{z}_i = \lambda_i z_i \tag{1}$$

The solutions (other than zero) of this system are 1-dimensional complex submanifolds of $\mathbb{C}^n$, which form the leaves of a foliation of $\mathbb{C}^n \backslash 0$. Such a leaf $\mathcal{L}$ is called a *Poincaré* leaf if the origin belongs to the closure of $\mathcal{L}$. Otherwise it is called a *Siegel* leaf. One says that Λ is in the *Poincaré domain* if the origin is not in the convex hull of the λ_i. In that case all leaves are Poincaré. If the origin is in the convex hull of the λ_i one says that Λ is in the *Siegel domain.* In this case there are both Poincaré and Siegel leaves. (See, e.g., [**C-K-P**]).

Let $\mathcal{S}$ be the union of all Siegel leaves of this system, which is empty if Λ is in the Poincaré domain and is an open dense set if Λ is in the

Siegel domain. The complement of $\mathcal{S}$ is a union of coordinate subspaces, corresponding to the subconfigurations of Λ which are in the Poincaré domain.

There is an action of $\mathbb{C}$ on $\mathcal{S}$ given by the flow of system (1). There is also an action of $\mathbb{C}^*$ on $\mathcal{S}$ given by scalar multiplication in $\mathbb{C}^n$. These two actions commute, giving a free action of $\mathbb{C}\times\mathbb{C}^*$ on $\mathcal{S}$ by holomorphic transformations. The corresponding quotient will be our manifold

$$N = N(\Lambda) = \mathcal{S}/\mathbb{C}\times\mathbb{C}^*.$$

To see that N is a compact Hausdorff manifold it is convenient to use the following alternative description: It can be shown ([**C-K-P**]) that on every Siegel leaf there is a unique point which is closest to the origin. The set of those points coincides with the set

$$M = \{z \in \mathbb{C}^n\backslash 0 |\, \Sigma\lambda_i z_i \bar{z}_i = 0\}$$

and therefore we can identify M with $\mathcal{S}/\mathbb{C}$. M is a cone from the origin so its quotient by the radial action of $\mathbb{R}^+$ can be identified with

$$M_1 = M \cap S^{2n-1} = \{z \in \mathbb{C}^n |\, \Sigma\lambda_i z_i \bar{z}_i = 0, \Sigma z_i \bar{z}_i = 1\}$$

which is compact. The equations describing M_1 are independent, as a result of condition (WH), therefore M_1 is a compact, smooth manifold.

To obtain N from M_1 it only remains to divide by the scalar action of S^1:

$$N = M_1/S^1$$

and N is also a compact, smooth manifold. N clearly inherits a complex structure from $\mathcal{S}$.

Observe that $N(\lambda_1, \ldots, \lambda_n)$ is biholomorphically equivalent to

$$N(\alpha\lambda_1 + \beta, \ldots, \alpha\lambda_n + \beta)$$

if $\alpha \neq 0$ and β are such that both configurations have the same $\mathcal{S}$: actually all orbits of the first action are orbits of the second action. In other words, two such collections of eigenvalues give equivalent manifolds if they are affinely equivalent in $\mathbb{C}$. Thus we must reduce effectively to an $(n-2)$-dimensional space of complex structures which is always smooth; we call this space a *reduced deformation space.*

To describe the topology of N we will use the following known facts about the topology of M_1: First observe that the smooth topological type of M_1 (as well as that of N) does not change if we vary continuously the parameters Λ as long as we do not violate condition (WH) in the process. It is shown in [**LdM2**] that the parameters Λ can always be so deformed until they occupy the vertices of a regular k-gon in the unit circle, where $k = 2l + 1$ is an odd integer, every vertex being occupied by one or more of the λ_i. Therefore the topology of M_1 (and that of N also) is totally described by this final configuration, which can be specified by the multiplicities of those vertices, that is, by the partition

$$n = n_1 + \cdots + n_k.$$

Observe that different partitions give different open sets S and therefore also different reduced deformation spaces. It is clear that if we permute cyclically the numbers n_i we obtain again the same manifolds and deformation spaces, but it follows from the next result that the cyclic order is relevant for their description.

It is shown in [**LdM2**] that the topology of M_1 is given as follows: Let $d_i = n_i + n_{i+1} + \cdots + n_{i+l-1}$ for $i = 1, \ldots, k$ (the subscripts being taken modulo k). Let also

$$d = \min\{d_1, \ldots, d_k\}.$$

These numbers determine the topology of M_1:

Theorem 1. (1) *If $k = 1$ then $M_1 = \varnothing$.*
(2) *If $k = 3$ then $M_1 = S^{2n_1-1} \times S^{2n_2-1} \times S^{2n_3-1}$.*
(3) *If $k = 2l + 1 > 3$ then M_1 is diffeomorphic to the connected sum of the manifolds $S^{2d_i-1} \times S^{2n-2d_i-2}$, $i = 1, \ldots, k$.*

The proof of parts (1) and (2) is quite direct, while the proof of part (3) is long and complicated ([**LdM2**]). In what follows we shall only use the fact that the integral homology groups of M_1 coincide with those of the above described connected sum and the fact that M_1 is $(2d-2)$-connected. The homology calculations (and part (2) of Theorem 1) were first obtained by C.T.C. Wall ([**Wa**]). Thus our results will be independent of [**LdM2**] and will provide a simplified proof of some of the cases of Theorem 1.

3. The manifolds N are not symplectic.

Theorem 2. *For $n > 3$, the manifold $N = N(\Lambda)$ is a compact, complex manifold that does not admit a symplectic structure.*

Proof. Recall that for $k = 1$ the manifold M_1 is empty. In general we have that M_1 lies in the sphere S^{2n-1} and that N sits inside the complex projective space $\mathbb{C}P^{n-1}$ (but not as a holomorphic submanifold), so we have an inclusion of S^1-bundles:

$$\begin{array}{ccc} M_1 & \rightarrow & S^{2n-1} \\ \downarrow & & \downarrow \\ N & \rightarrow & \mathbb{C}P^{n-1} \end{array}$$

We will prove first that the inclusion of N can be deformed down into a projective subspace of low dimension $d-1$, but not lower:

Lemma. *The above inclusion of S^1-bundles embeds homotopically in the following sequence of bundle maps:*

$$\begin{array}{ccccccc} S^{2d-1} & \rightarrow & M_1 & \rightarrow & S^{2d-1} & \rightarrow & S^{2n-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}P^{d-1} & \rightarrow & N & \rightarrow & \mathbb{C}P^{d-1} & \rightarrow & \mathbb{C}P^{n-1} \end{array}$$

where the composition of the bottom arrows is homotopic to the natural inclusion.

Proof of the Lemma. If we put d coordinates $z_i = 0$ we obtain a new manifold $M_1(\Lambda')$ where Λ' is a configuration of eigenvalues that is concentrated in $l+1$ consecutive vertices of the regular $(2l+1)$-gon. This configuration being in the Poincaré domain, it follows that the above manifold is empty.

This means that the original $M_1(\Lambda)$ does not intersect a linear subspace of $\mathbb{C}^n$ of codimension d and that correspondingly N does not intersect an d-codimensional projective subspace of $\mathbb{C}P^{n-1}$. Then the inclusion of N in $\mathbb{C}P^{n-1}$ can be deformed into a complementary projective subspace of dimension $d-1$, which gives the middle bundle map.

Now, M_1 being $(2d-2)$-connected, it follows that $M_1 \to N$ is a universal S^1-bundle for spaces of dimension less than $2d-1$ (see [**S**], §19) and therefore the Hopf bundle over $\mathbb{C}P^{d-1}$ admits a classifying map into it, which gives the first map in the bottom row. The composition of the bottom maps also classifies this Hopf bundle and is therefore homotopic to the natural inclusion, so the Lemma is proved.

From the description of M_1 it follows that M_1 is simply connected, except for the cases $k=3$, $d=n_1=1$. In these cases the S^1-action on $M_1 = S^1 \times S^{2n_2-1} \times S^{2n_3-1}$ can be concentrated on the first factor, and therefore N is diffeomorphic to $S^{2n_2-1} \times S^{2n_3-1}$. Unless $n_2 = n_3 = 1$ we have that $H^2(N) = 0$ and N is not symplectic.

In all the other cases we have that $d > 1$ and M_1 is 2-connected. From the cohomology Gysin sequence of the fibration $M_1 \to N$ it follows that $H^2(N) = \mathbb{Z}$ generated by the Euler class e. However, it follows from the Lemma that

$$e^{d-1} \neq 0, \qquad e^d = 0$$

so this class does not go up to the top cohomology group $H^{2n-4}(N)$, and it follows again that N is not symplectic, and Theorem 2 is proved.

Nevertheless, observe that N is a real algebraic submanifold of $\mathbb{C}P^{n-1}$ since it is the regular zero set of the (non holomorphic) function $g : \mathbb{C}P^{n-1} \to \mathbb{R}^2$ defined by

$$g([z_1, \ldots, z_n]) = \frac{\Sigma \lambda_i z_i \bar{z}_i}{\Sigma z_i \bar{z}_i}$$

This implies that the normal bundle of N in $\mathbb{C}P^{n-1}$ is trivial. Observe also that the map $\mathbb{C}P^{d-1} \to N$ in the Lemma is homotopic to an embedding (since, by definition of d, we have $n \geq 2d+1$ and therefore the dimension of N is greater than twice the dimension of $\mathbb{C}P^{d-1}$), whose normal bundle is then stably equivalent to the normal bundle of $\mathbb{C}P^{d-1}$ in $\mathbb{C}P^{n-1}$.

4. Some old complex structures

In the cases where M is not simply connected (i.e., when $k=3$ and $d = n_1 = 1$), the complex structure on N can be described in terms of the defining parameters by identifying it with previous descriptions of these known manifolds:

(a) Elliptic curves

When $n = k = 3$ the manifold N is diffeomorphic to the torus $S^1 \times S^1$. To identify the corresponding complex structure, observe that in this case $\mathcal{S} = (\mathbb{C}^*)^3$. The mapping $\exp : \mathbb{C}^3 \to \mathcal{S} = (\mathbb{C}^*)^3$ given by

$$\exp(\zeta_1, \zeta_2, \zeta_3) = (e^{\zeta_1}, e^{\zeta_2}, e^{\zeta_3})$$

can be used to identify $N(\lambda_1, \lambda_2, \lambda_3)$ with the quotient of $\mathbb{C}$ by the lattice generated by $\lambda_3 - \lambda_2$ and $\lambda_1 - \lambda_2$. So we have that

$N(\lambda_1, \lambda_2, \lambda_3)$ *is biholomorphically equivalent to the elliptic curve with modulus*

$$\frac{\lambda_3 - \lambda_2}{\lambda_1 - \lambda_2}.$$

Observe that in this case we obtain all complex structures on the torus. By chosing adequately the order of the λ_i we obtain a mapping from the Siegel domain (dynamical systems) to the Siegel upper half-plane in $\mathbb{C}$ (complex function theory).

(b) Generalized Hopf manifolds

When $n_1 = n_2 = 1$ the manifold N is diffeomorphic to $S^1 \times S^{2n_3-1}$. Here the mapping $\exp : \mathbb{C}^2 \times (\mathbb{C}^{n_3}\backslash 0) \to \mathcal{S} = (\mathbb{C}^*)^2 \times (\mathbb{C}^{n_3}\backslash 0)$ given by

$$\exp(\zeta_1, \zeta_2, \zeta) = (e^{\zeta_1}, e^{\zeta_2}, \zeta)$$

can be used to identify N with the quotient of $\mathbb{C}^{n_3}\backslash 0$ by the action of $\mathbb{Z}$ defined by the multipliers

$$\alpha_i = \exp\left(2\pi i \frac{\lambda_{2+i} - \lambda_2}{\lambda_1 - \lambda_2}\right), : i = 1, \dots, n_3.$$

In this case we obtain all complex structures on $S^1 \times S^{2n_3-1}$ having $\mathbb{C}^n\backslash 0$ as universal cover when there is no resonance among the α_i. But in the resonant case we do not obtain all such complex structures since we do not obtain the non-linear resonant cases of Haefliger. It is clear that, in order to obtain the latter, one must look at the resonant non-linear versions of equation (1).

(c) Generalized Calabi-Eckmann manifolds

When $n_1 = 1$ and n_2, n_3 are both greater than 1 we have seen that the manifold N is diffeomorphic to $S^{2n_2-1} \times S^{2n_3-1}$. Here the mapping $\exp : \mathbb{C} \times (\mathbb{C}^{n_2} \backslash 0) \times (\mathbb{C}^{n_3} \backslash 0) \to \mathcal{S} = \mathbb{C}^* \times (\mathbb{C}^{n_2} \backslash 0) \times (\mathbb{C}^{n_3} \backslash 0)$ given by

$$\exp(\zeta, \zeta_1, \zeta_2) = (e^{\zeta}, \zeta_1, \zeta_2)$$

can be used to identify N with the quotient of $(\mathbb{C}^{n_2} \backslash 0) \times (\mathbb{C}^{n_3} \backslash 0)$ by the action of $\mathbb{C}$ defined by the linear differential equation with eigenvalues $\lambda'_i = 2\pi i(\lambda_i - \lambda_1)$, $i = 2, \ldots, n$. This is exactly the construction of the Loeb-Nicolau complex structure corresponding to a linear system of equations of Poincaré type ([**L-N**]).

Again we obtain all their examples of complex structures on $S^{2n_2-1} \times S^{2n_3-1}$ when there is no resonance among the λ'_i. But, once more, we do not obtain the non-linear resonant structures.

Observe that in their construction only the quotients of the eigenvalues of the system are relevant for the definition of the complex structure on N, so once again only the quotients $\frac{\lambda_i - \lambda_k}{\lambda_j - \lambda_k}$ of our original eigenvalues count (in accordance with the observation made in section 2 that affinely equivalent configurations of eigenvalues with the same $\mathcal{S}$ give the same complex structure) and that they are actually *moduli* of that complex structure. We will prove in section 6 that for $d \geq 3$ the same result is true.

5. Some new complex structures

In all the other cases (i.e., when M_1 is simply connected) we obtain new complex structures on manifolds. An intermediate situation is given by the cases $k = 3$, with $n_1 = 2$, n_2 and n_3 even, where one can show, using the fact that each $\mathbb{C}^{n_i}$ can be considered as a quaternionic vector space, that N is diffeomorphic to $\mathbb{C}P^1 \times S^{2n_2-1} \times S^{2n_3-1}$. It is easy to see that in some cases N can be identified with the product of $\mathbb{C}P^1$ with one of the Loeb-Nicolau complex structures on $S^{2n_2-1} \times S^{2n_3-1}$. But in other cases there is no simple way to stablish such an identification, and it is plausible that these give new complex structures.

When $k = 3$, $n_1 > 2$ we definitely get a manifold which is not a product, but a twisted fibration over $\mathbb{C}P^{n_1-1}$. In fact, N clearly fibers over $\mathbb{C}P^{n_1-1}$ with fiber $S^{2n_2-1} \times S^{2n_3-1}$. This fibration does have a section (recall that we are assuming that $n_1 = d$ is not bigger than the other n_i) which is homotopic to the map $\mathbb{C}P^{n_1-1} \to N$ constructed in the Lemma in section 3. But, by the observation and the end of that section, the normal bundle of $\mathbb{C}P^{n_1-1}$ in N is stably equivalent to the normal bundle of $\mathbb{C}P^{n_1-1}$ in $\mathbb{C}P^{n-1}$. This bundle is non-trivial for $n_1 > 2$, being the sum of $n - n_1$ copies of the Hopf bundle and therefore its first Pontryagin class is $(n-n_1)\alpha^2$. (We owe this observation to Elías Micha). We therefore have:

Theorem 3. *When $3 \leq n_1 \leq n_2 \leq n_3$ there is a non-trivial $(S^{2n_2-1} \times S^{2n_3-1})$-fibration over $\mathbb{C}P^{n_1-1}$ with an $(n-2)$-dimensional space of complex structures.*

We will see in section 6 that all these structures are in fact non-equivalent and that for $n_1 \geq 4$ the family is universal when all the λ_i are distinct.

When $k > 3$ we get new complex structures on manifolds. We will give the complete description of the underlying real smooth manifold only in the case where all $n_i = 1$ (so $n = k = 2l + 1$), where the computations and arguments are simpler. To do this we can assume as before that the λ_i are the n-th roots of unity: $\lambda_i = \rho^i$, ρ a primitive root.

In that case M_1 is a stably parallelizable $(2n - 3)$-manifold with homology in the middle dimensions only, where it is free of rank n:

$$H_{n-2}(M_1) = H_{n-1}(M_1) = \mathbb{Z}^n$$

It follows from the Gysin sequence of the fibration $M_1 \to N$ (and from the order of its Euler class found in section 3) that N has homology only in dimensions $2i$, $i = 1, \ldots, n-2$ where it is free of rank 1, and in dimension $2l - 1$ where it is free of rank $2l$.

On the other hand, M_1 is the boundary of a manifold Q constructed as follows: Let

$$Z = \{z \in \mathbb{C}^n | \Sigma Re(\lambda_i) z_i \bar{z}_i = 0, \Sigma z_i \bar{z}_i = 1\}.$$

Z is diffeomorphic to $S^{2l-1} \times S^{2l+1}$ (since the defining quadratic form has index $2l$) and is the union of two manifolds with boundary

$$Q^{\pm} = \{z \in \mathbb{C}^n | \Sigma Re(\lambda_i) z_i \bar{z}_i = 0, \pm \Sigma Im(\lambda_i) z_i \bar{z}_i \geq 0, \Sigma z_i \bar{z}_i = 1\}$$

whose intersection is M_1.

The involution of $\mathbb{C}^n$ which interchanges the coordinates z_i and z_{n-i} preserves Z and M_1, and interchanges Q^+ with Q^-. Therefore these two are diffeomorphic and M_1 is an equator of Z.

Let $Q = Q^+$. It follows now easily from the Mayer-Vietoris sequence of the triple $(S^{2l-1} \times S^{2l+1}, Q, Q^-)$ that $H_i(Q) = 0$ for $i \neq 2l-1, 2l$, in which case it is free of rank $l+1$ and l, respectively, and that $H_i(M_1) \to H_i(Q)$ is always surjective. Q is also simply connected by Van Kampen's Theorem. The Hurewicz and Whitehead Theorems now show that all homology classes in Q can be represented by spheres which for dimensional reasons can be assumed to be embedded in M by Whitney's imbedding theorem. (This is enough to show, using the h-cobordism theorem, that M_1 is a connected sum, as described in Theorem 1. Cf. [**LdM2**] and the argument used below. It is shown in [**LdM2**] that these facts are true in general, by a detailed description of all homology classes in M_1).

The S^1 scalar action leaves Q invariant, so the quotient $R = Q/S^1$ is a compact manifold with boundary $\partial R = N$. Now the fibration $Q \to R$ again embeds in a diagram like the one in the Lemma of Section 3. It follows now from the cohomology Gysin sequence of the fibration $Q \to R$ that $H_{2i}(R) = \mathbb{Z}$, $i = 0, \ldots, l-1$ and $H_{2l-1}(R) = \mathbb{Z}^l$, all other homology groups being trivial.

Now we can embed, by the Lemma of section 3, $\mathbb{C}P^{l-1}$ in R representing all even dimensional homology classes, and l disjoint $(2l-1)$-spheres with trivial normal bundle representing the generators of the corresponding homology group of R (since all these classes come from Q and are therefore spherical, and their normal bundles are again stably equivalent to the trivial normal bundle of S^{2l-1} in $\mathbb{C}P^{n-1}$). Taking a tubular neighborhood of these manifolds and joining them by tubes we get a manifold with boundary R' whose inclusion in R induces isomorphisms in homology groups. It follows from the h-cobordism theorem ([**Mi**]) that $N = \partial R$ is diffeomorphic to $\partial R'$ which is a connected sum of simple manifolds. These are l copies of $S^{2l-1} \times S^{2l-1}$ and the boundary of the tubular neighborhood of $\mathbb{C}P^{l-1}$ in R. By the remark at the end of section 3 we know that the normal bundle of this inclusion is stably equivalent to the normal bundle of $\mathbb{C}P^{l-1}$ in $\mathbb{C}P^{2l}$. We have therefore proved the following

Theorem 4. *For every $l > 1$ there is a $(2l-1)$-dimensional space of complex structures on the connected sum of $\mathbb{C}P^{l-1}\tilde{\times}S^{2l}$ and l copies of $S^{2l-1} \times S^{2l-1}$, where $\mathbb{C}P^{l-1}\tilde{\times}S^{2l}$ denotes the total space of the S^{2l}-bundle over $\mathbb{C}P^{l-1}$ stably equivalent to the spherical normal bundle of $\mathbb{C}P^{l-1}$ in $\mathbb{C}P^{2l}$.*

We will see in section 6 that for $l \geq 3$ all these structures are in fact non-equivalent and that for $l \geq 4$ the family is universal.

Observe that for $l = 2$ we get a manifold which is close, but not equal, to the one constructed by Kato mentioned in the introduction, where the first summand is a product. Both manifolds had been considered before, from the point of view of group actions, by Goldstein and Lininger ([**G-L**]).

In general, these complex structures are very symmetric, in the sense that we can still find holomorphic actions of large groups on them (Cf. [**LdM1**]). In particular, there is an action of the complex, noncompact, $(n-2)$-torus $(\mathbb{C}^*)^{n-2}$ on them with a dense orbit. In this sense, our manifolds behave as toric varietes.

6. Rigidity and Versality

Recall that the complex structure on $N(\Lambda)$ does not vary within the affine equivalence class of Λ. We show now that the converse is true in most of the cases. These include in particular all cases with $k > 5$. It is plausible that the result is true in general.

Theorem 5. *Let $n = n_1 + \cdots + n_k$ be an ordered partition of n with $d \neq 2$. Then two collections of eigenvalues corresponding to this partition give holomorphically equivalent manifolds N if, and only if, they are affinely equivalent.*

Proof. The sufficiency of the condition was observed above. For the necessity, if $d = 1$ we are in the Calabi-Eckmann case, and this was shown by Loeb and Nicolau ([**L-N**], proposition 12). For $d > 2$ we follow their argument:

Let $V = \mathcal{S}/\mathbb{C}^*$ which is an open subset of $\mathbb{C}P^{n-1}$. Then the complement of V in $\mathbb{C}P^{n-1}$ is a union of projective subspaces whose smallest codimension is d. By the results of Scheja [**Sc**] we have that

$$H^i(V, \mathcal{O}) = H^i(\mathbb{C}P^{n-1}, \mathcal{O}) \ for\ i \leq d-2$$

where $\mathcal{O}$ denotes the sheaf of holomorphic functions on a manifold. The second cohomology groups were computed by Serre and are $\mathbb{C}$ in dimension 0 and trivial otherwise (see e.g. [**G-H**] p.118).

Now, let $\mathcal{O}^{inv}$ be the kernel of the map $\mathcal{O} \to \mathcal{O}$ given by the Lie derivative along the vector field ξ which generates the $\mathbb{C}$ action on V, so we have an exact sequence of sheaves:

$$0 \to \mathcal{O}^{inv} \to \mathcal{O} \xrightarrow{L_\xi} \mathcal{O} \to 0$$

The associated cohomology exact sequence shows that, for $d \geq 3$, $H^1(V, \mathcal{O}^{inv}) = \mathbb{C}$, but this group can be identified with $H^1(N, \mathcal{O})$. Therefore this group is also $\mathbb{C}$ and since it classifies the principal $\mathbb{C}$-bundles over N, any two non-trivial principal $\mathbb{C}$-bundles over N differ by a scalar factor.

Let N_1, N_2 be two such manifolds which are holomorphically equivalent and consider a biholomorphism $\phi : N_1 \to N_2$. Over each N_i there is a principal $\mathbb{C}$-bundle $V_i \to N_i$, where the total space V_i is in both cases V, but is foliated in two different ways by the projectivized leaves of each system. We have to lift ϕ to an equivalence of the principal $\mathbb{C}$-bundles V_i, which amounts to finding an equivalence between V_1 and $\phi^* V_2$. Now V_1 and $\phi^* V_2$ are non-trivial $\mathbb{C}$-bundles (otherwise they would have sections, N_i would embed holomorphically in $\mathbb{C}P^{n-1}$ and would be a Kähler manifold, recall [**We**], p.182). By the previous computation these differ by a scalar factor and there is an equivalence between V_1 and V_2 preserving the leaves of the foliations. By Hartog's Theorem this equivalence extends to one of $\mathbb{C}P^{n-1}$ into itself which must then necessarily be linear since the group of biholomorphisms of $\mathbb{C}P^{n-1}$ is the corresponding projective linear group. But then it follows easily that the corresponding eigenvalues must be affinely equivalent.

Observe that we have actually proved that any holomorphic equivalence between two such manifolds N_i extends to a linear automorphism of $\mathbb{C}P^{n-1}$. In the case all the λ_i are different it follows by the same argument that the action of $(\mathbb{C}^*)^{n-2}$ described at the end of section 5 gives the whole group of automorphisms of the corresponding N.

Theorem 5 says that when $d \neq 2$ the reduced deformation space of N injects into its versal deformation space. For $d = 1$ the question of whether the reduced deformation space is universal or not depends on the existence of resonances among the λ_i (see [**Hae**], [**L-N**]). For $d \geq 4$ the situation is simpler and only depends on the condition that all the λ_i be different:

Theorem 6. *Let $n = n_1 + \cdots + n_k$ be an ordered partition of n with $d \geq 4$. Let Λ be a collection of eigenvalues corresponding to this partition and assume that all λ_i are different. Then the corresponding reduced deformation space of $N(\Lambda)$ is universal.*

Proof. Following again [**L-N**] we consider the exact sequences of sheaves over V:

$$0 \to \Theta^{inv} \to \Theta \xrightarrow{L_\xi} \Theta \to 0$$

$$0 \to \mathcal{O}^{inv}\xi \to \Theta^{inv} \to \Theta_b \to 0$$

where Θ denotes the sheaf of holomorphic vector fields on a manifold and Θ^{inv} and Θ_b are defined by these sequences. Now again by Scheja ([**Sc**]) we have

$$H^i(V,\Theta) = H^i(\mathbb{C}P^{n-1},\Theta) \; for \; i \leq d-2$$

$H^0(\mathbb{C}P^{n-1},\Theta)$ is the space of holomorphic global vector fields on $\mathbb{C}P^{n-1}$ (all of which are linear) and can be identified with the space of

$n \times n$ matrices modulo the scalar ones. For $i > 0$, $H^i(\mathbb{C}P^{n-1}, \Theta) = 0$.

The first sequence above gives a cohomology exact sequence for $d \geq 4$:

$$0 \to H^0(\Theta^{inv}) \to H^0(\Theta) \xrightarrow{L_\xi} H^0(\Theta) \to H^1(\Theta^{inv}) \to 0.$$

Since ξ corresponds to the diagonal matrix with entries λ_i and these are different, the kernel and cokernel of L_ξ can be identified with the space of diagonal matrices modulo the scalar ones, so $H^1(\Theta^{inv})$ is a space of dimension $n-1$. The class of ξ in this vector space is non-zero.

From the exact sequences of sheaves we have the diagram:

$$\begin{tikzcd} H^0(\Theta_b) \arrow[r] & H^1(\mathcal{O}^{inv}) \arrow[r] & H^1(\Theta^{inv}) \arrow[r] & H^1(\Theta_b) \arrow[r] & 0 \\ & H^0(\mathcal{O}) \arrow[u, "\cong"] \arrow[r] & H^0(\Theta) \arrow[u] & & \end{tikzcd}$$

where the two middle horizontal maps are induced by multiplication by ξ. Since the lower one is injective by the above remark, it follows that so is the upper one and that $H^1(\Theta_b)$ is of dimension $n-2$.

Now it is easy to see that $H^i(\Theta_b)$ is isomorphic to $H^i(N, \Theta)$. It follows that $H^1(N, \Theta)$ is of dimension $n-2$. Since it contains the versal (Kuranishi) deformation space of N (see [**Su**] p. 160), we obtain a map from the reduced deformation space into it, which is injective by Theorem 5. Now, an injective holomorphic map between smooth spaces of the same dimension must be regular and it follows that the Kuranishi space is the whole $H^1(N, \Theta)$ and that the reduced space is itself a versal deformation space of N. But, using again the fact that no two structures in this space are equivalent, it follows that it is actually universal and Theorem 6 is proved.

It follows from the above computations that $H^0(N, \Theta)$ has also dimension $n-2$. (This also follows from the remark after Theorem 5 that the group of automorphisms of N has dimension $n-2$). Since this dimension is the same for all complex structures of the family with all λ_i different, a theorem of Wavrik (cf [**Su**], pp. 160-161) can also be used to prove that the Kuranishi family is universal.

It also follows from the above computations that if $d \geq 5$, even without the condition that the λ_i be different, then $H^2(N,\Theta) = 0$ and that therefore $H^1(N,\Theta)$ is the versal deformation space of N and the Kuranishi space is smooth. But, without the condition that all the λ_i be different, its dimension is bigger than $n-2$ and the reduced family only injects as a proper subfamily of the versal one.

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