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Fixed Points and Entropy for non Solvable Dynamics on $\mathbb{C}, 0$

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— Dedicated to the memory of R. Mañé

Abstract. This work generalizes for any non-solvable pseudo-groups of $\mathrm{Diff}_{0,\mathbb{C}}$ the existence of fixed points arbitrarly close to the origin already proved in the generic case [G-M,Wi1]. An elementary proof of the Sherbakov-Nakai's density theorem [Na] is added with moreover some more precision about the derivative of a germ sending a point close to an other one. At last the topological entropy and the entropy of fixed points are strictly positive for such pseudo-groups as in [Wi] and [G-M, Wi2].

Keywords: Entropy, Fixed Point, Foliation, Germ, Holomorphic Function, Pseudo-group.

Introduction

Any finite collection of germs of holomorphic diffeomorphisms fixing the origin generates a pseudo-group. The orbit of any complex number is the set of all possible images of this complex number by the germs of the pseudo-group, while this complex number remains in the domain of the germs. Let the tangent group be the multiplicative sub-group of $(\mathbb{C}^*, .)$ generated by the derivatives of the generator germs in 0. The tangent group contains a first topological information. Cerveau and Moussu [**Ce,Mo**] prove that if the tangent group is dense (generic case), any orbit is dense close to 0. Sherbakov and Nakai prove the property of sectorial density theorem for any non solvable pseudo-group. Franck Loray classifies pseudo-groups with discrete orbits [**Lo1**].

The second problem is the existence of periodic points arbitrarly

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close to the origin. Even for the iteration of one germ, there exists such points [**P-M**]. Some example of existence [**Ya**] and the generic existence of such fixed points [**G-M,Wi1**] are a first step for some more general results.

Thirdly, dynamical invariants of such pseudo-groups have to be evaluated. By example, the topological entropy of pseudo-group is one. It is defined by Ghys, Langevin and Walczak [**G,L,W**]. The notion of entropy of fixed point, evoqued by Mañe [**Ma**], Katok [**K, H**], Gomez-Mont and the author [**G.M,Wi2**] is a second one. In the generic case, theses entropies are strictly positive, and the entropy of fixed points is larger or equal to the topological entropy [**G.M,Wi2**]. We prove here three results for any non solvable pseudo-group:

- the existence of fixed points arbitrarly close to 0,
- the strict positivity of the topologieal entropy on any compact disk centered at 0,
- the strict positivity of entropy of fixed point on any compact disk centered at 0.

The initial idea of proof of the existence of fixed points was esquissed by Paulo Sad during a discussion in Rennes. A first proof of Julio Rebelo [**Re**], using the notion of Nakai's field, is submitted by Michel Bellard, Isabelle Liousse and Franck Loray [**B,Li,Lo**]. Nevertheless this proof is not sufficient to get the thermodynamical results of this paper.

The existence and abundance of such fixed points is equivalent to the existence and abundance of loops on leaves of germs of singular non dicritical holomorphic foliations of $\mathbb{C}^2$, as any pseudo-group is realized as projective holonomy of such foliations [**L-N**], [**Ca, L-N**], [**Ca, Sa**].

Some notations and definitions

We fix $a_1 < a_2$ two strictly positive integers and we fix g_1 and g_2 two germs a_1-flat and a_2-flat:

$$g_j(z) = \lambda_j z + \mu_j z^{a_j+1} + \dots, \quad \lambda_j, \mu_j \neq 0, \ \ j = 1, 2.$$

We denote by H_1 the set $\{Id, g_1, g_2, g_1^{-1}, g_2^{-1}\}$. We denote H the set of all compositions of element of H_1. Let n be a positive integer. We call

H_n the subset of H containing all compositions of n germs of H_1. Let $\alpha > 0$ be fixed, smaller than any radius of convergence of any germ of H_1.

Definition 1. *Domain of a germ of H_n.*

Let j be smaller than n and $f = f_j \circ f_{j-1} ... \circ f_1$, $f_k \in H_1$, $k = 1, 2, ..., j$ an element of H_n. Let z_0 be a non zero complex number. Then we define the partial images of z_0 by f the $j+1$-uplet $(z_0, z_1, ..., z_j)$ defined by induction : $z_k = f_k(z_{k-1})$, $0 \leq k \leq j$. Then the domain of f, denoted $Dom(f)$ is the compact $\{z_0$ such that $\mid z_k \mid \leq \alpha, \ 0 \leq k \leq j\}$. This domain depends on the external radius α, but this real number is supposed to be fixed and invariant.

Definition 2. *Fixed point of H.*

Let f be a germ of H, $f \neq id$. A fixed point z of f is a non zero complex number z such that $z \in Dom(f)$ and $f(z) = z$. If f is in H_n then z is a fixed point of index n.

Theorem 1. *For any pair f, g of germs of* $\mathrm{Diff}_{0,\mathbb{C}}$ *generating a non solvable pseudo-group there exist fixed points arbitrarly close to* 0.

Sketch of the proof. We suppose without any restriction that f and g are tangent to identity with different degrees of flatness [**Na**]. The angular derivative of iterations of any germ tangent to identity is firstly assymptotically estimated. We deduce that if $f^n(z)$ and $g^m(z)$ have equivalent modulus, the sizes of the images of a small ball centered at z by these iterations are not equivalent. Hence exists a germ of H sending z close to itself with arbitrarly large derivative. By this way occurs a configuration of fixed point [**G-M-Wi1**], i.e a couple (f, O) where f is a germ of H, O an open disk of $\mathbb{C}$, included in $Dom(f)$, and verifying $\overline{O} \subset f(O)$ or $\overline{f(O)} \subset O$. In this condition f admits a fixed point in the open disk O (Figure 1).

Proof. Let us recall briefly some properties of the angular derivative [**Wi**], [**G-M,Wi1**]. For any germ f of $\mathrm{Diff}_{0,\mathbb{C}}$ and any z in the disk of convergence of f the angular derivative of f in z is denoted $\Delta f(z)$ and is $zf'(z)/g(z)$. Its real part is $\partial/\partial\theta \, \mathrm{Arg}(f(z))$ where Arg is the argument (justification of the qualificative "angular"). This differential operator is compatible with the composition of germs:

$$\Delta(f \circ g)(z) = \Delta g(z)\ \Delta f(g(z)).$$

We remember first some properties of iteration of germs tangent to identity.

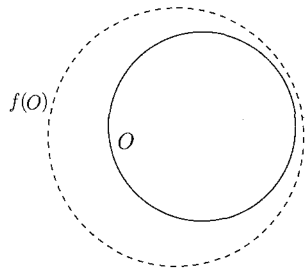

Figure 1

Theorem. ([Le].) *Given g an a-flat germ tangent to identity there exists α strictly positive such that there exist $2a$ open sectors of the open disk $\{|\ z\ |<\alpha\}$ such that for any z in such sector, one at least among the two sequences $(g^n(z))_{n\in\mathbb{N}}$ and $(g^{-n}(z))_{n\in\mathbb{N}}$ converges to 0 and remains in the initial sector of z.*

Any orbit draws petal and the reunion of petals is a flower with $2a$ petals. The top of a petal gets the maximum modulus of the orbit. It occurs where the difference between the argument of z and the argument of the first non-linear term of $g(z)$ is close to $\pm\pi/2$. (Figure 2).

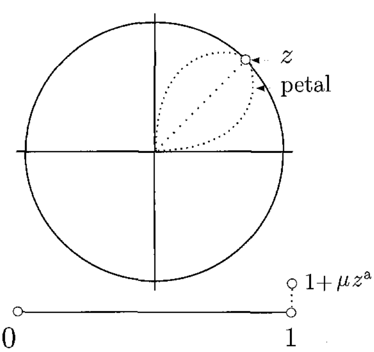

Figure 2

Follows now the main argument of this paper. It is an asymptotic property of the angular derivative.

Lemma 1.1. *Let g be a germ tangent to identity, not equal to identity, and z such that the sequence $(g^n(z))_{n\in\mathbb{N}}$ converges to 0. Then the modulus of the angular derivative $\Delta g^n(z)$ is asymptotically comparable to n^{-1}.*

Remark. The degree of flatness of g does not occur in this estimation.

Proof. Here "asymptotically comparable to n^{-1}" means that there exists some constant $K > 1$ depending on z ang g such that for sufficiently large n we have:

$$K^{-1} \leq \mid n\Delta g^n(z) \mid \leq K$$

Let a be an integer and let $g(z) = z + \mu z^{a+1} + \ldots$, $\mu \neq 0$ be an a-flat germ. As $\lim_{n\to 0} g^n(z)$ is 0, the modulus $\mid z \mid$ is supposed small and the sequence $(\mid g^n(z) \mid)_{n\in\mathbb{N}}$ decreases to 0. Using Nakai's estimations of iterations of germs, the coordinate z is changed in $\tilde{z} = z^{-a}$ and by conjugacy the germ g is transformed in a quasi translation $\tilde{g}$, (i.e. $\tilde{g}(\tilde{z}) = \tilde{z} - a\mu + c_1 \tilde{z}^{1/a} + \ldots$ where c_1 is some complex number). The n-th iteration of $\tilde{g}$ verifies

$$\tilde{g}^n(\tilde{z}) = \tilde{z} - na\mu + R_n(\tilde{z}).$$

The quantity $R_n(\tilde{z})$ is controlled : $R_n(\tilde{z}) = O(\ln(n))$ if $a = 1$, and $R_n(\tilde{z}) = O(n^{1-1/a})$ if $a > 1$. If n is large enough then the dominant term is $-na\mu$ in any case. As $\tilde{g}^n(\tilde{z})$ is equal to $(g^n(z))^{-a}$ the estimation of the angular derivative of g in $g^n(z)$. Let us remember the first coefficient of the angular derivative:

$$\Delta g(z) = \frac{z(1 + (a+1)\mu z^a + \ldots)}{z(1 + \mu z^a + \ldots)} = 1 + a\mu z^a + \ldots \ .$$

This implies a convenient control of the angular derivative of g in $g^n(z)$.

$$\begin{aligned} \Delta g(g^n(z)) &= 1 + a\mu(g^n(z))^a + \ldots, \\ &= 1 + \frac{a\mu}{\tilde{z} - na\mu + R_n(\tilde{z})} + \ldots \ . \end{aligned}$$

As $-na\mu$ is the dominant term of the denominator there is a simplification by $a\mu$ in the previous fraction. Then $\Delta g(g^n(z))$ is asymptotically equivalent to $1 - 1/n$. This point kills any dependance on the degree of flatness. The logarithm of this angular derivative is equivalent to $-1/n$. The partial sums of two positive asymptotically equivalent divergent series are asymptotically equivalent. Therefore the partial sums of index m of the divergent series $\{ \ln(\Delta g(g^n(z))) \}_{n\in\mathbb{N}}$ are asympotically equivalent to $-\ln(m)$. As the angular derivative is compatible with the composition of germs, the exponential of the previous partial sums are $\Delta g^m(z)$. They are comparable to m^{-1} if m is sufficiently large. □

If n is sufficently large then $\tilde{g}^n$ is relatively close to $-na\mu$. It implies that the modulus of $g^n(z)$ is close to $\mid a\mu \mid^{-1/a} \; n^{-1/a}$, then depends on the degree of flatness. The dependance of the modulus of $g^n(z)$ on the degree of flatness of g and the simultaneous independance of the angular derivative with this degree of flatness will generate local hyperbolicity of some special compositions. Let ε be a strictly positive real such that the compact ball $\{\mid y - z \mid\leq \varepsilon\}$, denoted $B(z, \varepsilon)$, is in the open sector containing z.

Lemma 1.2. *If n is large enough, then $g^n(B(z,\varepsilon))$ contains a compact ball centered at $g^n(z)$ with radius asymptotically comparable to $(\varepsilon/\alpha)n^{-1-1/a}$ and is contained in a ball with same center and equivalent radius.*

Proof. Let x be fixed in $B(z, \varepsilon)$. As in the proof of lemma 1.1 the modulus of $g^n(x)$ is comparable to $n^{-1/a}$. Then as $\mid x \mid$ is close to α, the rate $\mid x/g^n(x) \mid$ is comparable to $n^{1/a}/\alpha$. As the derivative $(g^n)'(x)$ is by definition of the angular derivative the product $\Delta g^n(x) \; (g^n(x)/x)$. The estimation of Lemma 1.1 implies that the derivative $(g^n)'(x)$ is asymptotically comparable to $(1/\alpha)n^{-1-1/a}$ and this estimation is uniform on the compact ball $B(z, \varepsilon)$. □

For the final step four germs in H are used. They are denoted f_1, f_2, f_3 and f_4 and possess four different degrees of flatness $k_1 < k_2 < k_3 < k_4$. The coefficient of degree $k_j + 1$ of f_j is denoted μ_j, $j = 1, 2, 3, 4$. The existence of these four germs is the consequence of the non-solvability of the pseudo-group [**Lo1**], [**Na**]. Each germ will have a specific utility in the following construction: f_2 and f_3 generates some hyperbolicity, f_1 is used for large correction of some trajectory, f_4 is used for some fine correction. We suppose now that the argument of z verifies simultaneously three conditions:

$$\mathrm{Arg}(\mu_j z^{k_j}) \in \mathrm{Arg}(z) + \pi + [\pm\frac{\pi}{4k_j}], \quad j = 2, 3. \qquad \text{(i, ii)}$$

These two conditions assure that the sequences $(\mid f_j^n(z) \mid)_{n\in\mathbb{N}}$, $j = 2, 3$ decrease to 0. The petals corresponding to the iteration of f_2 and f_3 arrive to 0 with two different tangencies. The difference of angle is smaller than $2\pi(1/k_2 - 1/k_3)$ (Figure 3).

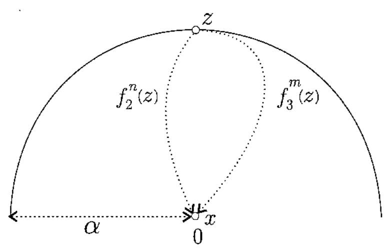

Figure 3

The third condition is

$$\mathrm{Arg}(\mu_1 z^{k_1}) \in \mathrm{Arg}(z) \pm \frac{\pi}{2} + [\pm\frac{\pi}{4k_1}]. \qquad \text{(iii)}$$

It implies that for any complex in the sector between the two branchs petals, g_1 changes more the argument than the radius. Therefore the first iterates of y by g_1 draw the top of a petal (Figure 4).

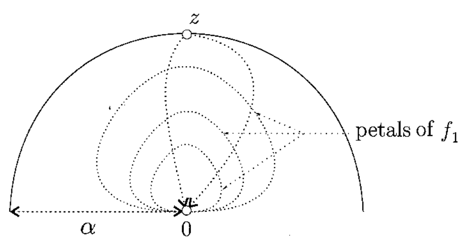

Figure 4

The complex z is fixed with modulus α. The germ f_3 is iterated from z until the asymptotic estimations of lemmas 1.1 and 1.2 hold. Let

N_3 be the number of iterations of f_3 and β the modulus of $f_3^{N_3}(z) = z_3$. The lemma 1.1 implies that β^{-k_3} and N_3 are comparable. The same result induces that the angular derivative of $f_3^{N_3}$ in z is comparable to $1/N_3$.

The sequence $f_1^n(z_3)$ progresses in the direction of the petal $\{f_2^m(z)\}_{m\in\mathbb{N}}$. At any new iteration of f_1 an argument comparable to β^{k_1} is added. The condition (iii) on the argument of z implies that the modulus of any iteration are in bounded rate while $f_1^n(z_3)$ remains between the two petals $\{f_2^m(z)\}_{m\in\mathbb{N}}$ and $\{f_3^p(z)\}_{p\in\mathbb{N}}$. There exists a convenient power of iteration N_1 of f_1 such that the argument of $f_1^{N_1}(z_3)$ is $O(\beta^{k_1})$-close to the argument of some point $g_2^{n_2}(z)$, and the distance $\mid f_2^m(z) - f_1^{N_1} \circ f_3^{N_3}(z) \mid$ is minimal for $m = n_2$. The integer N_1 is comparable to β^{-k_1} and we denote z_1 the point $f_1^{N_1}(z_3)$.

The condition (iii) implies that $\mid f_2^{n_2}(z) \mid$ is comparable to β. Then Lemma 1.1 implies that the angular derivative $\Delta f_2^{n_2}(z)$ is comparable to $1/n_2$ in modulus. Moreover Lemma 1.2 gives more precise information: The image $f_2^{n_2}(B(z,\varepsilon))$ is contains a ball centered at $f_2^{n_2}(z)$ with radius comparable $\varepsilon/\alpha n_2^{-1-1/k_2}$. As n_2 is comparable to β^{-k_2} and N_3 to β^{-k_3}, as β is arbitrarly small, the radius of the ball contained in $f_2^{n_2}(B(z,\varepsilon))$ is arbitrarly much more than the radius of the ball containing $f_3^{N_3}(B(z,\varepsilon))$.

The angular derivative of f_1 is bounded by $\exp(K\beta^{k_1})$ on all points of the orbit of $f_1^n(z_3)$. Follows that the angular derivative $\Delta f_1^{N_1}(z_3)$ is bounded by $\exp(K\beta^{k_1})^{O(\beta^{-k_1})}$, and then it is uniformly bounded. As the moduli of $f_1^{N_1}(z_3)$ and z_3 are both comparable to β, their rate is uniformly bounded, and the derivative $(f_1^{N_1})'(z_3)$ is also uniformly bounded. Then the image $f_1^{N_1} \circ f_3^{N_3}(B(z,\varepsilon))$ is contained in a ball centered at z_1 with radius equivalent to $(\varepsilon/\alpha)N_3^{-1-1/k_3}$. As z_1 is only $O(\beta^{k_1+1})$-close to $f_2^{n_2}(z)$, the inclusion

$$f_1^{N_1} \circ f_3^{N_3}(B(z,\varepsilon)) \subset f_2^{n_2}(B(z,\varepsilon))$$

is not yet proved.

This construction just needs some use of the fine correction germ f_4 to be completed. As f_4 is more tangent to identity that the other germs, we can suppose $k_4 > (k_2+1)(k_3+1)$. Hence, there exists at least one petal of f_4 between the petals of $(f_2^m(z))_{m\in\mathbb{Z}}$ and $(f_3^p(z))_{p\in\mathbb{Z}}$. As the angular progression of $f_1^n(z_3)$ is comparable to the small quantity β^{k_1}, some point $f_1^{N_{1,1}}(z_3)$ $0 \le N_{1,1} \le N_1$ is close to the top of this petal of f_4. Then the argument of $f_1^{N_{1,1}}(z_3)$ is corrected by some iterations of f_4, without sensitive perturbation of modulus. The collection of points $f_4^q(f^{N_{1,1}}(z_3))$ is constitued with points with $0(\beta^{k_4})$-calibrated differences of argument. The iteration of these points by powers of f_1 does not separe their arguments or their moduli because f_1 does not separe points in this region. Then exists a composition φ_4 of f_1 and f_4 such that the argument of $\varphi_4(z_3)$ is $O(\beta^{k_4})$-close to the argument of some point $f^{N_2}(z)$, where $\mid f_2^n(z) - \varphi_4 \circ f_3^{N_3}(z) \mid$ is minimal for $n = N_2$ and the integer N_2 is necessary comparable to n_2 (Figure 5).

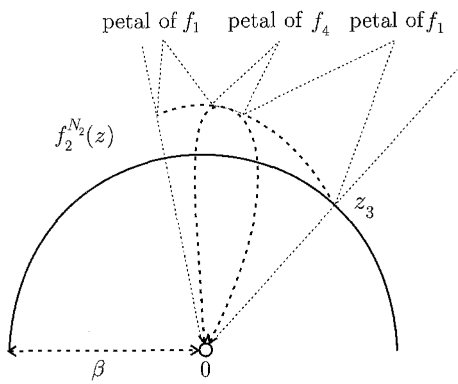

Figure 5

This last estimation of distance and the previous estimations of radius induce the configuration of fixed point implied by the inclusion

$$\varphi_4 \circ f_3^{N_3}(B(z,\varepsilon)) \subset f_2^{N_2}(B(z,\varepsilon)). \qquad \square$$

Let us denote f the previous composition of H and x the previous fixed point which is in the compact ball $B(z,\varepsilon)$. The construction of f with the four germs f_1, f_2, f_3 and f_4 implies that the modulus of $f'(x)$ is arbitrarly small if the internal radius β is arbitrarly small. As a corollary of the theorem we give now an elementary proof of the

Sherbakov-Nakai's density theorem with moreover some freedom in the choice of the derivative. Let H be a non solvable pseudo-group, f_1, f_2, f_3 and f_4 four germs as above and α a radius small enough to get the dynamic of petal for any germ f_j, $j = 1, 2, 3, 4$. Let us remember that k_j is the degree of tangency of f_j.

Corollary. *For any pair x, y of points in a same sector of f_1 and α-close to 0, there exists a germ f of H such that x is in the domain of f, and $f(x)$ is arbitrarly close to y, and $\mid f'(x) \mid$ is either arbitrarly large, arbitrarly small or arbitrarly close to* 1.

Proof. Given $\eta > 0$ we consider the two compact ball $B(x,\eta)$ and $B(y,\eta)$. Then the images $f_2^{N_2}(B(x,\eta))$ and $f_3^{N_3}(B(y,\eta))$ have different size if $\mid f_2^{N_2}(x) \mid$ and $\mid f_3^{N_3}(y) \mid$ are small and equivalent. These images are in the same sector of f_1. As above, there exists a convenient composition of f_1 and f_4 such that $\varphi(f_3^{N_3}(B(y,\eta)))$ is included in $f_2^{N_2}(B(x,\eta))$. Therefore $f_2^{-N_2} \circ \varphi \circ f_3^{N_3}(y)$ is η-close to x and the derivative of this germ in y is arbitrarly small.

The same construction with a large derivative needs one computational detail. We can suppose without any restriction that the moduli of x and y are equivalent, and that the small radius β is smaller than these two moduli. Then the derivative in y of this previous composition is equivalent to N_2/N_3, or $\beta^{1/k_2 - 1/k_3}$. The image

$$f_2^{N_2}(B(x, \eta(\beta^{1/k_2-1/k_3})^2))$$

contains a ball with radius equivalent to $\eta(\beta/\alpha)(\beta^{1/k_2-1/k_3})^2)$. The image $f_3^{N_3}(B(y,\eta)$ is contained in a ball with radius equivalent to $\eta(\beta/\alpha)\beta^{1/k_3})$. A convenient composition of f_1 and f_4 sends the center of one ball $O(\beta^{k_4+1})$-close to the center of the other ball. If we have

$$\beta^{2/k_2-2/k_3} < \beta^{1/k_3} \text{ i.e. } k_3 > \frac{3}{2}k_2$$

then exists a germ of H sending y η-close to x with arbitrarly large derivative. This condition can be always satisfied because there exist, in a non solvable pseudo-group, germs with arbitrarly large degree of tangency to the identity.

To send x close to y with a derivative close to 1 in modulus we take a intermediary point ζ in the convenient sector of f_1, and send x close to z with a large derivative, using the small radius β and ζ close to y with a small derivative, using the small radius β'. Adjusting the two small radii β and β' it is possible to compense almost exactly the largeness and smallness of the two previous derivatives. □

The following remark is a justification to give an other proof of the sectorial density of orbits. Nakai's construction needs three germs, f, g and their commutator. The present construction needs a fourth one. The main consequence is some possible election of the derivative of a germ sending a point arbitrarily close to an other. It follows a direct consequence of the corrolary. This statement is just a bit more precise than the Sherbakov-Nakai's density theorem.

Given η and K strictly positive, there exists an integer $N_{\eta,K}$ depending on η and K such that for any couple (x, y) in the same sector, there exists a germ of H with length smaller than $N_{\eta,K}$ such that

$$x \in \mathrm{Dom}(f), \quad | f(x) - y | < \eta \text{ and } | f'(x) | \leq K.$$

Two notions of entropy are now defined before the proof of some new results of non nullity of entropies. These definitions refer to [**G,L,W**], [**Ma**] and [**K, H**], [**Wi**] and [**G-M,Wi2**].

Definition 3. Topological entropy of pseudo-groups of germs of $\mathrm{Diff}_{0,\mathbb{C}}$.

Let H be a pseudo-group of $\mathrm{Diff}_{0,\mathbb{C}}$ with a finite generator set H_1, containing id and stable by inversion. Fixed $0 < \varepsilon, \beta < \alpha$, α smaller than any radius of convergence of any germ of H_1, and n a positive integer, two complex numbers x and y are $(\alpha, \beta, n, \varepsilon)$-separated if and only if there exists g in H_n such that the partial images of x and y by g are in the compact crown $\{\beta \leq | z | \leq \alpha\}$, and the distance between $g(x)$ and $g(y)$ is at least ε. A part A of $\{\beta \leq | z | \leq \alpha\}$ is $(\alpha, \beta, n, \varepsilon)$-separated if and only if two distinct points of A are $(\alpha, \beta, n, \varepsilon)$-separated. Remark that the cardinal of A is necessarily finite. It is a consequence of the theorem of Bolzano Weierstrass. The maximum cardinal of any

$(\alpha, \beta, n, \varepsilon)$-separated set is denoted $\mathcal{N}(\alpha, \beta, n, \varepsilon)$. The topological entropy of the pseudo-group H relatively to its generator set H_1 and restricted to $\{\beta \leq \mid z \mid \leq \alpha\}$ and denoted $h_{top}(H, H_1, \alpha, \beta)$ is

$$h_{top}(H, H_1, \alpha, \beta) = \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \ln(\mathcal{N}(\alpha, \beta, n, \varepsilon)).$$

Definition 4. Entropy of fixed point of pseudo-groups of $\mathrm{Diff}_{0,\mathbb{C}}$.

Fixed H, H_1, $0 < \beta < \alpha$ and n as above, a complex number x is an (α, β, n)-fixed point if and only if there exists g in $H_n - \{id\}$ such that the partial images of x by g are in $\{\beta \leq \mid z \mid \leq \alpha\}$ and g fixes x. The cardinal of the set of (α, β, n)-fixed points is denoted $\mathcal{M}(\alpha, \beta, n)$. The entropy of fixed point of the pseudo-group H relatively to its generator set H_1 and restricted to $\{\beta \leq \mid z \mid \leq \alpha\}$ and denoted $h_{fix}(H, H_1, \alpha, \beta)$ is

$$h_{fix}(H, H_1, \alpha, \beta) = \limsup_{n \to +\infty} \frac{1}{n} \ln(\mathcal{M}(\alpha, \beta, n)).$$

We claim the two last results. Their proof is a direct consequence of Theorem 1 and a general lemma about separation. Let H be a non solvable pseudo-group.

Theorem 2. *The topological entropy $h_{top}(H, H_1, \alpha, \beta)$ is strictly positive for any radius α if β is small enough.*

Theorem 3. *The entropy of fixed point $h_{fix}(H, H_1, \alpha, \beta)$ is strictly positive for any radius α if β is small enough.*

Proof of theorem 2. The idea refers to [**G-M, Wi2**]. We know by the demonstration of the theorem 1 that there exist a complex z with modulus α, a real ε strictly positive, and a real $\Lambda > 2$ such that for any y in $\{\mid z - x \mid \leq \varepsilon\}$, exists g_y such that

$$\begin{aligned} B(z, \varepsilon) &\subset \mathrm{Dom}(g_y), \\ \mid g_y(y) - z \mid &\leq \varepsilon, \\ \mid g_y'(y) \mid &\geq \Lambda. \end{aligned}$$

As the derivative $g_y'(y)$ can be chosen arbitrarly large or small in modulus, this one is fixed larger larger than 3. Then by continuity of g_y, there exists a small open ball $B^o(y, \varepsilon_y)$ $(= \{\mid t - y \mid < \varepsilon_y\})$ where

$$\forall \ x_1, x_2 \ \in \ B^o(y, \varepsilon_y), \ \mid \frac{g_y(x_1) - g_y(x_2)}{x_1 - x_2} \mid > 2.$$

The compact ball $B(z, \varepsilon)$ is covered not with the open balls $B^o(y, \varepsilon_y)$, but with open balls $B^o(y, \varepsilon_y/2)$. There exists a finite sub-covering by the theorem of Bolzano-Weiertstrass. Let ε_0 be the smaller radius of the balls of this finite sub-covering, and N its cardinal. The center of the open balls of the finite sub-covering are denoted $y_1, y_2, , ..y_N$, their radius are now $\varepsilon_1/2$, $\varepsilon_2/2$, ... , $\varepsilon_N/2$ and f_1, f_2, ... , f_N the locally expansive germs associated to y_1, y_2, ... , y_N. Let L be the maximal length of the compositions f_1, f_2, ... , f_N. Let β be the minimal modulus of any partial image of $B(z, \varepsilon)$ by any germ f_1, f_2, ... , f_N. Assume z_1 and z_2 are in the compact ball $B(z, \varepsilon)$ and assume $\mid z_1 - z_2 \mid = \varepsilon_0 2^{-n}$. Then exists a integer j_1 in $\{1, 2, ..., N\}$ such that

$$\mid z_1 - y_{j_1} \mid \ \leq \ \varepsilon_{j_1}/2 \text{ and } \mid z_2 - z_1 \mid \ \leq \ \varepsilon_{j_1}/2.$$

The triangular inequality implies that $\mid z_2 - y_{j_1} \mid$ is smaller than ε_{j_1} and then $\mid f_{j_1}(z_1) - f_{j_1}(z_2) \mid$ is strictly larger than $\varepsilon 2^{-n+1}$. An immediate induction proves that z_1 and z_2 are $(\alpha, \beta, Ln, \varepsilon_0)$-separated. The maximal cardinal of ε-separated set included in the unit square is approximatively $1/\varepsilon^2$. It implies

$$\mathcal{N}(\alpha, \beta, Ln, \varepsilon_0) \ \geq \ \frac{\varepsilon^2}{2} \frac{1}{2\varepsilon_0^2} 2^{2n}.$$

This minoration gives immediatly

$$h_{top}(H, H_1, \alpha, \beta) \ \geq \ \frac{2 \ln(2)}{L} \ > 0 \qquad \square$$

The chore of the previous argument is as follows: the minoration of the distance between two points induces their mutual separation by the pseudo-group. Hence we get a minoration of the maximal cardinal of seperated set, and hence of the topological entropy. Symmetrically if two points are separated by the pseudo-group, then their mutual distance is minorated, and that implies a majoration of the topological entropy. This is a general result valid for any pseudo-group of $\mathrm{Diff}_{0,\mathbb{C}}$.

Lemma 2. **([Wi].)** *For any pseudo-group of* $\mathrm{Diff}_{0,\mathbb{C}}$ *exists a radius* α *small enough, a constant* K *and an integer* d *only depending on the generator set such that for any inner radius* $\beta < \alpha$, *for any pair* x, y *of the compact crown* $\{\beta \leq\mid z \mid\leq \alpha\}$ *such that* $\mid x - y \mid\leq (\varepsilon\beta/\alpha)\exp(-nK\alpha^d)$, x *and* y *are not* $(\alpha, \beta, n, \varepsilon)$*-separated by the pseudo-group.*

Proof. Let $\tilde{H}_1$ be a finite collection of germs $\{\tilde{g}_1^{\pm 1}, ...\tilde{g}_k^{\pm 1}, id\}$ such that $\tilde{g}_j(z) = \lambda_j z + \mu_j z^{d+1} + ...$, where $\mu_j \neq 0$ for some index $j \in \{1, 2, ..., k\}$. If the radius α is small enough then we have for any complex numbers x and y of modulus smaller than α

$$\exp(-K\alpha^d) \leq \mid \frac{\tilde{g}_j^{\pm 1}(x)}{\lambda_j^{\pm 1} x} \mid, \mid \frac{\tilde{g}_j^{\pm 1}(x) - \mid \tilde{g}_j^{\pm 1}(y)}{\lambda_j^{\pm 1}(x - y)} \mid \leq \exp(+K\alpha^d),$$

where K is some constant number depending only on $\tilde{H}_1$. Let $0 < \varepsilon, \beta < \alpha$ be real numbers and x and y (n, ε)-separated by $\tilde{H}$, g a germ of H_n separating x and y, such that all partial images of x and y by g remain in the compact crown $\{\beta \leq\mid z \mid\leq \alpha\}$. The majoration of any rate of increasing $(\tilde{g}_j(x) - \tilde{g}_j(y))/(x - y)$ implies the minoration of $\mid x - y \mid$ by $\varepsilon \mid g'(0) \mid \exp(-nK\alpha^d)$. The control of $\tilde{g}_j(x)/\lambda_j x$ induces the minoration of $\mid g'(0) \mid$ by $\beta/\alpha \exp(-nK\alpha^d)$. Finally the distance $\mid x - y \mid$ is larger than $(\varepsilon\beta/\alpha)\exp(-2nK\alpha^d)$. By contraposition, if the distance between x and y is smaller than the previous bound then they are not $(\alpha, \beta, n, \varepsilon)$-separated. □

Proof of theorem 3

Fixed points are directly constructed from separated points, using the statement of density which is a more technical version of the Sherbakov-Nakai's density theorem. Moreover the main idea of [**G-M, Wi2**], based of the exponential control of the distance between two separated points is largely the chore of the proof. The real number η is now fixed. There exists an integer N_η such that, for any couple x, y in a same sector, a germs of H_{N_η} sends x η-close to y with a derivative less than 1 in modulus. Lemma 2 induces that this derivative is also more than $\beta/\alpha \exp(-N_\eta K\alpha^d)$, where K and d depend only on the generator set

$\{f_1, f_2, f_3, f_4\}$.

Let m be an integer (think it is very large, more precisely $m >> L, N_\eta$), x and y be in the initial compact ball $B(z, \varepsilon)$ and verify moreover and $\mid x-y \mid= \eta 2^{-m}$, $m >> N_\eta$. As in the proof of theorem 2 the complex numbers x and y are $(\alpha, \beta, mL, \eta)$-separated. Moreover the sequence of partial images of x and y can be qualitatively and quantitatively described. Firstly some iterations of some k_2-flat germs f_2 put x and y close to 0, in some neighbourough of the circle centered at 0 with radius β. Then some combination of germs f_1 and f_4 respectively less and more tangent to identity than f_2 changes the argument of the previous point. At last the point comes back close to the circle of radius α by some iterations of a k_3-flat germ f_3. The number of elementary composition of this first part of the partial images is less than L. Then we compose by identity until the length of the composition is exactly L. This composition with length exactly equal to L is denoted elementary cycle.

The distance between corresponding partial images of x and y during this part of the trajectory called elementary cycle is always at most equivalent to the distance between their final images $\mid x_L - y_L \mid$, because with the estimations of Lemma 1.2, the partial images of x and y are closer one to the other when they are close to the inner circle of radius β. It follows that the sum of the distances between corresponding partial images during an elemetary cycle is dominated by $O(L \mid x_L - y_L \mid)$, because it is a sum of at most L terms and each term is at most equivalent to $\mid x_L - y_L \mid$. Let us recall the inequality.

$$\mid x_L - y_L \mid \; > \; 2 \mid x - y \mid .$$

Let $n \leq m$ be the minimal index of separation of x and y, i.e x and y are (nL, η)-separated but not $((n-1)L, \eta)$-separated by such composition of elementary cycle. Then the separation does not occur by any concatenation of $n-1$ elementary cycles. The three following technical lemmas show that the local action of separating composition is very comparable to an homothecy with exponentially large rate.

Lemma 3.1. *The sum*

$$\sum_{0\leq j\leq (n-1)L} \mid x_j - y_j \mid$$

is dominated by $2L\eta(\alpha/\beta)\exp(LK\alpha^d)$.

Proof. The mutual distance between corresponding partial images of x and y during the last elementary cycle is dominated by

$$\eta(\alpha/\beta)\exp(LK\alpha_d)$$

where d and K depend only on H_1. The sum of the distances between the last elementary cycle is dominated by L multilplied by the previous distance. The distance between $x_{(n-2)L}$ and $y_{(n-2)L}$ is dominated by some $\eta/2$ because $\mid x_{(n-1)L} - y_{(n-1)L} \mid$ is larger than $2 \mid x_{(n-2)L} - y_{(n-2)L} \mid$ by definition of an elementary cycle. Therefore, the sum of mutual distances between corresponding partial images of x and y during the penultian elementary cycle is smaller than $N\eta/2(\alpha/\beta)\exp(LK\alpha_d)$. An immediate induction gives

$$\begin{aligned}\sum_{0\leq j\leq (n-1)L} \mid x_j - y_j \mid &\leq L\eta(\alpha/\beta)\exp(LK\alpha_d)(1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n-1}}),\\ &\leq 2L\eta(\alpha/\beta)\exp(LK\alpha_d)\end{aligned}$$

□

If the process of separation is completed by a last elementary cycle between $x_{(n-1)L}$ and x_{nL}, $y_{(n-1)L}$ and y_{nL}, then the L last distances are dominated by $\eta(\alpha/\beta)\exp(LK\alpha^d)$. As η can be chosen arbitrarly small, the same result persists, and the sum of mutual distances between the corresponding partial images of x and y during the whole process of separation (i.e. the n elementary cycles) remains arbitrarly small if η is arbitrarly small. More precisely it is at most comparable to η. The composition of H_{nL} separating x and y is now denoted g, and $g = g_{nL} \circ g_{nL-1} \circ \ldots \circ g_1$ where g_j is a germ of $\{f_1^{\pm 1}, f_2^{\pm 1}, f_3^{\pm 1}, f_4^{\pm 1}, id\}$. Twin consequences (lemma 3.2 and 3.3) of this fact are now accessible.

Lemma 3.2. *If η is small enough, then $g'(x)/g'(y)$ is $O(\eta)$-close to* 1.

Remark. The previous rate is a complex number, and the statement has to be understood as a result in $\mathbb{C}$, not only a result about modulus of the rate $g'(x)/g'(y)$.

Proof. The derivative $g'(x)$ is the product of all derivative $g'_j(x_{j-1})$, $j \in \{1, 2, nL\}$. The difference $g'_j(x_{j-1}) - g'_j(y_{j-1})$ is a convergente series of valuation at last one with complex coefficients μ_l in factor with $x^l_{j-1} - y^l_{j-1}$. The factor $x_{j-1} - y_{j-1}$ is common to any polynomial $x^l_{j-1} - y^l_{j-1}$ because we have the identity

$$X^l - Y^l = (X - Y)(X^{l-1} + X^{l-2}Y + ... + Y^{l-1})$$

As the moduli of x_j and y_{j-1} are smaller than α, the modulus of

$$(x^{l-1}_{j-1} + x^{l-2}_{j-1}y_{j-1} + ...y^{l-1}_{j-1})$$

is smaller than $l\alpha^{l-1}$ and the modulus of $x^l_j - y^l_j$ is smaller than $\mid x_{j-1} - y_{j-1} \mid (l\alpha^l)$. Therefore the series

$$\sum_{l\geq 1} \frac{\mu_l(x^l_{j-1} - y^l_{j-1})}{x_{j-1} - y_{j-1}}$$

converges absolutely. Hence $g'_j(x_{j-1}) - g'_j(y_{j-1})$ is the product of $x_{j-1} - y_{j-1}$ by some uniformly bounded holomorphic function.

The derivative $g'_j(x_{j-1})$ is close to 1 because $g'_j(0)$ is exactly 1, thus

$$\frac{g'_j(x_{j-1}) - g'_j(y_{j-1})}{g'_j(x_{j-1})}$$

is also the product of $x_{j-1} - y_{j-1}$ by some uniformly bounded holomorphic function. Therefore, since the identity

$$\frac{g'_j(x_{j-1}) - g'_j(y_{j-1})}{g'_j(x_{j-1})} = 1 - \frac{g'_j(y_{j-1})}{g'_j(x_{j-1})}$$

holds, it follows

$$\frac{g'_j(x_{j-1})}{g'_j(y_{j-1})} = 1 + (x_{j-1} - y_{j-1})\Phi(x_{j-1}, y_{j-1}).$$

where Φ is uniformly bounded and holomorphic. Since for X close to 0 the map $1 + X$ is an approximation of the map $\exp(X)$ we prefer an exponential expression of the previous identity :

$$g_j'(x_{j-1})/g_j'(y_{j-1}) = exp((x_{j-1} - y_{j-1})\Psi(x_{j-1}, y_{j-1}))$$

where Ψ is uniformly bounded and holomorphic. Let B be an uniform upperboud valid for any function Ψ on the domain $\{| X | \leq \alpha\} \times \{| Y | \leq \alpha\}$. Hence $f'(x)/f'(y)$ is the product of all elementary terms $g_j'(x_{j-1})/g_j'(y_{j-1})$ and consequently

$$\frac{g'(x)}{g'(y)} = \exp(\sum_{1 \leq j \leq nL} (x_{j-1} - y_{j-1})\Psi(x_{j-1}, y_{j-1})).$$

The majoration of the sum of mutual distances of corresponding partial images of x and y of Lemma 3.1 and the uniform upperbound of any function Ψ induces that the rate $g'(x)/g'(y)$ is the exponential of some quantity, absolutely majorated by $2LB\eta \exp(LK\alpha^d)$. Hence $g'(x)/g'(y)$ is $O(\eta)$-close to 1. □

Lemma 3.3. *If η is small enough, then $(g(x) - g(y)/(g'(x)(x - y))$ is $O(\eta)$-close to* 1.

Proof. The rate of increasing $(g(x) - g(y)/(x - y)$ can also be written as the product of elementary terms depending on x_j and y_j. Then the following identity holds :

$$\frac{g(x) - g(y)}{g'(x)(x - y)} = \prod_{1 \leq j \leq nL} \frac{x_j - y_j}{g_j'(x_{j-1})(x_{j-1} - y_{j-1})}.$$

Some Taylor's formula applied to each elementary term of the product shows once more that each elementary term is $O(| x_{j-1} - y_{j-1} |)$-close to 1. The end of the proof is similar to the previous one. □

Since g separates x and y, the rate $(g(x) - g(y))/(x - y)$ is at least 2^m is modulus. If η is chosen small enough, then Lemma 3.2 induces that $g'(x)$ is larger than 2^{m-1} in modulus. Hence the germ g^{-1} send the open disk centered at $g(x)$ with radius η in an open set contained in the open disk centered at $g(x)$ with radius $\eta 2^{-m+1} \exp(+2LK\alpha^d)$ by lemma 2, 3.2 and 3.3. Using now the corrollary about density of pseudo-orbits, there exists a germ $\tilde{g}$ such that $| \tilde{g}(x) - g(x) | \leq \eta/2$. Moreover the derivative $\tilde{g}'(x)$ is less than 1 in modulus.

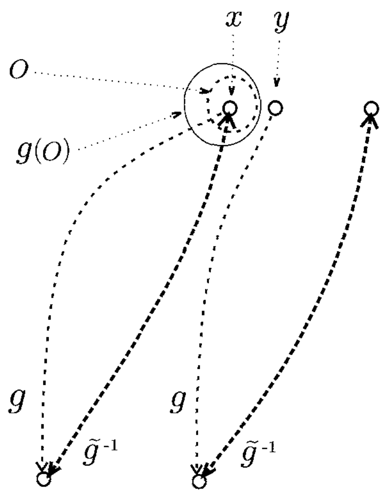

Figure 6

Then the image of $g^{-1}(B(g(x),\eta))$ by $\tilde{g}$ is contained in an open disk centered $\eta/2$-close to $g(x)$ with radius strictly smaller than $\eta/2$ if m is large enough. Then we get a configuration of fixed point and $\tilde{g}\circ g^{-1}$ admits a fixed point close to x

The same construction for y gives also a fixed point close to y. But we can not assume now that the two fixed points are different. We refine the previous construction as follows, and precise that the symbol $\mathcal{E}$ is the function integral part. Given x and y (nL,η)-separated but not $((n-1)L,\eta)$-separated, and separating x and y we choose $\tilde{x}$ close enough to x so that $G(\tilde{x}$ is $(\mathcal{E}(\sqrt{n})L,\eta)$ separated from x, but not $((\mathcal{E}\sqrt{n}-1)L,\eta)$-separated from x. Given $\tilde{G}$ separating $G(x)$ from $G(\tilde{x})$, we construct as above a fixed point close to x. The symmetrical construction with such y and $\tilde{y}$ gives also a fixed point close to y. The two fixed points are necessary different because their images by G are different. As the cardinal of separated points increases at least exponentially with n, the cardinal of fixed points of index $n+\mathcal{E}\sqrt{n}+2N_\eta$ increases also exponentially with n. As n is the principal part of the previous sum, the result is proved. □

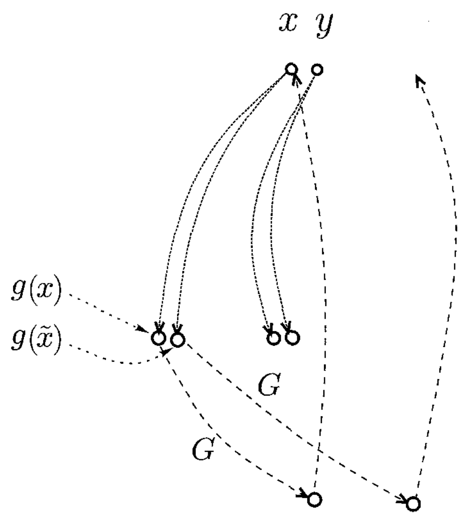

Figure 7

A numerical estimation of entropies

In the generic case (i.e the tangent group is dense in $(\mathbb{C}^*, .)$) the topological entropy is equivalent to α^d, where α is the radius and d the minimal degree of tangency to identity of the commutators of the generator germs [**Wi**]. As the condition of density disappears in the present hypothesis, this estimation is no longer valid. It only subsists the majoration of the entropy by some equivalent of α^d. It is a direct consequence of Lemma 2. As we have seen in Theorem 2 the topological entropy is larger than $\ln(2)/L$, and the integer L is now estimated relatively to α.

The integer L is the length of an elementary cycle. An elementary cycle is decomposed in three phases. During the first an initial point is transported from the circle of radius α to the circle of radius β by some iterations of the k_3-flat germ f_3. The number of iterations N_3 has to be large enough if we want that the estimation of the angular derivative $\Delta f_3^{N_3}$ of Lemma 1.1 holds. The change of variable give for z of modulus α $\tilde{z}$ of modulus α^{-k_3}. As in the demonstration of Lemma 1.1 we replace

$$1 + \frac{a\mu}{\tilde{z} - na\mu + R_n(\tilde{z})} + \dots$$

by $1-1/n$. The error in proportionnal to ε if n is $O(\tilde{z}/\varepsilon)$. The change of variable implies that $\mid \tilde{z} \mid$ is equal to α^{-k_3}. The partial sums $1/1+1/2+ \ldots 1/n$ and $1/(\tilde{z}+1)+1/(\tilde{z}+2)+\ldots+1/(\tilde{z}+n)$ are in a rate controlled by $1 \pm 2\varepsilon$ if

$$1-\varepsilon \;\leq\; \frac{\ln(n+\alpha^{-k_3})-\ln(\alpha^{-k_3})}{\ln(n)} \;\leq\; 1+\varepsilon.$$

Then n is larger than $\alpha^{-k_3/\varepsilon}$. The length L we need in the previous construction (especially for the proof of Theorem 1), is comparable to some power $\alpha^{k_j/\varepsilon}$. The real number ε is supposed to be small enough to get hyperbolicity when we compose some positive power of f_3 with some negative power of f_2. As minoration of the derivative of the contraction $f_3^{N_3}$ is $O(\beta^{1+3\varepsilon+k_3})$ and like the minoration of the derivative of $f_2^{-N_2}$ it is $O(\beta^{-1+3\varepsilon-k_2})$. It is sufficient for ε to be smaller than $(k_3-k_2)/10$, smaller than $1/10$ for example. Therefore the entropy is minorated by $O(\alpha^{10\sup\{k_j, j=1,2,3,4\}})$ of the previous estimation.

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