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Holomorphic Rank of Hypersurfaces with an Isolated Singularity

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—Dedicated to the memory of R. Mañé

Abstract. Let V be a germ at $0 \in \mathbb{C}^n, n \geq 3$, of hypersurface with an isolated singularity at 0. In this paper we prove that the maximal number of germs of vector fields in $V^* = V - 0$, which are linearly independent in all points of V^* is two. In the cases $n = 3, 4$ and of quasi homogeneous hypersurfaces ($\forall n \geq 3$), we prove that this number is one.

Keywords: Hypersurfaces, Rank, Vector fields.

1. Introduction

In this paper we consider the problem of finding the maximal number of holomorphic vector fields in a singular hypersurface with isolated singularity, which are linearly independent in all points.

Let M be a complex manifold of dimension m and $X_1, \ldots, X_k$ be holomorphic vector fields in M. We say that they are linearly independent (briefly l.i.) if for all p$\in$M the vectors $X_1(p), \ldots, X_k(p)$ are linearly independent. The rank of M (denoted by Rank(M)) is the maximum number of holomorphic l.i. vector fields in M. So, for instance Rank(M) ≥ 1, if there exists a holomorphic vector field in M without singularities.More specifically we will consider a germ at $0 \in \mathbb{C}^{n+1}$ of a analytic subset.

Definition 1. *Let V be a germ at $0 \in \mathbb{C}^{n+1}$ of a hypersurface with an isolated singularity at 0. Let $V(U)$ be a representative of V in a*

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neighborhood U of 0. We denote by $V^*(U)$ the set of smooth points of $V(U)$ (in our case $V^*(U) = V(U) - \{0\}$). The rank of V at 0 is by definition

$$\text{Rank}(V) = \max\{\text{Rank}(V^*(U)); V(U) \text{ is a representative of V}\}$$

Our main results are the following:

Theorem 1. *Let V be a germ at $0 \in \mathbb{C}^{n+1}$ of hypersurface with an isolated singularity at 0. Then $1 \leq \text{Rank}(V) \leq 2$. If V is quasi homogeneous, or if $n \leq 3$ then $\text{Rank}(V) = 1$.*

We would like to observe that recently J. Seade proved that V^* admits n C^∞ vector fields which are linearly independent (over $\mathbb{C}$) in all points of V^* ([Se]).

One of the main tools in the proof of Theorem 1 is the following:

Theorem 2. *Let $X_1, \dots, X_k$ be holomorphic vector fields in a complex manifold M of dimension $m \geq k$, and let $D(X_1, \dots, X_k) = D = \{p \in M; X_1(p), \dots, X_k(p)$ are linearly dependent$\}$.If D is not empty, then every irreducible component of D has dimension $\geq k-1$.*

In §2 we will prove Theorem 2 and in §3 Theorem 1.

Theorem 1 motivates the following problems:

Problem 1. *Generalize theorem 1, or give a counter example, for $n \geq 4$.*

Problem 2. *Calculate the rank of germs of analytic sets of codimension bigger than one, with an isolated singularity. The same for sets of codimension one, but with non isolated singular set.*

I would like to thank J. Seade ,who motivated me in the subject, for helpful conversations.

2. Proof of Theorem 2

Let $X_1, \dots, X_k$ be holomorphic vector fields in M and

$$D = D(X_1, \dots, X_k) = \{p \in M; X_1(p), \dots, X_k(p) \text{ are l.d.}\},$$

(where l.d. means "linearly dependent").

Suppose $D \neq \varnothing$ and let us prove that all components of D have dimension $\geq k-1$. It is enough to prove that for any $p \in D$ there exists a neighborhood V of p such that $\dim(D \cap V) \geq k-1$. Fix $p \in D$. By taking a holomorphic coordinate system around p, we can suppose that $p = 0 \in \mathbb{C}^m$ and that $X_1, \ldots, X_k$ are holomorphic vector fields in a neighborhood of 0.

Since $X_1(0), \ldots, X_k(0)$ are l.d., the subspace of $\mathbb{C}^m$ generated by them,

$$(X_1(0), \ldots, X_k(0)) \subset (e_1, \ldots, e_{k-1}) = E,$$

where $\{e_1, \ldots, e_m\}$ is a basis of $\mathbb{C}^m$. Let $r = \dim(X_1(0), \ldots, X_k(0))$.

1st case: $r = k-1$
In this case we can suppose that $(X_1(0), \ldots, X_{k-1}(0)) = E$, so that for x in a neighborhood V of 0 the set $\{X_1(x), \ldots, X_{k-1}(x), e_k, \ldots, e_m\}$ is a basis of $\mathbb{C}^m$. On the other hand we can write $X_k(x) = X_k^{'}(x) + X_k^{''}(x)$, where $X_k^{'}(x) \in (X_1(x), \ldots, X_{k-1}(x))$ and $X_k^{''}(x) \in (e_k, \ldots, e_m)$. Now, observe that $D \cap V = \{x;\ X_k^{''}(x) = 0\}$. Therefore $D \cap V$ is defined by $m-k+1$ equations, which implies that $\dim(D \cap V) \geq k-1$.

2nd case: $r < k-1$
We can suppose that

$$(X_1(0), \ldots, X_k(0)) = (X_1(0), \ldots, X_r(0)) = (e_1, \ldots, e_r).$$

Observe that for $\lambda \neq 0$ the vectors

$$X_1(0), \ldots, X_r(0), X_{r+1}(0) + \lambda.e_{r+1}, \ldots, X_{k-1}(0) + \lambda.e_{k-1}$$

are l.i., whereas the vectors

$$X_1(0), \ldots, X_r(0), X_{r+1}(0) + \lambda.e_{r+1}, \ldots, X_{k-1}(0) + \lambda.e_{k-1}, X_k(0)$$

are l.d.. Let V be a neighborhood of 0 and $\epsilon > 0$ be such that for $x \in V$ and $0 <\mid \lambda \mid< \epsilon$, the vectors

$$X_1(x), \ldots, X_r(x), X_{r+1}(x) + \lambda.e_{r+1}, \ldots, X_{k-1}(x) + \lambda.e_{k-1}$$

are l.i.. Let

$$D(\lambda) = \{x \in V;\ X_1(x), \ldots, X_r(x), X_{r+1}(x) + \lambda.e_{r+1}, \ldots, X_{k-1}(x) + \lambda.e_{k-1}, X_k(x) \quad \text{are l.d. } \}.$$

It follows from the first case that all components of $D(\lambda)$ have dimension $\geq k-1$ for $0 < \lambda < \epsilon$. By a classical result on hyperplane sections, we must have $\dim(D \cap V) \geq k-1$. □

3. Proof of Theorem 1

3.1 – Preliminary results.

Let V be a germ at $0 \in \mathbb{C}^{n+1}$ of hypersurface with an isolated singularity at 0. Let f be a germ at $0 \in \mathbb{C}^{n+1}$ of holomorphic function with an isolated singularity at 0 such that $V = f^{-1}(0)$. We will use the following notations:

1 – f_j for the partial derivative $\partial f/\partial x_j$, $j = 1, \ldots, n+1$.

2 – e_j for the vector $\partial/\partial x_j$, $j = 1, \ldots, n+1$, where $(x_1, \ldots, x_{n+1})$ is a fixed coordinate system around $0 \in \mathbb{C}^{n+1}$.

3 – $\wedge$ for the exterior product of forms or vectors.

Let us fix representatives of V and f in a polydisc $P^{n+1} = P$ around 0, which we will call still V and f. We will use the notations $V^* = V \backslash \{0\}$, $P^* = P \setminus \{0\}$, $\mathcal{O}(V^*)$ for the ring of holomorphic functions on V^* and $\chi(V^*)$ for the set of holomorphic vector fields on V^*.

Lemma 1. *If $n \geq 2$, then any vector field $X \in \chi(V^*)$ can be extended to a holomorphic vector field $\tilde{X}$ on P. The vector field $\tilde{X}$ must satisfy the following relations:*

a) *$\tilde{X}(f) = g.f$, where g is holomorphic on P.*

b) *$\tilde{X}(0) = 0$.*

Proof. It is well known that any $h \in \mathcal{O}(V^*)$ can be extended to a holomorphic function $\tilde{h}$ on P (cf. [G]). Denote by $x = (x_1, \ldots, x_{n+1})$ a coordinate system in $\mathbb{C}^{n+1}$ and set

$$X_j = X(x_j \mid_{V^*}) \in \mathcal{O}(V^*), \quad j = 1, \ldots, n+1$$

Let $h_j \in \mathcal{O}(P)$ be a holomorphic extension of X_j, $j = 1, \ldots n+1$. It is not difficult to see that the vector field $\tilde{X} = \sum_{j=1}^{n+1} h_j.e_j$ is a holomorphic extension of X.

Let us prove a). Observe first that $\tilde{X}$ is tangent to V^*. This implies that the function $h = \tilde{X}(f) = \sum_{j=1}^{n+1} f_j.h_j$ vanishes on V^*. Since f is irreducible we must have $\tilde{X}(f) = g.f$ for some g$\in \mathcal{O}(P)$.

Let us prove b). Suppose by contradiction that $\tilde{X}(0) \neq 0$. In this case, after a change of variables near 0 we can suppose that $\tilde{X} = e_1$. From a) we get $\partial f/\partial x_1 =$ g.f. As the reader can check easily, this implies that the x_1-axis is contained in the singular set of V (unless 0 is not a singular point of V), which is a contradiction. □

Corollary 1. *Let $V \subset \mathbb{C}^{n+1}$, be as above, where $n \geq 2$. Then* Rank$(V) \leq$ 2.

Proof. Let X_1, X_2, X_3 be vector fields in $\chi(V^*)$. It follows from Lemma 1 that they can be extended to holomorphic vector fields on P, which we call still X_1, X_2, X_3, and such that $X_j(0) = 0$, $j = 1, 2, 3$. Since $0 \in D(X_1, X_2, X_3)$ we get that $\dim(D(X_1, X_2, X_3)) \geq 2$, so that $\dim(D(X_1, X_2, X_3) \cap V) \geq 1$, because $\dim(V) = n$. This implies that the vector fields cannot be l.i. on V^*. □

Now we consider the case of vector fields which have f as first integral.

Definition 2. *We say that X is a first integral for f if $X(f) = 0$. We will set*

$$\mathcal{I}(V) = \{Y \in \chi(V^*)\ ; Y \text{ can be extended to a first integral } \tilde{Y} \text{ of } f\ \}$$

Corollary 2. *Let $X, Y \in \mathcal{I}(V)$. Then X and Y are not l.i. on V^*.*

Proof. Let $\tilde{X}$ and $\tilde{Y}$ be extensions of X and Y respectively, which are first integrals of f. Since $\tilde{X}(f) = \tilde{Y}(f) = 0$, $\tilde{X}$ and $\tilde{Y}$ are tangent to the level hypersurfaces of f, f$^{-1}(c)$, $c \in \mathbb{C}$, small.

Since $0 \in D(\tilde{X}, \tilde{Y})$, we get that $D(\tilde{X}, \tilde{Y})$ contains a non constant holomorphic curve γ such that $\gamma(0) = 0$. If $\gamma \subset V$ we are done. Suppose $\gamma \not\subset V$. In this case γ cuts the hypersurfaces f$^{-1}(c)$ for small $| c |> 0$. Since $\tilde{X}$ and $\tilde{Y}$ are both tangent to these hypersurfaces we get that $\dim(D(\tilde{X}, \tilde{Y}) \cap f^{-1}(c)) \geq 1$, for small $| c |> 0$. This implies that $\dim(D(\tilde{X}, \tilde{Y})) \geq 2$. Therefore $\dim(D(\tilde{X}, \tilde{Y}) \cap V) \geq 1$, which implies the Corollary. □

Lemma 2. Rank$(V) \geq 1$.

Proof. Let us suppose first that n is odd, so that n+1=2k. In this case the vector field X on P defined by

$$X = f_2.e_1 - f_1.e_2 + \cdots + f_{2k}.e_{2k-1} - f_{2k-1}.e_{2k} = \sum_{j=1}^{k}(f_{2j}.e_{2j-1} - f_{2j-1}.e_{2j})$$

is tangent to V (because X(f)=0) and in some neighborhood U of 0 it vanishes only at 0. This proves the lemma in this case.

Let us suppose now that n is even, so that $n+1 = 2k+1$. It is well known that there exists a hyperplane E through $0 \in \mathbb{C}^{n+1}$ such that 0 is an isolated singularity of the restriction $f\,|_{P\cap E}$. After a linear change of variables we can suppose that $E = \{x_{2k+1} = 0\}$. Consider the vector field

$$X = \sum_{j=1}^{k}(f_{2j}.e_{2j-1} - f_{2j-1}.e_{2j})$$

Since $X(f) = 0$, X is tangent to V. It is enough to prove that for some neighborhood U of 0 we have Sing$(X)\cap V \cap U = \{0\}$. Suppose by contradiction that X vanishes in points of V^* arbitrarily near 0. This implies that $V \cap \text{Sing}(X)$ contains a non constant holomorphic curve $\gamma(t) = (x_1(t), \ldots, x_{2k+1}(t))$ such that $\gamma(0) = 0$. Now, $X(\gamma(t)) = 0$ implies that $\partial f/\partial x_j(\gamma(t)) \equiv 0$ for all $j = 1, \ldots, 2k$. Since $f(\gamma(t)) \equiv 0$ we get

$$0 \equiv \sum_{j=1}^{2k+1} f_j(\gamma(t)).x_j'(t) = f_{2k+1}(\gamma(t)).x_{2k+1}'(t)$$

This implies that $x_{2k+1}'(t) \equiv 0$, because 0 is an isolated singularity for f. It follows that the curve γ is contained in the hyperplane $\{x_{2k+1} = 0\} = E$, which is a contradiction, because 0 is an isolated singularity for $f\,|_{E\cap P}$ and $f_1, \ldots, f_{2k}$ vanish along γ. □

In the next results we will use the so called "De Rham's division theorem", which we state bellow (cf. [M]).

De Rham's division theorem. *Let $0 \in P$ be a polydisk in $\mathbb{C}^m$ and ω be a holomorphic 1-form with an isolated singularity at 0. If η is a* *holomorphic p-form in P, $1 \leq p \leq m-1$, such that*

$$\omega \wedge \eta = 0$$

then there exists a holomorphic $(p-1)$-form β such that

$$\eta = \omega \wedge \beta$$

As a consequence we obtain the following:

Lemma-3. *Let $V \subset P \subset \mathbb{C}^{n+1}$ and f be as before and η be a holomorphic p-form on P, where $1 \leq p \leq n-1$. Then $\eta\mid_{V*} = 0$ if, and only if,*

$$\eta = df \wedge \theta + f.\mu$$

where θ is a holomorphic $(p-1)$-form and μ is a holomorphic p-form.

Proof. It is not difficult to see that $\eta = df \wedge \theta + f.\mu$ implies that $\eta\mid_{V*} = 0$. We leave the proof for the reader.

Let us suppose that $\eta\mid_{V*} = 0$.Fix $x \in V^*$. It follows from $\eta\mid_{V*} = 0$ that $df_x \wedge \eta_x = 0$, so that $df \wedge \eta = f.\alpha$, for some (p+1)-form α. This implies that $df \wedge \alpha = 0$, and so by De Rham's Theorem we have that $\alpha = df \wedge \beta$, for some p-form β, because $p+1 \leq n$. From this we get $df \wedge (\eta - f.\beta) = 0$. Again by De Rham's Theorem we get $\eta = df \wedge \theta + f.\beta$, for some $(p-1)$-form θ. □

Corollary. *Let $V \subset P$ and f be as before. Let X and Y be holomorphic vector fields on P, tangent to V^*, and $\Omega = dx_1 \wedge \cdots \wedge dx_{n+1}$. Then*

$$i_X\ i_Y\ (\Omega) = df \wedge \theta + f.\mu$$

where θ is a (n-2)-form and μ a (n-1)-form.

Proof. Immediate from the fact that $i_X\ i_Y\ (\Omega)\mid_{V*} = 0$. □

Lemma 4. *Let $0 \in V \subset P \subset \mathbb{C}^{n+1}$ and V^* be as before. If $n \geq 3$ then $H^1(V^*, \mathcal{O}) = 0$.*

Proof. Let $U_j = \{x \in P;\ f_j(x) \neq 0\}$ and $V_j = U_j \cap V^*$, $1 \leq j \leq n+1$ and consider the coverings $\mathcal{U} = (U_j)_{1 \leq j \leq n+1}$ and $\mathcal{V} = (V_j)_{1 \leq j \leq n+1}$ of P^* and V^* respectively. Since the $U_{j's}$ and $V_{j's}$ are Stein $\mathcal{U}$ and $\mathcal{V}$ are Leray coverings, and so it is enough to prove that $H^1(\mathcal{V}, \mathcal{O}) = 0$. We will set $V_{ij} = V_i \cap V_j$, $V_{ijk} = V_i \cap V_j \cap V_k$, $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$.

Let $\mathcal{G}$=$(\mathrm{g}_{ij})_{V_{ij}}$ be a cocycle in $C^1(\mathcal{V},\mathcal{O})$. Since the $U_{j's}$ are Stein we can extend g_{ij} to $\tilde{g}_{ij} \in \mathcal{O}(U_{ij})$ (cf.[**G**]). Consider the coboundary $\mathcal{G}$=$(\mathrm{g}_{ijk} = \tilde{g}_{ij} + \tilde{g}_{jk} + \tilde{g}_{ki})_{U_{ijk}}$ in $C^2(\mathcal{U},\mathcal{O})$. Now, $\mathrm{g}_{ijk} \mid_{V_{ijk}} = 0$, so that g_{ijk}=f.h_{ijk}, where $\mathcal{H}$=$(\mathrm{h}_{ijk})_{U_{ijk}}$ is a cocycle in $H^2(\mathcal{U},\mathcal{O})$. Since $n \geq 3$ we have $H^2(P^*,\mathcal{O}) = 0$ (cf.[**F**]), and so $\mathcal{H} = \delta(\mathcal{K})$ for some $\mathcal{K} \in C^1(\mathcal{U},\mathcal{O})$, which means that $h_{ijk} = k_{ij} + k_{jk} + k_{ki}$, $k_{ij} \in \mathcal{O}(V_{ij})$. This implies that $\mathcal{L} = (l_{ij} = \tilde{g}_{ij} - f.k_{ij})_{V_{ij}}$ is a cocycle in $C^1(\mathcal{U},\mathcal{O})$. On the other hand, $H^1(P^*,\mathcal{O}) = 0$, and so $\mathcal{L} = \delta(\mathcal{M})$, for some $\mathcal{M} = (m_j)_{U_j} \in C^0(\mathcal{U},\mathcal{O})$. If we set $\mathcal{N} = (n_j = m_j|_{V_j})_{V_j}$, then it is not difficult to see that $\mathcal{G} = \delta(\mathcal{N})$. This proves Lemma 4. □

Lemma 5. *Let V and V^* be as before. If $n \geq 2$ then there exists a holomorphic n-form ν on V^* such that $\nu_p \neq 0$ $\forall p \in V^*$.*

Remark. It is possible to prove that ν extends to P if, and only if, 0 is a smooth point of V.

Proof of Lemma 5. Let us consider the coverings $\mathcal{U} = \{U_j;\ j = 1,\ldots,n+1\}$ and $\mathcal{V} = \{V_j;\ j = 1,\ldots,n+1\}$, used in the proof of Lemma 4. Let ν_j be the n-form in U_j defined by $\nu_j = (f_j)^{-1}\ i_{e_j}(\Omega)$, where $\Omega = dx_1 \wedge \cdots \wedge dx_{n+1}$. We have,

$$\Omega = f_j^{-1}.dx_1 \wedge \cdots \wedge dx_{j-1} \wedge df \wedge \cdots \wedge dx_{n+1} = df \wedge \nu_j$$

From the above relation we get that $df \wedge (\nu_i - \nu_j) = 0$ on $U_i \cap U_j$, and so $\nu_i \mid_{V_i \cap V_j} \equiv \nu_j \mid_{V_i \cap V_j}$. This implies that we can define a n-form ν on V^* such that $\nu \mid_{V_j} = \nu_j$. We leave for the reader the proof that ν does not vanishes on V^*. □

Corollary. *Let $X, Y \in \chi(P)$ be tangent to V and θ,μ be such that*

$$i_X\ i_Y\ (\Omega) = df \wedge \theta + f.\mu$$

where $\Omega = dx_1 \wedge \cdots \wedge dx_{n+1}$.. Then

$$\theta \mid_{V^*} =\ i_X\ i_Y\ (\nu)$$

where ν is as in Lemma 5.

Proof. Since X and Y are tangent to V we have X(f)=g.f and Y(f)=h.f, g,h$\in \mathcal{O}(P)$. Now for $j = 1, \dots, n+1$ we have $\Omega = df \wedge \nu_j$ (see the proof of Lemma 5), so that

$$i_Y\ (\Omega) = \text{h.f.}\nu_j\ -\ \text{df} \wedge (i_Y\ \nu_j) \quad \Rightarrow$$

$$i_X\ i_Y\ (\Omega) = \text{h.f.}(i_X\ \nu_j)\ -\ \text{g.f.}(i_Y\ \nu_j)\ +\ \text{df} \wedge (i_X\ i_Y\ \nu_j)$$

This implies that on V we have

$$\text{df} \wedge \theta = \text{df} \wedge (i_X\ i_Y\ \nu_j) \quad \Rightarrow \quad \text{df} \wedge (\theta\ -\ i_X\ i_Y(\nu_j)) = 0$$

Since $\nu\ |_{V_j} = \nu_j$, it follows from the above relation that $\theta\ |_{V^*} = i_X\ i_Y\ (\nu)$, as we wished. □

3.2 – Proof of Theorem 1 in the cases n=2 and n=3

3.2.1 – Proof of Theorem 1 in the case n=2.

Let $X, Y \in \chi(V^*)$ and suppose by contradiction that they are l.i. on V^*. It follows from Lemma 1 that X and Y extend to holomorphic vector fields on P, which we call still X and Y. Since X(0)=Y(0)=0, we get that $0 \in D(X, Y)$, and so $D(X, Y)$ contains some non constant holomorphic curve $\gamma(t)$ such that $\gamma(0) = 0$. Observe that the curve $\gamma \not\subset V$.

Now consider the 1-form $i_X\ i_Y\ (\Omega)$ where $\Omega = dx_1 \wedge dx_2 \wedge dx_3$. It follows from the Corollary of Lemma 3 that there exists a holomorphic function g and a 1-form μ such that

$$i_X\ i_Y\ (\Omega) = g \cdot df + f \cdot \mu \tag{1}$$

Assertion- $g(0) \neq 0$ - In fact, it follows from the Corollary of Lemma 5 that $g\ |_{V^*} =\ i_X\ i_Y\ (\nu)$, where ν is as in Lemma 5. This implies that $\forall x \in V^*$ we have $g(x) \neq 0$, because X and Y are l.i. on V^*. It follows that $g(0) \neq 0$, because $g(0) = 0$ would imply that $g^{-1}(0) \cap V^* \neq \varnothing$.

Now, X and Y are l.d. along γ, and so (1) implies that

$$g_\gamma \cdot df_\gamma + f_\gamma \mu_\gamma = i_{X_\gamma}\ i_{Y_\gamma}\ \Omega = 0 \tag{2}$$

If we set $F(t) = f(\gamma(t))$, $G(t) = g(\gamma(t))$, we get from (2) that,

$$G(t)\ F'(t) = G(t)\ df_{\gamma(t)} \cdot \gamma'(t) = -F(t)\mu_{\gamma(t)} \cdot \gamma'(t) = F(t)\ k(t) \tag{3}$$

Since $G(0) \neq 0$ we can divide both members of (3) by G(t) (for $| t |< \epsilon, \epsilon$ small), getting $F'(t) = h(t)\ F(t)$, where $h(t) = k(t)/G(t)$. It follows that

$$F(t) = F(0) \exp(\int_0^t h(s)\ ds) = 0$$

so that the curve $\gamma \subset V$, which is a contradiction. □

3.2.2 – Proof of Theorem 1 in the case n=3.

Let $X, Y \in \chi(V^*)$ and suppose by contradiction that they are l.i. on V^*.The idea is to prove that there exists $Z \in \chi(P)$ such that $Z(f) =$ 1+g.f, where g$\in \mathcal{O}(P)$, which is not possible if 0 is singular point of f.

As before consider the 1-form $i_X\ i_Y\ (\Omega)$ where $\Omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$. Let θ and μ be such that $i_X\ i_Y\ (\Omega) = df \wedge \theta + f.\mu$. Set $\theta = \sum_{i=1}^4 \theta_i.dx_i$, and for $p \in P$, $K_p = \ker(\theta_p)$.

Assertion 1. $\forall p \in V^*$ we have $\theta_p \neq 0$ and $K_p \perp T_p(V^*)$ (where $\perp$ means transversal), so that $df_p \wedge \theta_p \neq 0$.

Proof. It follows from the Corollary of Lemma 5 that $\theta\ |_{V^*} = \ i_X\ i_Y\ (\nu)$. Since X and Y are l.i. on V^* this implies that $\forall p \in V^*$ we have $\theta_p \neq 0$ and that $K_p \cap T_pV^*$ is the subspace of T_pV^* generated by X(p) and Y(p). Therefore $K_p \perp T_p(V^*)$ as we wished.

Assertion 2. There exist functions $Z_1, \dots, Z_4 \in \mathcal{O}(V^*)$ such that

$$\sum_{i=1}^{4} f_i.Z_i \ = 1 \ \text{ and } \ \sum_{i=1}^{4} \theta_i.Z_i \ = 0$$

This means in other words that the vector field Z defined along V^* by $Z = \sum_{i=1}^4 Z_i.e_i$ satisfies the following relations

$$i_Z\ (df)\ = 1 \ \text{ and } \ i_Z\ \theta\ = 0$$

This will imply the result, because if we extend the functions $Z_{i's}$ to functions $h_1, \dots, h_4 \in \mathcal{O}(P)$ then we get a vector field W=$\sum_{i=1}^4 h_i.e_i$ on P such that i_W(df)=1+g.f, as desired.

Proof of assertion 2. For all $p \in V_j$ we have $df_p \wedge \theta_p \neq 0$. This implies locally the existence of the vector field Z, so that there is a covering

$\mathcal{W} = \{W_\alpha;\ \alpha \in A\}$ of V^* by open sets and a collection $\{Z_\alpha\}_{\alpha\in A}$ of vector fields such that for all $\alpha \in A$

$$i_{Z_\alpha}\ (df)\ = 1 \ \text{ and } \ i_{Z_\alpha}(\theta) = 0$$

Let $\alpha, \beta \in A$ be such that $W_{\alpha,\beta} = W_\alpha \cap W_\beta \neq \varnothing$ and consider $Z_{\alpha,\beta} = Z_\beta - Z_\alpha$. We have $i_{Z_{\alpha,\beta}}$df=0 and so $Z_{\alpha,\beta} \in \chi(W_{\alpha,\beta})$, that is, it is tangent to V^*. Since $i_{Z_{\alpha,\beta}}\ \theta = 0$ and $\theta\ |_{V^*} = \ i_X\ i_Y\ (\nu)$ we get that

$$Z_{\alpha,\beta} = g_{\alpha,\beta}.X + h_{\alpha,\beta}.Y \quad \text{,where } \ g_{\alpha,\beta} \ \text{ and } \ h_{\alpha,\beta} \in \mathcal{O}(W_{\alpha,\beta})$$

Now, since X and Y are l.i., the collections $\{g_{\alpha,\beta}\}_{W_{\alpha,\beta}\neq\varnothing}$ and $\{h_{\alpha,\beta}\}_{W_{\alpha,\beta}\neq\varnothing}$ are cocycles in $C^1(\mathcal{W}, \mathcal{O})$. Hence by Lemma 4 they are coboundaries, and so there exist collections $\{g_\alpha\}_{W_\alpha}$ and $\{h_\alpha\}_{W_\alpha}$ where $g_\alpha, h_\alpha \in \mathcal{O}(W_\alpha)$ such that

$$g_{\alpha,\beta} \ = \ g_\beta \ - \ g_\alpha \quad \text{and} \quad h_{\alpha,\beta} \ = \ g_\beta \ - \ g_\alpha$$

This implies that $Z_\alpha - g_\alpha \cdot X - h_\alpha \cdot Y = Z_\beta - g_\beta \cdot X - h_\beta \cdot Y$ on $W_{\alpha,\beta}$, so that we can define Z along V^* by

$$Z\ |_{W_\alpha} = \ Z_\alpha \ - \ g_\alpha.X \ - \ h_\alpha.Y$$

It is not difficult to see that $i_Z\ (df)\ = 1$ and $i_Z(\theta) = 0$, which proves assertion 2. □

Remark. The above argument could be applied in the general case, $n \geq 4$,if we could obtain the form θ in such a way that for any $p \in V^*$ the equations $i_{Z(p)}\ (df)\ = 1$ and $i_{Z(p)}\ (\theta)\ = 0$ were solvables,for some $Z(p) \in T_pV^*$. In this point we have used that θ is a 1-form if $n = 3$.

3.3 – Proof of Theorem 1 in the quasi homogeneous case

In this section we will suppose that V is quasi homogeneous, that is $V = f^{-1}(0)$, where f is quasi homogeneous. We say that $f\colon \mathbb{C}^m \to \mathbb{C}$ is quasi homogeneous if there are $k_1, \ldots, k_m, k \in \mathbb{N}$ such that $\forall t \in \mathbb{C}$ and $\forall (x_1, \ldots, x_m) \in \mathbb{C}^m$ we have

$$f(t^{k_1}.x_1, \ldots, t^{km}.x_m) = t^k.f(x_1, \ldots, x_m) \tag{1}$$

It is not difficult to see that a function f which satisfies condition (1) must be a polynomial. The following result,due to K. Saito, is known:

Saito's Theorem (cf. [S]). *Let f be a germ at $0 \in \mathbb{C}^m$ of holomorphic function, with an isolated singularity at 0. Then the following are equivalent:*

(a) *There exists a coordinate system $(x_1, \ldots, x_m)$ around 0, such that f in this coordinate system is a germ of a quasi homogeneous polynomial.*

(b) *There exists a germ of holomorphic vector field Z such that $Z(f) = f$.*

Moreover the coordinate system in (a) can be chosen in such a way that the vector field Z is linear and diagonal with all eigenvalues rational and positive.

For instance if f satisfies a relation like in (1) then the vector field Z can be chosen as $\sum_{i=1}^{m} \lambda_j.x_j.e_j$ where $\lambda_j = k_j/k$. Observe that if $Z' = Z + X$, where X is a first integral of f, then $Z'(f) = f$.

From now we fix the quasi homogeneous polynomial f in $\mathbb{C}^m$, where $m = n + 1$, with an isolated singularity at 0.

Given a neighborhood U of 0 and two holomorphic vector fields $X, Y \in \chi(U \cap V^*)$ we will say that they are l.d., if $D(X, Y)$ contains a non constant holomorphic curve $\gamma \subset V$ such that $\gamma(0) = 0$. Two germs X and Y of holomorphic vector fields at $0 \in V$ will be l.d. if they have representatives $\tilde{X}, \tilde{Y} \in \chi(U \cap V)$ which are l.d. (for some U). If they have representatives $\tilde{X}, \tilde{Y} \in \chi(U \cap V)$ which are l.i. in $U \cap V^*$ then we will say that they are l.i. on V.

We will use the following notations:

$$\chi_m = \text{ the set of germs of holomorphic vector fields at} 0 \in \mathbb{C}^m$$

$$\chi_{V^*} = \text{ the set of germs of holomorphic vector fields at} 0 \in V$$

$$\mathcal{D}(V) = \{X \in \chi_{V^*} \;; \forall Y \in \chi_{V^*} \text{ then } X \text{ and } Y \text{ are l.d.}\}$$

Lemma 6. *Let X, Y, A and B be germs of holomorphic vector fields at $0 \in V$, such that X and Y are l.i. on V. There exists $\epsilon > 0$ such that if $\mid s \mid, \mid t \mid < \epsilon$, then $X + s.A$ and $Y + t.B$ are l.i. on V.*

Proof. Let us consider representatives $\tilde{X}, \tilde{Y}, \tilde{A}$ and $\tilde{B}$ of the germs defined in a ball $U = B(0,r)$ around 0, in such a way that $\tilde{X}$ and $\tilde{Y}$ are l.i. on $U \cap V^*$. Consider extensions of these vector fields to a neighborhood U of $0 \in \mathbb{C}^{n+1}$, which we call again X, Y, A and B. Set $X_s = X + s.A$, $Y_t = Y + t.B$ and $D(s,t) = D(X_s, Y_t)$. Observe that $A = \{(p,s,t) \in U \times \mathbb{C}^2;\ p \in D(s,t)\}$ is an analytic subset of $U \times \mathbb{C}^2$. Since X and Y are l.i. on V, $D(0,0)$ is a curve passing through 0 and such that $D(0,0) \cap V^* = \varnothing$. On the other hand, if the Lemma was false, there would exist sequences $(r_n = (s_n, t_n))_n$ and $(p_n)_n$, such that $lim_n\, r_n = 0$, $p_n \in V^* \cap D(r_n)$ and $\mid p_n \mid = \epsilon$ for some small $\epsilon > 0$. We can assume that $(p_n)_n$ converges to some $p \in V^*$. By continuity, it follows that $p \in D(0,0) \cap V^*$, which is a contradiction. □

Lemma 7. *Let f be as before and $Z \in \chi_m$ be such that $Z(f) = u.f$, where u is a unity (that is $u(0) \neq 0$). Then $Z\mid_{V^*} \in \mathcal{D}(V)$.*

Proof. Dividing Z by u if necessary, we can suppose that u=1. Let $\tilde{X} \in \chi_{V^*}$ and let us prove that $\tilde{X}$ and Z are l.d.. Let us consider representatives of $\tilde{X}$ and Z , which we call still $\tilde{X}$ and Z, in a polydisk P around 0 and a holomorphic extension X of $\tilde{X}$. From Lemma 1 we have $X(f) = g.f$ for some holomorphic function g. Let $Y = X - g.Z$. It is not difficult to prove the following facts:

i) X and Z are l.d. on V if, and only if, Z and Y are l.d. on V.

ii) $Y(f) = 0$, so that Y is a first integral of f.

We need a Lemma. □

Lemma 8. *For $1 \leq i,j \leq m$ set $Y_{i,j} = f_j.e_i - f_i.e_j$. Let Y be a first integral of f. Then there exists a antisymetric matrix*

$$A = (a_{i,j})_{1\leq i\leq m}^{1\leq j\leq m}\ ,\ \text{where } a_{i,j} \in \mathcal{O}_m$$

such that

$$Y = \sum_{i,j} a_{i,j}.Y_{i,j}$$

Proof. Let $\Omega = dx_1 \wedge \cdots \wedge dx_m$ and consider the $n = m-1$ form $\omega = i_Y(\Omega)$. We have $df \wedge \omega = df(Y).\Omega = 0$, and so De Rham's Theorem implies that $\omega = df \wedge \theta$, where θ is a $n-1$ form. Set

$$\theta = \sum_{i,j=1}^{i,j=m} a_{i,j}.\alpha_{i,j},$$

where $\alpha_{i,j} = i_{e_j} i_{e_i}(\Omega)$ and $a_{i,j} = -a_{j,i}$. From $df \wedge \Omega = 0$ it is possible to prove that

$$df \wedge i_{e_j} i_{e_i}(\Omega) = i_{Y_{i,j}}(\Omega)$$

so that

$$i_Y(\Omega) = df \wedge \theta = \sum_{i,j} a_{i,j}.i_{Y_{i,j}}(\Omega)$$

which implies the Lemma. □

End of the proof of Lemma 7. Let Z,Y be such that $Z(f) = f$, $Y(f) = 0$, and $Y = \sum_{i,j} a_{i,j}.Y_{i,j}$, where $A = (a_{i,j})_{1 \le i \le m}^{1 \le j \le m}$. We will consider two cases.

1st case: $m = n + 1$ is even. Suppose first that the matrix $A(0) = (a_{i,j}(0))_{1 \le i \le m}^{1 \le j \le m}$ is non singular. In this case 0 is an isolated singularity for Y.

In fact

$$Y = \sum_{i=1}^{m} Y_i.e_i = \sum_{i,j} a_{i,j}.(f_j.e_i - f_i.e_j) \quad \Rightarrow \quad Y_i = 2.\sum_{j=1}^{m} a_{i,j}.f_j$$

Since $A(0)$ is non singular, then $A(p)$ is non singular for p in a neighborhood B of 0. This implies that if $p \in B$ is a singularity of Y, then $f_1(p) = \cdots = f_m(p) = 0$, and so $p = 0$.

Now, it follows from Theorem 2 that $D(Y, Z)$ contains a non constant holomorphic curve γ such that $\gamma(0) = 0$. It is enough to prove that $\gamma \subset V$. Let us prove this fact.

Since $\gamma \subset D(Y, Z)$ and 0 is an isolated singularity for Y, we have that for a small fixed $t \neq 0$ there exists $c \in \mathbb{C}$ such that:

$$Z_{\gamma(t)} = c.Y_{\gamma(t)} \quad \Rightarrow \quad f(\gamma(t)) = df_{\gamma(t)}.Z_{\gamma(t)} = c.df_{\gamma(t)}.Y_{\gamma(t)} = 0$$

so that $\gamma \subset V$,as we wished.

Now suppose that $A(0)$ is singular. Suppose by contradiction that Y and Z are l.i. on V. Since m is even there exists a antisymetric matrix

$K = (k_{i,j})_{1\leq i\leq m}^{1\leq j\leq m}$ such that for small $s \neq 0$ the matrix $A(0) + s.K$ is non singular. We leave the proof of this fact for the reader. Set

$$Y_s = \sum_{i,j}(a_{i,j} + s.k_{i,j}).Y_{i,j} = Y + s.W$$

It follows from Lemma 6 that if s is small enough then Z and Y_s are l.i. on V. On the other hand this contradicts the fact that $A(0) + s.K$ is non singular for $s \neq 0$. This contradiction implies that Z and Y are l.d. on V.

2nd case: $m = n + 1$ is odd, say $m = 2k + 1$. In this case $A(0)$ is singular because it is antisymetric. Let us suppose first that $A(0)$ has rank $m - 1 = 2k$. Fix a neighborhood B of 0 such that $A(p)$ has rank $2k$ for any $p \in B$.

Consider the 2-vector $\Theta = \sum_{i,j} a_{i,j}.e_i \wedge e_j$. Since $A(p)$ has rank $2k$ for any $p \in B$, the $2k$-vector Θ^k does not vanishes on B. It follows that there exists a 1-form $\omega = \sum \omega_i \, dx_i$ such that

$$\text{(2)} \quad \Theta^k = \sum_{i=1}^{m}(-1)^{i+1}\,\omega_i.e_1\wedge\ldots\wedge e_{i-1}\wedge e_{i+1}\wedge\ldots\wedge e_m = i_\omega(e_1\wedge\ldots\wedge e_m)$$

where ω does not vanishes on B.

Observe that ω and Θ satisfy the following properties:

(a)$i_\omega(\Theta) = \sum_{i,j} a_{i,j}(\omega_i.e_j - \omega_j.e_i) = 2\sum_{i,j} a_{i,j}.\omega_i.e_j = 0$.

(b)$i_Y(\omega) = 0$.

(c)$i_{df}(\Theta) = -Y$.

In fact, (c) follows from $Y = \sum_{i,j} a_{i,j}.Y_{i,j}$. Let us prove (a). For any fixed $p \in B$ there is a base $\mathcal{V} = (v_1, \ldots, v_{2k+1})$ of $\mathbb{C}^m$ such that $\Theta = v_1\wedge v_2 + \cdots + v_{2k-1}\wedge v_{2k}$. Let $(\alpha_1, \ldots, \alpha_{2k+1})$ be the dual basis of $\mathcal{V}$. We have $\Theta^k = k!.v_1\wedge\ldots\wedge v_{2k}$. Since $v_1\wedge\ldots\wedge v_{2k+1} = \lambda.e_1\wedge\ldots\wedge e_{2k+1}$, where $\lambda \neq 0$ we get from (2) that $\omega_p = c.\alpha_{2k+1}$, where $c \neq 0$. This implies (a). It is easy to see that (a) implies (b). We leave the proof of this fact for the reader.

Now, since $\omega_0 \neq 0$ there exists a vector $e \in \mathbb{C}^m$ such that $\omega_0.e \neq 0$. By taking a smaller B, if necessary, we can suppose that $\omega_p.e \neq 0$ for all

$p \in B$. Consider the analytic sets

$$E = \{p \in B \ ; \ Z(p) \wedge Y(p) \wedge e = 0\} \quad \text{and} \quad F = \{p \in B \ ; \ \omega_p.Z(p) = 0\}$$

Observe that F has codimension one, E has dimension ≥ 2 (Theorem 2) and $0 \in E \cap F$. This implies that $\Sigma = E \cap F$ has dimension ≥ 1, and $0 \in \Sigma$. Therefore Σ contains a non constant holomorphic curve γ such that $\gamma(0) = 0$. The following assertion will finish the proof:

Assertion - $\Sigma \subset V \cap D(Y, Z)$

Proof. Let $p \in \Sigma$. Since $Z(p) \wedge Y(p) \wedge e = 0$, it follows that there are $a, b, c \in \mathbb{C}$, not all zero, such that

$$a.Z(p) + b.Y(p) + c.e = 0 \qquad (*)$$

Since $p \in F$ and $\omega_p.Y(p) = 0$, if we apply ω_p to (*) we get $c.\omega_p.e = 0 \Rightarrow c = 0$, so that $a.Z(p) + b.Y(p) = 0$, which implies that $p \in D(Y, Z)$. Suppose first that $a \neq 0$. In this case if we apply df_p to $a.Z(p) + b.Y(p) = 0$ we get $a.f(p) = a.df_p.Z(p) = 0$, and so $p \in V \cap D(Y, Z)$ as we wished.

Let us consider the case $a = 0$. In this case we must have $Y(p) = 0$. On the other hand (b) implies that:

$$i_\omega(e_1 \wedge \ldots \wedge e_m) = \Theta^k \ \Rightarrow \ i_{df}(i_\omega(e_1 \wedge \ldots \wedge e_m)) = -k.Y \wedge \Theta^{k-1}$$

The above relation implies that if $Y(p) = 0$ then $df_p \wedge \omega_p = 0$, and so $df_p = \alpha.\omega_p$. Hence $f(p) = df_p.Z(p) = \alpha.\omega_p.Z(p) = 0$, because $p \in F$. Therefore $p \in V \cap D(Y, Z)$, as we wished.

Let us suppose now that the rank of $A(0)$ is less than $2k$. The proof is similar to that we have done in the case m even. Suppose by contradiction that Y and Z are l.i. on V. Since m is odd there exists a antisymetric matrix $K = (k_{i,j})_{1 \leq i \leq m}^{1 \leq j \leq m}$ such that for small $s \neq 0$ the matrix $A(0) + s.K$ has rank $2k$. We leave the proof of this fact for the reader. Set

$$Y_s \ = \ \sum_{i,j}(a_{i,j} + s.k_{i,j}).Y_{i,j} \ = \ Y + s.W$$

It follows from Lemma 6 that if s is small enough then Z and Y_s are l.i. on V. On the other hand this contradicts the fact that $A(0) + s.K$ has rank $2k$. This contradiction implies that Z and Y are l.d. on V. This ends the proof of Lemma 7.

End of the proof of Theorem 1.

Let $\tilde{X}, \tilde{Y} \in \chi_{V*}$ and let us prove that they are l.d. on V. Suppose by contradiction that they are l.i. on V. Consider holomorphic extensions X and Y of $\tilde{X}$ and $\tilde{Y}$ respectively. We have $X(f) = g.f$ and $Y(f) = h.f$. It follows from Lemma 7 that $g(0) = 0$ and $h(0) = 0$. Let Z be such that $Z(f) = f$. Lemma 6 implies that there exists $\epsilon > 0$ such that if $s \leq \epsilon$ than $Y_s = Y + s.Z$ and X are l.i. on V. On the other hand $Y_s(f) = u.f$ where $u = g + s$, so that $u(0) = s \neq 0$. Hence Lemma 7 implies that $Y_s \in \mathcal{D}(V)$, which is a contradiction. This ends the proof of Theorem 1.

□

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