Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol. 29, N. 2, 229-251

Quasi-conformal mapping theorem and bifurcations

Robert Roussarie

—Dedicated to the memory of R. Mañé

Abstract. Let H be a germ of holomorphic diffeomorphism at $0 \in \mathbb{C}$. Using the existence theorem for quasi-conformal mappings, it is possible to prove that there exists a multivalued germ S at 0, such that $S(ze^{2\pi i}) = H \circ S(z)$ (1). If H_λ is an unfolding of diffeomorphisms depending on $\lambda \in (\mathbb{C}, 0)$, with $H_0 = Id$, one introduces its ideal $\mathcal{I}_H$. It is the ideal generated by the germs of coefficients $(a_i(\lambda), 0)$ at $0 \in \mathbb{C}^k$, where $H_\lambda(z) - z = \sum a_i(\lambda) z^i$. Then one can find a parameter solution $S_\lambda(z)$ of (1) which has at each point z_0 belonging to the domain of definition of S_0, an expansion in series $S_\lambda(z) = z + \sum b_i(\lambda)(z - z_0)^i$ with $(b_i, 0) \in \mathcal{I}_H$, for all i.

This result may be applied to the bifurcation theory of vector fields of the plane. Let X_λ be an unfolding of analytic vector fields at $0 \in \mathbb{R}^2$ such that this point is a hyperbolic saddle point for each λ. Let $H_\lambda(z)$ be the holonomy map of X_λ at the saddle point and $\mathcal{I}_H$ its associated ideal of coefficients. A consequence of the above result is that one can find analytic intervals σ, τ, transversal to the separatrices of the saddle point, such that the difference between the transition map $D_\lambda(z)$ and the identity is divisible in the ideal $\mathcal{I}_H$. Finally, suppose that X_λ is an unfolding of a saddle connection for a vector field X_0, with a return map equal to identity. It follows from the above result that the Bautin ideal of the unfolding, defined as the ideal of coefficients of the difference between the return map and the identity at any regular point $z \in \sigma$, can also be computed at the singular point $z = 0$. From this last observation it follows easily that the cyclicity of the unfolding X_λ is finite and can be computed explicitly in terms of the Bautin ideal.

Keywords: Quasi-conformal, bifurcations, mapping theorem.

I. Introduction and results.

Let $H(z)$ be a germ of holomorphic diffeomorphism at $0 \in \mathbb{C}$, such that $H(0) = 0$. One can ask the following question: is there a germ of analytic multivalued mapping $S(z)$, with $S(0) = 0$, such that the new value of

Received 13 March 1998.

$S(z)$ obtained after one turn around the origin is equal to $H \circ S(z)$? Writing $S(ze^{2\pi i})$ for this second determination of S, one can write:

$$S(ze^{2\pi i}) = H \circ S(z) \quad \text{with} \quad S(0) = 0. \tag{1}$$

Let us formulate this question in the universal covering $\pi = \exp$: $\widetilde{\mathbb{C}} \longrightarrow \mathbb{C} - \{0\}$. Let be $\widetilde{z} = \widetilde{x} + i\widetilde{y}$ the coordinate in $\widetilde{\mathbb{C}}$. We will call neighborhood of $\widetilde{0} = \{\widetilde{x} = -\infty\}$, any open set of $\widetilde{\mathbb{C}}$ of the form $\{\widetilde{z} \mid \widetilde{x} < \rho(\widetilde{y})\}$ where ρ is some continuous function of $\widetilde{y}$. The germ H lifts as a diffeomorphism $\widetilde{H} : \widetilde{\mathbb{D}} \rightarrow \widetilde{\mathbb{C}}$, where $\widetilde{\mathbb{D}}$ is the neighborhood of $\widetilde{0}$ equal to $\pi^{-1}(\mathbb{D} - \{0\})$ and $\mathbb{D}$ is some disk centered at $0 \in \mathbb{C}$ on which H is defined. The multivalued mapping S lifts as a diffeomorphism $\widetilde{S} : \widetilde{W} \rightarrow \widetilde{\mathbb{C}}$, where $\widetilde{W}$ is a neighborhood of $\widetilde{0}$ and must verify:

$$\widetilde{S}(\widetilde{z} + 2\pi i) = \widetilde{H} \circ \widetilde{S}(\widetilde{z}) + 2\pi i \quad \text{with} \quad \widetilde{S}(\widetilde{0}) = \widetilde{0}. \tag{$\widetilde{1}$}$$

The last condition means that the real part of $\widetilde{S}(\widetilde{z})$ tends to $-\infty$ when $\widetilde{x}$ tends to $-\infty$ and $\widetilde{y}$ remains in some arbitrary compact subset of $\mathbb{R}$.

This question was indirectly solved by Pérez-Marco and Yoccoz in [**P.Y**] by finding a local analytic vector field on $\mathbb{C}^2$, near a hyperbolic saddle point, with a germ of holonomy equals to H, above a loop around the singular point on the unstable manifold : the transition map D from a section to the stable manifold toward a section to the unstable manifold is a solution of the equation (1).

Here, we want to present a direct construction for the solution of (1) (or ($\widetilde{1}$)), based on the quasi-conformal mapping theorem of Ahlfors-Bers. This proof has the advantage to be simpler than the one by Pérez-Marco and Yoccoz because we will remain in the complex dimension 1. Moreover, we are interested in parameter families as we will explain now and our proof easily extends to parameter families.

We suppose that λ is a parameter near $0 \in \mathbb{C}^\ell$ and we consider a germ of analytic parameter family of diffeomorphisms $H_\lambda(z)$ at $0 \in \mathbb{C}$, with $H_\lambda(0) = 0$ for all λ. This germ of family will be represented by an analytic mapping $H_\lambda(z) : \mathbb{D} \times P \rightarrow \mathbb{C}$ where $\mathbb{D}$ is a disk centered at $0 \in \mathbb{C}$ and P is a neighborhood of $0 \in \mathbb{C}^\ell$. For any $\lambda \in P$, $z \rightarrow H_\lambda(z)$ is an holomorphic diffeomorphism of $\mathbb{D}$ into $\mathbb{C}$, with $H_\lambda(0) = 0$.

For analytic families of functions, one can define the following ideal of functions in the parameter space:

Definition 1. **(Ideal of coefficients.)** *Let $(f_\lambda)_\lambda$ be an analytic family of functions defined on $U \times P \subset \mathbb{C} \times \mathbb{C}^\ell$, $f_\lambda(z) : U \times P \longrightarrow \mathbb{C}$, where U is a connected open set in $\mathbb{C}$ and P an open neighborhood of $0 \in \mathbb{C}^\ell$, the parameter space. Let z_0 be any point in U and let us consider the series:*

$$f_\lambda(z) = \sum_{i=0}^{\infty} a_i(\lambda)(z - z_0)^i.$$

The ideal of coefficients of the family $\mathcal{I}(f_\lambda)$, is the ideal generated by the germs of coefficients $\widetilde{a}_i = (a_i, 0)$, in the ring $\mathcal{O}_0(\mathbb{C}^\ell)$ of analytic germs of functions at $0 \in \mathbb{C}^\ell$.

It is easy to obtain the following property and alternative definition for $\mathcal{I}(f_\lambda)$:

Proposition 1. *The ideal $\mathcal{I}(f_\lambda)$ is independent of the choice of the base point z_0. Moreover it is also generated by the germs of the functions: $\lambda \longrightarrow f_\lambda(z)$, when $z \in U$.*

Proof. The first assumption was proved in [$\mathbf{R}_1$]. Let be $\mathcal{I}$ the ideal generated by the germs $\lambda \longrightarrow f_\lambda(z_1)$, for all $z_1 \in U$. The function $\lambda \longrightarrow f_\lambda(z_1)$ is the constant coefficient of the expansion of $f_\lambda(z)$ in $(z - z_1)$. Then, it follows from the first assumption that $\mathcal{I} \subset \mathcal{I}(f_\lambda)$. Now, because the ideal $\mathcal{I}$ is closed it contains the germ

$$\lambda \longrightarrow \frac{df_\lambda}{dz}(z_0) = \lim_{z \longrightarrow z_0} \frac{f_\lambda(z) - f_\lambda(z_0)}{z - z_0},$$

and by an induction on i, it contains also any mapping

$$\lambda \longrightarrow \frac{\partial^i f_\lambda}{\partial z^i}(z_0).$$

This implies that $\mathcal{I}(f_\lambda) \subset \mathcal{I}$ and the equality between the two ideals. □

Remark. Obviously, the ideal $\mathcal{I}(f_\lambda)$ is attached to the germ of family $(f_\lambda, (0,0))$ at $(0,0) \in \mathbb{C} \times \mathbb{C}^\ell$.

We can now formulate the existence result for the equation (1) and a family H_λ:

Theorem 1. *Let $H_\lambda(z)$ be a germ of analytic parameter family of diffeomorphisms at $0 \in \mathbb{C}$, with $H_\lambda(0) = 0$ for any λ and $H_0(z) \equiv z$. Then there exists a solution $\widetilde{S}_\lambda(\widetilde{z})$ of the equation $(\widetilde{1})$, defined and analytic on $\widetilde{W} \times P$, where $\widetilde{W}$ is some neighborhood of $\widetilde{0}$, and P is chosen small enough. Moreover, for any $\widetilde{z} \in \widetilde{W}$, the map $\lambda \longrightarrow (\widetilde{S}_\lambda(\widetilde{z}) - \widetilde{z})$ has a germ at $0 \in C^\ell$ in the ideal $\mathcal{I}(H_\lambda - \mathrm{Id})\Big($i.e., in the ideal of coefficients of the germ of family $(H_\lambda(z) - z)_\lambda\Big)$.*

Remarks. **1.** A stronger result could be a parameter version of the theorem of Pérez-Marco, Yoccoz: does there exist an analytic parameter family of vector fields,

$$X_\lambda = -x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + P_\lambda$$

where any coefficient of the field P_λ is in the ideal $\mathcal{I}(H_\lambda - \mathrm{Id})$?

2. The size of the domain $\widetilde{W}$ which is defined by the function ρ depends on the property of H_λ, because this domain can be obviously obtained as the union of the iterations by H_λ of a strip $\widetilde{W}_0 = \{\widetilde{x} + i\widetilde{y} \mid \widetilde{x} \le A, 0 \le \widetilde{y} \le 2\pi\}$. Let us suppose that P is compact. It is easy to see that one can always take $\rho(\widetilde{y}) = -c_1 - c_2 \mid \widetilde{y} \mid$ for some c_1, $c_2 > 0$. It is possible to have a more precise and useful estimation when one allows the domain $\widetilde{W}$ to depend on λ, i.e. defined by a continuous function $\rho_\lambda(\widetilde{y})$ depending on λ. Let $\Sigma \subset P$ be the subset defined by $\mid H'_\lambda \mid = 1$. Using the proof of Il'yashenko in [**I**] (see also [$\mathbf{R}_2$] for instance), it is easy to find a continuous function $\rho_\lambda(\widetilde{y})$ which verifies $\rho_\lambda(\widetilde{y}) \ge -c_3(1 + \mid \widetilde{y} \mid)^{\frac{1}{2}}$, for all $\lambda \in \Sigma$, where c_3 is some positive constant. The multivalued function $S_\lambda(z)$ is then quasi-analytic in the sense of [**I**], when $\lambda \in \Sigma$. This means that S_λ is determined by its Dulac formal series at $z = 0$ when $\lambda \in \Sigma$ (it is easily proved that such series exists for any λ).

Let $(X_\lambda)_\lambda$ be an holomorphic family of vector fields near a saddle point at $0 \in \mathbb{C}^2$. One can suppose that X_λ is defined in some neighborhood B of $0 \in \mathbb{C}^2$, for $\lambda \in P$, some neighborhood of $0 \in \mathbb{C}^\ell$ and that the local unstable manifold W^u and the local stable one W^s of X_λ are independent of λ. Let σ, τ be two sections, transversal respectively to W^s and W^u. Let H_λ be the holonomy map on τ, above a loop in

$W^u - \{0\}$, based at the point $\tau \cap W^u$. The transition map D_λ from σ to τ is a multivalued germ on σ at the point $\sigma \cap W^s$. If we identify σ and τ with a disk $\mathbb{D} \subset \mathbb{C}$ centered at 0 and the points $\sigma \cap W^s, \tau \cap W^u$ with 0, the map $\widetilde{D}_\lambda$ verifies the equation $(\widetilde{1})$:

$$\widetilde{D}_\lambda(\widetilde{z} + 2\pi i) = \widetilde{H}_\lambda \circ \widetilde{D}_\lambda(\widetilde{z}) + 2\pi i, \quad \widetilde{D}_\lambda(\widetilde{0}) = \widetilde{0}.$$

The theorem 1 can be used to obtain a "good" parametrization of the transversal σ, well-adapted to the holonomy map H_λ:

Theorem 2. *If $\sigma \simeq \mathbb{D}$ and P are chosen small enough, there exists an analytic change of coordinate $C_\lambda(z) : \mathbb{D} \times P \to \mathbb{D}$ on the transversal $\sigma \simeq \mathbb{D}$, depending analytically on λ and verifying $C_\lambda(0) = 0$, such that the germ of the function $\lambda \longrightarrow (D_\lambda \circ C_\lambda(z) - z)$ belongs to $\mathcal{I}(H_\lambda - \mathrm{Id})$ for all z $\Big($ this means that the ideal $\mathcal{I}(D_\lambda \circ C_\lambda - \mathrm{Id})$ is contained in the holonomy ideal: $\mathcal{I}(H_\lambda - \mathrm{Id})\Big)$.*

Let us consider now a real analytic family of vector fields which unfolds a saddle connection. We suppose that X_0 has a saddle singular point at $0 \in \mathbb{R}^2$ with a saddle connection Γ made by the coincidence of one stable and one unstable separatrix. The family X_λ is an unfolding of X_0, defined near Γ, for λ near $0 \in \mathbb{R}^\ell$. Here, we are interested to unfolding of infinite codimension. This means that the return map for X_0, along Γ is equal to identity. In $[\mathbf{R}_1]$, it was proved that the number of limit cycles (isolated closed orbits) which bifurcate from Γ, for λ near 0, is always finite. This bound of the number of limit cycles is called the cyclicity of the germ (X_λ, Γ) and denoted $Cycl\ (X_\lambda, \Gamma)$ (see $[\mathbf{R}_1]$, $[\mathbf{R}_2]$ for a precise definition). Then the result in $[\mathbf{R}_1]$ was that $Cycl\ (X_\lambda, \Gamma) < \infty$.

Let σ and τ be two analytic sections to the stable and unstable local separatrices of X_0, contained in Γ. Let $\sigma^+ \simeq [0, X[$ be the half-section on the side on which the return map is defined along Γ, with $0 \simeq \sigma^+ \cap \Gamma$. We can suppose that, for $\lambda \in P \subset \mathbb{R}^\ell$, some neighborhood of $0 \in \mathbb{R}^\ell$, the Dulac map $D_\lambda(x)$ (transition from σ to τ near the saddle point) as well as the regular transition $R_\lambda(x)$, for $-X_\lambda(x)$, above the regular arc of Γ between σ and τ, are defined from $\sigma^+ \times P$ to τ. The limit cycles of X_λ

near Γ are in one to one correspondence with the roots of the equation:

$$\Delta_\lambda(x) = D_\lambda(x) - R_\lambda(x) = 0. \tag{2}$$

The cyclicity $Cycl\ (X_\lambda, \Gamma)$ is equal to the minimum of the number of roots of (2) in σ^+, for $\lambda \in P$, if P and σ^+ are chosen small enough.

The proof of the finite cyclicity in [$\mathbf{R}_1$] is based on a (x,ω)-expansion of order k, k large enough, of Δ_λ:

$$\Delta_\lambda(x) = \sum_{i=0}^{k} \beta_i(\lambda)[x^i + \cdots] + \sum_{j=1}^{k} \alpha_j(\lambda)[x^j\omega + \cdots] + \psi_k(x,\lambda). \tag{3}$$

Here β_i and α_j are analytic coefficients independent of the order k;

$$\omega(x,\lambda) = \frac{x^{-\alpha_1(\alpha)} - 1}{\alpha_1(\lambda)}$$

where $\alpha_1(\lambda) = 1 + \dfrac{\lambda_2}{\lambda_1}(\lambda)$ $\Big(\lambda_2(\lambda) < 0 < \lambda_1(\lambda)$ are the eigenvalues of X_λ at the saddle point 0 ; $\alpha_1(0) = 0\Big)$. The sums $+\cdots$ are finite polynomial expansions in the monomials $x^\ell\omega^s$ of order strictly larger than the first monomial in the bracket, and $\psi_k(x,\lambda)$ is a remainder of class $\mathcal{C}^k$ which is $k-$flat at $x = 0$ for any λ :

$$\psi_k(0,\lambda) = \cdots = \frac{\partial^k\psi_k}{\partial x^k}(0,\lambda) = 0.$$

The expansion (3) can be written for $(x,\lambda) \in \sigma^+ \times V_k$ (where V_k is a neighborhood of 0 in P, depending on k).

One defines the ideal $\mathcal{I}_0$ associated to the unfolding (X_λ) as the ideal generated by the germs of coefficients α_i, β_j in $\mathcal{O}_0\ (\mathbb{C}^\ell)$.

In [$\mathbf{R}_1$], one has also defined the Ideal of Bautin $\mathcal{I}$ of the unfolding (X_λ) as the ideal generated by the coefficients of the family $(\Delta_\lambda(x))_\lambda$ for $x \in]0, X[$ (the regular points of σ^+). Recall that this ideal is generated by the coefficients of the expansion of the real analytic map $\Delta_\lambda(x)$ at any point $x \neq 0$, or equivalently by the germs $\lambda \to \Delta_\lambda(x)$ for $x \in \sigma^+ - \{0\}$.

Let $\mathcal{I}_H$ be as above the ideal $\mathcal{I}(H_\lambda(z) - z)$ associated to the holonomy map at the saddle point. For a real family $(X_\lambda)_\lambda$ the three ideals $\mathcal{I}_H$, $\mathcal{I}_0$, $\mathcal{I}$ are real,i.e. generated by real analytic functions and in [$\mathbf{R}_1$] it was proved that: $\mathcal{I}_H \subset \mathcal{I}_0 \subset \mathcal{I}$. Now, as a direct consequence of theorem 2, one can prove the following:

Theorem 3. $\mathcal{I}_0 = \mathcal{I}$: *for any* $x_0 \notin 0$, *the germ* $\{\lambda \longrightarrow \Delta_\lambda(x_0)\}$ *belongs to the ideal* $\mathcal{I}_0$.

Finally, a simple argument of division of the map $\Delta_\lambda(x)$ in the ideal $\mathcal{I}_0$ implies the finite cyclicity and, moreover gives an explicit bound for $\mathcal{C}ycl\ (X_\lambda, \Gamma)$:

Theorem 4. *The cyclicity of* (X_λ, Γ) *is finite. Moreover, let be* ℓ *the smallest integer such that the first* ℓ *coefficients in the list* β_0, α_1, β_1, $\alpha_2 \dots$ *generate the ideal* $\mathcal{I}_0$. *Then,* $\mathcal{C}ycl\ (X_\lambda, \Gamma) \leq \ell$.

Remark. The finite cyclicity was already established in $[\mathbf{R}_1]$, using the division in the ideal $\mathcal{I}$. But because it was not possible to control the factors in the division, no explicit bound was obtained for it.

In the second paragraph, one gives the proof of theorem 1, based on the Ahlfors-Bers theorem and deduce from it in the third paragraph, the theorem 2 about the division of the Dulac transition in the holonomy ideal. In the last paragraph one establishes the theorems 3,4 for the cyclicity of unfoldings of saddle connections.

This paper follows from discussions with Christiane Rousseau about the possibility to apply a Khovanskii method to the study of bifurcations of infinite codimension, during a stay in the Centre de Mathématiques de l'Université de Montreal in january 1997. I thank her for the kind invitation.

II. Construction of multivalued mappings.

We suppose that $H_\lambda(z)$ is a family of holomorphic diffeomorphisms of $\mathbb{C}$, defined on $\mathbb{D} \times P$, where $\mathbb{D}$ is a disk centered at $0 \in \mathbb{C}$, and P is a compact neighborhood of $0 \in \mathbb{C}^\ell$, the parameter space. Moreover, one assumes that $H_\lambda(0) = 0$ for $\forall \lambda$ and that $H_0(z) \equiv z$. Let be $\pi = \exp\ :\ \widetilde{\mathbb{C}} \longrightarrow \mathbb{C} - \{0\}$, the universal covering map.

The family $H_\lambda(z)$ lifts on $\widetilde{\mathbb{C}}$, as a family of diffeomorphisms $\widetilde{H}_\lambda(\tilde{z})$ with $\tilde{z} \in \widetilde{\mathbb{D}} = \pi^{-1}(\mathbb{D})$ and $\lambda \in P$. For each $\lambda \in P$, the diffeomorphism

$\tilde{z} \longrightarrow \widetilde{H}_\lambda(\tilde{z})$ verifies:

$$\widetilde{H}_\lambda(\tilde{z}+2\pi i) = \widetilde{H}_\lambda(\tilde{z}) + 2\pi i. \tag{4}$$

We will write $\widetilde{H}_\lambda(\tilde{z}) = \tilde{z} + \tilde{h}_\lambda(\tilde{z})$, with $\tilde{h}_\lambda$ a $2\pi i$-periodic function such that $\tilde{h}_\lambda(\tilde{0}) = 0$ and $\tilde{h}_0(\tilde{z}) \equiv 0$.

We are looking for a holomorphic family of diffeomorphisms $\widetilde{S}_\lambda(\tilde{z})$, defined for $\tilde{z} \in \widetilde{W}$, some neighborhood of $\tilde{0}$ in $\widetilde{\mathbb{D}}$, and $\lambda \in P$, some restricted compact neighborhood of $0 \in \mathbb{C}^\ell$, which must be solution of the equation $(\tilde{1})$:

$$\widetilde{S}_\lambda(\tilde{z}+2\pi i) = \widetilde{H}_\lambda \circ \widetilde{S}_\lambda(\tilde{z}) + 2\pi i \ , \quad \widetilde{S}_\lambda(\tilde{0}) = \tilde{0}.$$

Moreover, we want that the germs of maps $\lambda \longrightarrow S_\lambda(\tilde{z}) - \tilde{z}$ at $\lambda = 0$, belong to the ideal of coefficients $\mathcal{I}_H = \mathcal{I}(H_\lambda - \mathrm{Id})$.

The idea of the construction of $\widetilde{S}_\lambda(\tilde{z})$ is as follows. First, we will construct a smooth solution $\widetilde{\Sigma}_\lambda(\tilde{z})$ of the equation $(\tilde{1})$. In fact $\widetilde{\Sigma}_\lambda(\tilde{z})$ will be smooth in $\tilde{z}$, but holomorphic in λ. As a consequence $\widetilde{\Sigma}_\lambda(\tilde{z})$ will be quasi-conformal in $\tilde{z}$, uniformly in P, and using the Ahlfors-Bers theorem, it will be possible to construct a family $\varphi_\lambda(z)$ of quasi-conformal homeomorphisms near $0 \in \mathbb{C}$, with $\varphi_\lambda(0) = 0$, such that $\widetilde{S}_\lambda(\tilde{z}) = \widetilde{\Sigma}_\lambda \circ \tilde{\varphi}_\lambda(\tilde{z})$ will be a holomorphic solution of $(\tilde{1})$ ($\tilde{\varphi}_\lambda$ is the lift of φ_λ on $\widetilde{\mathbb{C}}$). We will choose for φ_λ a normalized solution of a Beltrami equation on S^2, in such a way that $\widetilde{S}_\lambda(\tilde{z})$ will verify the required analyticity and division properties in function of the parameter λ.

1. Construction of a smooth solution $\widetilde{\Sigma}_\lambda(\tilde{z})$ of $(\tilde{1})$.

We choose η, $0 < \eta < \pi$ and $A \in \mathbb{R}$ and consider the region:

$$\widetilde{W}_0 = \{\tilde{z} \in \widetilde{\mathbb{C}} \mid \tilde{x} \leq A \ , \quad -\eta \leq \tilde{y} \leq 2\pi + \eta\}$$

which is partitioned in three strips:

$$\tilde{b} = \{\tilde{z} \in \widetilde{W}_0 \mid -\eta \leq \tilde{y} \leq \eta\} \ , \quad \tilde{b}_2 = \{\tilde{z} \in W_0 \mid \eta \leq \tilde{y} \leq 2\pi - \eta\}$$

and

$$\tilde{b}_3 = \{\tilde{z} \in \widetilde{W}_0 \mid 2\pi - \eta \leq \tilde{y} \leq 2\pi + \eta\}.$$

We suppose that A is chosen small enough such that $\widetilde{W}_0 \subset \widetilde{\mathbb{D}}$. We define a map $\widetilde{\Sigma}_\lambda(\widetilde{z})$, for $(\widetilde{z},\lambda) \in \widetilde{W}_0 \times P$ (for a neighborhood P small enough), by: $\widetilde{\Sigma}_\lambda(\widetilde{z}) = \widetilde{z}$ for $\widetilde{z} \in \widetilde{b}$, $\widetilde{\Sigma}_\lambda(\widetilde{z}) = \widetilde{H}_\lambda(\widetilde{z})$ for $\widetilde{z} \in \widetilde{b}_3$ and by taking an isotopy between these two definitions in the strip $\widetilde{b}_2$. More explicitly, we choose a smooth real function $\varphi(\widetilde{y}) : \mathbb{R} \longrightarrow [0,1]$ such that $\varphi(\widetilde{y}) \equiv 0$ for $\widetilde{y} \leq \eta$, $\varphi(\widetilde{y}) \equiv 1$ for $\widetilde{y} \geq 2\pi - \eta$ and we define $\widetilde{\Sigma}_\lambda(\widetilde{z})$ by:

$$\widetilde{\Sigma}_\lambda(\widetilde{z}) = \widetilde{z} + \varphi(\widetilde{y})\widetilde{h}_\lambda(\widetilde{z}) \tag{5}$$

for $\widetilde{z} = \widetilde{x} + i\widetilde{y} \in \widetilde{W}_0$. One choose A small enough such that $\widetilde{\Sigma}_\lambda(\widetilde{W}_0) \subset \widetilde{\mathbb{D}}$, $\forall \lambda \in P$.

We can extend the definition (5) by $\widetilde{H}_\lambda$-iterations. First we define a domain $\widetilde{W}$, neighborhood of $\widetilde{0}$ by iterating $\widetilde{W}_0$:

$$\widetilde{W} = \left\{ \begin{array}{l} \widetilde{z} \in \widetilde{\mathbb{C}} \mid \exists n \in \mathbb{Z} \text{ such that } \widetilde{z} - 2\pi i n \in \widetilde{W}_0 \text{ and} \\ \widetilde{H}_\lambda^k \circ \widetilde{\Sigma}_\lambda(\widetilde{z} - 2\pi i n) \in \mathbb{D} \text{ for } 1 \leq k \leq n, \ \forall \lambda \in P \end{array} \right\}. \tag{6}$$

Next we define $\widetilde{\Sigma}_\lambda(\widetilde{z})$ for $(\widetilde{z},\lambda) \in \widetilde{W} \times P$, by:

$$\widetilde{\Sigma}_\lambda(\widetilde{z}) = \widetilde{H}_\lambda^n \circ \widetilde{\Sigma}_\lambda(\widetilde{z} - 2\pi i n) + 2\pi i n. \tag{7}$$

if $\widetilde{z} - 2\pi i n \in \widetilde{W}_0$.

It is clear, from the construction of $\widetilde{\Sigma}_\lambda$ on $\widetilde{W}_0$, that this definition is coherent (independent of the choice of n) and that $\widetilde{\Sigma}_\lambda(\widetilde{z})$ verifies the equation $(\widetilde{1})$ and the continuity property $\widetilde{\Sigma}_\lambda(\widetilde{0}) = \widetilde{0}$.

Next, because $\widetilde{h}_\lambda(\widetilde{0}) = 0$ (i.e. $\widetilde{h}$ tends to 0 when $\widetilde{y}$ tends to $-\infty$), it is clear that if A is small enough and for a fixed P, $\widetilde{\Sigma}_\lambda(\widetilde{z})$ is a smooth diffeomorphism of $\widetilde{W}$ into $\widetilde{\mathbb{C}}$ for $\forall\lambda \in P$, which is holomorphic in λ.

In fact, $\widetilde{\Sigma}_\lambda(\widetilde{z})$ is an uniform family of smooth quasi-conformal mappings. Recall that if $f(z)$ is a smooth diffeomorphism of an open domain $U \subset \mathbb{C}$ into $\mathbb{C}$, preserving the orientation, one defines its dilatation coefficient to be the function $\mu(z) : U \to \mathbb{C}$, equal to $\mu(z) = \frac{\bar{\partial} f(z)}{\partial f(z)}$, where $df(z) = \bar{\partial} f(z) d\bar{z} + \partial f(z) dz$. We will say that f is quasi-conformal if $\exists k < 1$ such that $\mid \mu(z) \mid \leq k$ for all $z \in U$. A smooth family of diffeomorphisms $f_\lambda(z)$, with $(z,\lambda) \in U \times P$, is said to be an (uniform) family of quasi-conformal mappings if $\exists k < 1$ such that $\mid \mu(z,\lambda) \mid \leq k$ for $(z,\lambda) \in U \times P$, where $\mu(z,\lambda) = \frac{\bar{\partial} f_\lambda(z)}{\partial f_\lambda(z)}$.

Here, as $\widetilde{y} = \frac{1}{2i}(\widetilde{z} - \bar{\widetilde{z}})$, one has that:

$$\bar{\partial}\ \widetilde{\Sigma}(\widetilde{z}) = -\frac{1}{2i}\ \varphi'(\widetilde{y})\ \widetilde{h}_\lambda(\widetilde{z})\ , \tag{8}$$

$$\partial\widetilde{\Sigma}(\widetilde{z}) = 1 - \frac{1}{2i}\ \varphi'(\widetilde{y})\widetilde{h}_\lambda(\widetilde{z}) + \varphi(\widetilde{y})\partial\widetilde{h}_\lambda(\widetilde{z}). \tag{9}$$

Then, as $\widetilde{h}_\lambda(\widetilde{z}) = O(\exp(\widetilde{z}))$, uniformly in $\lambda \in P$, it is clear that, given any k, $0 < k < 1$, one can choose A small enough such that the dilatation coefficient $\widetilde{\mu}(\widetilde{z}, \lambda)$ of $\widetilde{\Sigma}_\lambda(\widetilde{z})$ will verify $|\ \widetilde{\mu}(\widetilde{z}, \lambda)\ | \leq k$ for $(\widetilde{z}, \lambda) \in \widetilde{W_0} \times P$.

Next, because the dilatation coefficient of a diffeomorphism is invariant by holomorphic conjugacy, the dilatation coefficient of $\widetilde{\Sigma}(\widetilde{z}, \lambda)$ is $2\pi i$-periodic and verify the same inequality for any $(\widetilde{z}, \lambda) \in \widetilde{W} \times P$.

2. The construction of the mapping $\varphi_\lambda(z)$.

We recall that the Ahlfors-Bers theorem addresses the inverse problem : given a function $\mu(z)$ on U, called a Beltrami field, find a mapping $f(z)$ which has $\mu(z)$ as dilatation coefficient. We need now to be more precise on the properties of μ and f.

Definition 2. *Let be $U \in \mathbb{C}$ a simply connected open set. A Beltrami field $\mu(z) : U \longrightarrow \mathbb{C}$ is any bounded measurable function with L^∞-norm:*

$$\text{Sup}\ \{|\ \mu(z)\ |\ \ |\ z \in U\} = \|\mu\|_\infty < 1.$$

As in the case of the dilatation coefficient of a diffeomorphism, one interprets μ as defining of a field of ellipses (defined up to scalar homotheties). One has a natural action of the diffeomorphisms on Beltrami fields (see [**L**]). If $G : U \longrightarrow V$ is a diffeomorphism, and $\mu : U \longrightarrow \mathbb{C}$ a Beltrami-field on U, we will write $G^*(\mu)$, the image of μ by the diffeomorphism G.

For instance, the dilatation coefficients $\widetilde{\mu}(\widetilde{z}, \lambda)$ of $\widetilde{\Sigma}(\widetilde{z}, \lambda)$, on $\widetilde{W}$, is invariant by the translation $\widetilde{z} \longrightarrow \widetilde{z} + 2\pi i$. It induces a Beltrami field $\mu(z, \lambda)$ on $W - \{0\}$ for $\forall \lambda \in P$, where $W = \pi(\widetilde{W_0}) \cup \{0\}$, because $|\ \mu(\pi(\widetilde{z}), \lambda)\ | = |\ \widetilde{\mu}(\widetilde{z}, \lambda)\ | \leq k < 1$.

Of course $\mu(z, \lambda)$ is discontinuous at $0 \in \mathbb{C}$, but because $\mu(z, \lambda)$ is smooth on $W - \{0\}$ and bounded by k, it defines a Beltrami field on

W, which verifies also: $||\mu||_\infty \leq k < 1$ (here $||\ .\ ||_\infty$ is the norm in $L^\infty(W \times P)$).

Moreover $\mu_z : \lambda \longrightarrow \mu(z,\lambda)$ is holomorphic in P for any $z \neq 0$ and it follows from the formulas (8), (9) that the germ of μ_z at $\lambda = 0$ belongs to the ideal of coefficients $\mathcal{I}_H$ of $h_\lambda(z) = H_\lambda(z) - z$.

Take a disk $\mathbb{D}$, centered at $0 \in \mathbb{C}$ and such that $\overline{\mathbb{D}} \subset \text{int } W$. We choose some smooth function $\psi(z) : \mathbb{C} \longrightarrow [0,1]$ such that $\psi(z) \equiv 1$ for $z \in \mathbb{D}$ and $\psi(z) \equiv 0$ for $z \in \mathbb{C} - \mathbb{D}'$ where $\mathbb{D}'$ is a second disk chosen such that: $\overline{\mathbb{D}} \subset \text{int } \mathbb{D}' \subset \overline{\mathbb{D}}' \subset \text{int } W$. We consider the Beltrami field $\mu'(z,\lambda) = \psi(z)\mu(z,\lambda)$ which coincides with μ on $\mathbb{D}$, is defined on the whole Riemann sphere S^2 for all $\lambda \in P$ and is equal to 0 in a neighborhood of $\infty \in S^2$. Moreover, μ' keeps the properties of μ in relation to λ.

We want now to apply the Ahlfors-Bers theorem to the Beltrami field μ'. First we recall the definition:

Definition 3. *An homeomorphism $f : S^2 \longrightarrow S^2$ is said quasi-conformal if and only if the partial derivatives ∂f, $\bar{\partial} f$ exist and satisfy in the L^2 local sense an equation $\bar{\partial} f - \mu \partial f = 0$ where $\mu(z)$ is in $L^\infty(S^2)$ with $||\mu||_\infty \leq k$ for some $k < 1$ (i.e., μ is a Beltrami field).*

The mapping theorem of Ahlfors and Bers says inversely that given some μ Beltrami field μ, there exists a quasi-conformal f which verifies the Beltrami-equation:

$$\bar{\partial} f - \mu \partial f = 0 \tag{10}$$

in L^2-local sense (see [**A**] for instance). This implies that the dilatation coefficient of f is defined and is equal to μ almost everywhere. Moreover if $\mu \in L_0^\infty(S^2)$, i.e. if μ has a compact support in $\mathbb{C}$, there exists a unique solution F_μ of (10), such that $F_\mu(z) - z = o(\frac{1}{z})$.

This unique solution can be construct explicitly as follows, using singular integrals (see [**L**]). For $\omega(z) \in \mathcal{C}_0^\infty(S^2)$ (space of smooth functions with compact support in $\mathbb{C}$), one defines, the two following operators:

$$T\omega(z) = -\frac{1}{\pi} \int\int_{\mathbb{C}} \frac{\omega(s)}{s - z}\, d\xi \wedge d\eta \tag{11}$$

and:

$$H\omega(z) = \lim_{\varepsilon \to 0} \; -\frac{1}{\pi} \int\int_{|z-s|>\varepsilon} \frac{\omega(s)}{\mid s - z \mid^2} \, d\xi \wedge d\eta \tag{12}$$

for $s = (\xi, \eta) \in \mathbb{C}$.

The operator H can be extended continuously in $L^p(S^2)$ for $p > 1$. Then, taking $\mu \in L_0^\infty(S^2)$, $\|\mu\|_\infty < 1$, one can define inductively in $L^p(S^2)$ a sequence of functions $(\varphi_i)_i$:

$$\varphi_1 = \mu \; , \quad \varphi_n = \mu H \varphi_{n-1} \quad n \geq 2 \tag{13}$$

which converges in $L^p(S^2)$ if μ verifies:

$$\|H\|_p \; \|\mu\|_\infty < 1 \tag{14}$$

which is always possible, if $\|\mu\|_\infty \leq k$, small enough.

The following representation formula was first established by Bojarski (see [**L**]) for the solution F_μ of Beltrami equation (10):

$$F_\mu(z) = z + \sum_{i=1}^{\infty} T\varphi_i(z) \tag{15}$$

where the φ_i are defined by (13). The series is absolutely and uniformly convergent in the Riemann sphere.

We are going to use this representation formula, to deduce the following properties of $f_\lambda(z) = f(z, \lambda) = F_{\mu'}(z)$, the solution associated to $\mu'(z, \lambda)$.

Proposition 5. *The family of quasi-conformal mappings $f_\lambda(z)$ associated to the Beltrami field $\mu'(z, \lambda)$ by the representation formula (15) is holomorphic in $b \times P$, where b is the sector:*

$$b = \pi(\widetilde{b}) = \{z \mid \; \mid Arg \; z \mid < \eta \; , \; \mid z \mid > 0\}.$$

Moreover, for each $z \in b$, the germ: $\lambda \longrightarrow (f_\lambda(z) - z)$ at $0 \in \mathbb{C}^\ell$, belongs to the ideal $\mathcal{I} = \mathcal{I}(H_\lambda(z) - z)$.

Proof. The support of $\mu'(z, \lambda)$ in S^2 does not intersect the open sector b. Then, the functions $\varphi_i(z, \lambda)$ obtained by the formulas (13) are equal to zero on $b \times P$. As a consequence, the functions $T\varphi_i(z, \lambda)$ are holomorphic in $b \times P$, and, because the convergence of (14) is absolute and uniform on S^2, we obtain that the limit $f(z, \lambda)$ is also holomorphic on $b \times P$.

Now we want to prove that $\lambda \longrightarrow f_\lambda(z) - z$ belongs to the ideal $\mathcal{I}$ for $\forall z \in b$. Suppose that $\mathcal{I}$ is generated by the analytic functions $\Lambda_1(\lambda), \cdots, \Lambda_\ell(\lambda)$ on P. The function $\mu'(z,\lambda)$ is in L_0^∞ $(S^2 \times P)$. (The space of functions which are measurable and bounded on $S^2 \times P$, with a support compact in $\mathbb{C} \times P$; here Supp $\mu' \subset W \times P$). Moreover the germ $\lambda \longrightarrow \mu'(z,\lambda)$ is in the ideal $\mathcal{I}$ for all $z \in S^2$. Using a classical theorem for holomorphic ideals (see **[H]** for instance), it is possible to choose a sub-neighborhood P' of $0 \in \mathbb{C}^\ell$, $\bar{P}' \subset \text{int } P$, and a constant $M_1 > 0$ such that:

$$\mu'(z,\lambda) = \sum_{i=1}^{\ell} \Lambda_i(\lambda)\mu^i(z,\lambda) \tag{16}$$

on $S^2 \times P'$. The functions μ^i are holomorphic in λ, $\mu^i \in L_0^\infty(S^2 \times P)$ (with Supp $\mu^i \subset \text{Supp}\,\mu'$) and satisfy:

$$\|\mu^i\|_\infty \leq M_1 \|\mu'\|_\infty \tag{17}$$

where:

$$\|\mu\|_\infty = Sup\ \{|\ \mu(z,\lambda)\ |\ \ |\ \ (z,\lambda) \in S^2 \times P'\}.$$

Let be any function $g \in L_0^\infty(S^2 \times P')$ with Supp $g \subset W \times P'$. Let us write $g_\lambda(z) = g(z,\lambda)$. Then g_λ belongs to $L^p(S^2)$ for any $p \geq 1$ and any $\lambda \in P'$, and verifies:

$$\|g_\lambda\|_p \leq M_2 \|g\|_\infty$$

where $M_2 = M_2(W,p)$ is some constant depending on W and p. Recall that, to apply the formulas (13), (15) we have chosen k,p, such that the inequality (14) holds. Then there exists some constant $M_3 > 0$, such that at each step of the induction (13), one can write:

$$\varphi_{i,\lambda} = \sum_{j=1}^{\ell} \Lambda_j(\lambda)\ \varphi_{i,\lambda}^j \ \text{ with } \ \|\varphi_{i,\lambda}^j\|_p \ \leq M_3\ (\|H\|_p\ \|\mu'\|_\infty)^i. \tag{18}$$

These formula and inequality are obtained inductively. It is verified for $i = 1$ and $M_3 > M_1 M_2 / \|H\|_p$. Then one has:

$$\varphi_{i+1,\lambda} = \mu'_\lambda H \varphi_{i,\lambda} = \mu'_\lambda \sum_{j=1}^{\ell} \Lambda_j(\lambda) H \varphi_{i,\lambda}^j\ , \quad \text{for } \ i \geq 2 \tag{19}$$

and then:

$$\varphi_{i+1,\lambda} = \sum_{j=1}^{\ell} \Lambda_j(\lambda)\ \varphi^j_{i+1,\lambda} \tag{20}$$

with:

$$\varphi^j_{i+1,\lambda} = \mu'_\lambda H \varphi^j_{i,\lambda}.$$

Let us suppose by induction that $\varphi^j_{i,\lambda}$ verifies the inequality (18). One has:

$$\|\varphi^j_{i+1,\lambda}\|_p \ \leq\ \|\mu'_\lambda\|_\infty\ \|H\varphi^j_{i,\lambda}\|_p\ \leq\ \|\mu'_\lambda\|_\infty\ \|H\|_p\ \|\varphi^j_{i,\lambda}\|_p$$

which implies:

$$\|\varphi^j_{i+1,\lambda}\|_p\ \leq M_3\ (\|H\|_p\ \|\mu'\|_\infty)^{i+1}.$$

Now, the Hölder inequality gives:

$$\|T\varphi^j_i\|_\infty \leq c_p \ \mathrm{Sup}\ \{\|\varphi^j_{i,\lambda}\|_p \quad | \quad \lambda \in P'\}$$

for some constant c_p, depending on p. It follows that:

$$\|T\varphi^j_i\|_\infty \leq c'_p\ \Big(\|H\|_p\ \|\mu'\|_\infty\Big)^i \tag{21}$$

with $c'_p = M_3 c_p$.

Then the series $\sum\limits_i T\varphi^j_i$ is absolutely and uniformly convergent toward a continuous function $h_j(z,\lambda)$, which is holomorphic in λ. We have:

$$\begin{aligned} f(z,\lambda) &= z + \sum_{i=1}^{\infty} T\varphi_i(z,\lambda) \\ &= z + \sum_{i=1} T\Big(\sum_{j=1}^{\ell} \Lambda_j(\lambda)\varphi^j_i(z,\lambda)\Big). \end{aligned} \tag{22}$$

The inequality (21), permits to commute the summations in (22) and to obtain finally:

$$f(z,\lambda) = z + \sum_{j=1}^{\ell} \Lambda_j(\lambda) h_j(z,\lambda). \tag{23}$$

This last formula is a division of $f(z,\lambda) - z$ in the ideal $\mathcal{I}$. □

The solution $f(z, \lambda)$ does not preserve $z = 0$ in general. We need this condition to be able to lift f in $\widetilde{\mathbb{C}}$.

Lemma 6. *The function* $\lambda \longrightarrow f(0, \lambda)$ *is holomorphic and belong to the ideal* $\mathcal{I}$.

Proof. Let us choose a sequence $(z_i)_i \longrightarrow 0$ with $z_i \in b$. It follows from the proposition 5 that for any $z_i \in b$: $\lambda \longrightarrow f(z_i, \lambda)$ is holomorphic. Then, because $f(z, \lambda)$ is continuous in (z, λ), we have that the uniform limit $f(0, \lambda)$ of $f(z_i, \lambda)$ is holomorphic in λ and belongs to the closed ideal $\mathcal{I}$. □

Using the Lemma 6, we have that family $F(z, \lambda) = f(z, \lambda) - f(0, \lambda)$ verifies the same properties than f and fixes 0 : $F(0, \lambda) \equiv 0$ for $\forall \lambda \in P$.

For each $\lambda \in P$, $F(z, \lambda)$ is solution of the Beltrami-equation for μ', and then for $\mu(z, \lambda)$ on $\mathbb{D}$.

This means that $F(z, \lambda)$ sends the field of ellipses associated to μ on $\mathbb{D}$ on the field of circles on $F(\mathbb{D}, \lambda)$ which can be suppose equal to the fixed disk $\mathbb{D}$, using the uniformization theorem of Riemann.

The inverse mapping $\varphi(z, \lambda) = F^{-1}(z, \lambda)$ is also quasi-conformal from $\mathbb{D}$ to $\mathbb{D} \subset W$ and brings the trivial field of circles on the field of ellipses associated to $\mu(z, \lambda)$ on W. Let us recall the other properties of $\varphi(z, \lambda)$:

- $\varphi(0, \lambda) = 0$
- $\varphi(z, \lambda)$ is analytic in (z, λ) for $(z, \lambda) \in b' \times P$ where b' is some sector at 0, chosen so that $F(b, \lambda) \supset b'$ for all $\lambda \in P$.
- $\lambda \longrightarrow \varphi(z, \lambda)$ is in the ideal $\mathcal{I}$ for all $z \in b'$.

3. The analytic solution $\widetilde{S}_\lambda(\widetilde{z})$. (Proof of theorem 1)

The family of quasi-conformal mappings $\varphi_\lambda(z)$ lifts in a family of quasi-conformal mappings $\widetilde{\varphi}_\lambda(\widetilde{z})$, defined for $(\widetilde{z}, \lambda) \in \widetilde{\mathbb{D}} \times P$, with $\widetilde{\varphi}_\lambda(\widetilde{0}) = \widetilde{0}$.

As lift of an univalent mapping, $\widetilde{\varphi}_\lambda(\widetilde{z})$ verifies:

$$\widetilde{\varphi}_\lambda(\widetilde{z} + 2\pi i) = \widetilde{\varphi}_\lambda(\widetilde{z}) + 2\pi i. \tag{24}$$

Next, one can restrict $\widetilde{\varphi}_\lambda(\widetilde{z})$ to $\widetilde{V} \times P$, where $\widetilde{V}$ is some open neighborhood of $\widetilde{0}$ of the form $\widetilde{V} = \{\widetilde{x} \leq \rho(\widetilde{y})\}$ for some continuous function ρ chosen such that, for all $\lambda \in P$: $\widetilde{\varphi}_\lambda(\widetilde{V}) \subset \widetilde{W}$.

We define $\widetilde{S}_\lambda(\widetilde{z}) : \widetilde{V} \times P \longrightarrow \widetilde{\mathbb{C}}$ by :

$$\widetilde{S}_\lambda(\widetilde{z}) = \widetilde{\Sigma}_\lambda \circ \widetilde{\varphi}_\lambda(\widetilde{z}). \tag{25}$$

For any $\lambda \in P$, $\widetilde{\varphi}_\lambda$ sends the trivial field of circles in the field of ellipses associated to $\widetilde{\mu}(\widetilde{z}, \lambda)$ and next, by construction, $\widetilde{\Sigma}_\lambda(\widetilde{z})$ sends this field of ellipses in the trivial field of circles on $\widetilde{\mathbb{C}}$.

Then, the composition $\widetilde{S}_\lambda(\widetilde{z})$ sends the field of circles of $\widetilde{V}$ into the field of circles of $\widetilde{\mathbb{C}}$. This means that $\widetilde{S}_\lambda(\widetilde{\delta})$ is conformal as a function of $\widetilde{z}$, and then is holomorphic in $\widetilde{z}$ for any $\lambda \in P$, as a consequence of the Weyl lemma (see [**L**]).

Let $\widetilde{b}'$ be a strip around $\{\widetilde{y} = 0\}$ in $\widetilde{V}$ such that $\pi(\widetilde{b}') \subset b'$. We know that $\widetilde{\varphi}_\lambda(\widetilde{z})$ is analytic on $\widetilde{b}'$. Then, because $\widetilde{\varphi}_\lambda(\widetilde{b}') \subset \widetilde{b}$ and $\widetilde{\Sigma}_\lambda(\widetilde{z}) \equiv \widetilde{z}$ on $\widetilde{b}$, we have that $\widetilde{S}_\lambda(\widetilde{z})$ is analytic on $\widetilde{b}'$ for any $\lambda \in P$. As $\widetilde{z} \longrightarrow \widetilde{S}_\lambda(\widetilde{z})$ is analytic on $\widetilde{V}$, for all $\lambda \in P$ it follows by analytic continuation that $\widetilde{S}_\lambda(\widetilde{z})$ is analytic in $(\widetilde{z}, \lambda)$ on the whole set $\widetilde{V} \times P$.

We also know that for any $\widetilde{z} \in \widetilde{b}'$, the germ of the map $\lambda \longrightarrow \widetilde{\varphi}_\lambda(\widetilde{z}) - \widetilde{z}$ belongs to the ideal $\mathcal{I}$, and then the same is true for the germ of the map: $\lambda \longrightarrow \widetilde{S}_\lambda(\widetilde{z}) - \widetilde{z}$. Using the proposition 1, it follows immediately, that this is also the case for $\forall \widetilde{z} \in \widetilde{V}$.

Finally it remains to verify that $\widetilde{S}_\lambda(\widetilde{z})$ satisfies the equation $(\widetilde{1})$:

$$\begin{aligned} \widetilde{S}_\lambda(\widetilde{z} + 2\pi i) &= \widetilde{\Sigma}_\lambda \circ \widetilde{\varphi}_\lambda(\widetilde{z} + 2\pi i) \\ &= \widetilde{\Sigma}_\lambda\Big(\widetilde{\varphi}_\lambda(\widetilde{z}) + 2\pi i\Big). \end{aligned}$$

As $\widetilde{\Sigma}_\lambda$ is a solution of $(\widetilde{1})$, one has :

$$\widetilde{\Sigma}_\lambda\Big(\widetilde{\varphi}_\lambda(\widetilde{z}) + 2\pi i\Big) = \widetilde{H}_\lambda \circ \widetilde{\Sigma}_\lambda \circ \widetilde{\varphi}_\lambda(\widetilde{z}) + 2\pi i.$$

And then, finally:

$$\widetilde{S}_\lambda(\widetilde{z} + 2\pi i) = \widetilde{H}_\lambda \circ \widetilde{S}_\lambda(\widetilde{z}) + 2\pi i.$$

The continuity property $\widetilde{S}_\lambda(\widetilde{0}) = \widetilde{0}$ follows from the same property for $\widetilde{\varphi}_\lambda$ and $\widetilde{\Sigma}_\lambda$.

III. Division of the Dulac transition in the holonomy ideal.

We want to prove now the theorem 2. Let (X_λ) be an holomorphic family of vector fields, defined near a saddle point at $0 \in \mathbb{C}^2$. We can suppose that 0 is a saddle point of X_λ for $\forall \lambda \in P$, a compact neighborhood of 0 in the parameter space $\mathbb{C}^\ell$, and that the local stable and unstable manifolds are independent of λ. Let σ and τ be some analytic segments, transversal respectively to the local stable and to the local unstable manifold. One can suppose σ, τ to be parametrized by $z \in \mathbb{D} \subset \mathbb{C}$, near 0, and σ, P chosen small enough such that the Dulac transition map $D_\lambda(z)$ is defined from $b \times P$ into τ, where b is some angular sector at the origin of σ, with angle less than 2π. This map can be extended as a multivalued map, which can be lifted in a map $\widetilde{D}_\lambda(\widetilde{z})$ on some domain $\widetilde{W} \times P$ where $\widetilde{W}$ is a neighborhood of $\widetilde{0}$ in $\widetilde{\mathbb{C}}$. If $H_\lambda(z)$ is the holonomy germ on τ and $\widetilde{H}_\lambda(\widetilde{z})$ its lift on $\widetilde{\mathbb{C}}$, one has:

$$\widetilde{D}_\lambda(\widetilde{z} + 2\pi i) = \widetilde{H} \circ \widetilde{D}_\lambda(\widetilde{z}) + 2\pi i \ , \quad \widetilde{D}_\lambda(\widetilde{0}) = \widetilde{0}.$$

Then, $\widetilde{D}_\lambda(\widetilde{z})$ is a solution of equation $(\widetilde{1})$, associated to H_λ. Let $\widetilde{S}_\lambda(\widetilde{z})$ be the solution of the equation $(\widetilde{1})$ given by the theorem 1. One can suppose $\widetilde{W}$ and P chosen such that $\widetilde{S}_\lambda(\widetilde{z})$ is defined on $\widetilde{W} \times P$. As P is compact, one can also suppose that there exists a neighborhood $\widetilde{V}$ of $\widetilde{0}$ such that $\widetilde{V} \times P$ is contained in the image of $\widetilde{W} \times P$ by $\widetilde{D}_\lambda$, for all λ. The map $\widetilde{S}_\lambda \circ \widetilde{D}_\lambda^{-1}$ is then defined on $\widetilde{V} \times P$ and verify:

$$\widetilde{S}_\lambda \circ \widetilde{D}_\lambda^{-1} \ (\widetilde{z} + 2\pi i) = \widetilde{S}_\lambda \circ \widetilde{D}_\lambda^{-1}(\widetilde{z}) + 2\pi i \ \text{ and } \ \widetilde{S}_\lambda \circ \widetilde{D}_\lambda^{-1}(\widetilde{0}) = \widetilde{0}. \tag{26}$$

This means that $\widetilde{S}_\lambda \circ \widetilde{D}_\lambda^{-1}$ is the lift of some analytic diffeomorphism $C_\lambda(z)$, defined on $V \times P$, where V is a neighborhood of 0 in σ. This neighborhood V is contained in $\pi(\widetilde{V}) \cup \{0\}$ and $C_\lambda(0) = 0$ for $\forall \lambda \in P$ (this last property follows from $\widetilde{D}_\lambda(\widetilde{0}) = \widetilde{S}_\lambda(\widetilde{0}) = \widetilde{0}$).

If one introduces the multivalued map $S_\lambda(z)$ associated to $\widetilde{S}_\lambda(\widetilde{z})$, one can see $C_\lambda(z)$ as a change of coordinates in a neighborhood of $0 \in \sigma$, which transports $D_\lambda(z)$ into $S_\lambda(z)$:

$$S_\lambda(z) = D_\lambda \circ C_\lambda(z). \tag{27}$$

The map $S_\lambda(z)$ verifies the properties obtained in theorem 1. In particular for any z in the domain of definition of $S_\lambda(z)$, one has that the germ at $0 \in \mathbb{C}^\ell$ of the holomorphic function $\lambda \longrightarrow S_\lambda(z) - z$ belongs to the holonomy ideal $\mathcal{I}_H = \mathcal{I}(H_\lambda - \mathrm{Id})$.

IV. Bifurcations of homoclinic loops.

We want to prove now the theorems 3 and 4. Let us consider a real analytic family of vector fields (X_λ) which unfolds a saddle connection on $\mathbb{R}^2$. We suppose that X_0 has a saddle singular point at $0 \in \mathbb{R}^2$, with a saddle connection Γ, i.e some stable separatrix at 0 coincides with some unstable separatrix. The family (X_λ) is an unfolding of X_0, defined near Γ, for λ near $0 \in \mathbb{R}^\ell$. We suppose that the return map for X_0, along Γ, is equal to identity. In $[\mathbf{R}_1]$, it was proved that the cyclicity of (X_λ, Γ) is finite. We want to give now a new proof of this result, based on theorem 2 above.

First we have to explain how one looks at the limit cycles which bifurcate near Γ and to introduce the ideals in the parameter space on which will be based the proof of the finite cyclicity and its computation.

Let σ and τ be two analytic sections to the stable and unstable local separatrices of X_0, contained in Γ. Let be $\sigma^+ \simeq [0, X[\subset \mathbb{R}$, the half-section contained in σ, on which is defined the return map along Γ, with $0 \in [0, X[$ corresponding to the point $\sigma \cap \Gamma$. We can suppose that the Dulac transition map from σ^+ to τ is defined for any $\lambda \in P$. It is a parameter family $D_\lambda(x) : \sigma^+ \times P \longrightarrow \tau$. We suppose also that the regular transition defined by the flow of $-X_\lambda$ gives an analytic map $R_\lambda(x)$ from $\sigma \times P$ into τ. Then, the limit cycles of X_λ near Γ are in one to one correspondence with the roots of the equation:

$$\Delta_\lambda(x) = D_\lambda(x) - R_\lambda(x) \tag{28}$$

on σ^+, for $\lambda \in P$.

The cyclicity is equal to the minimum of the number of roots when the diameters of σ and P go to zero.

The structure of the Dulac map is deduced from the following Dulac normal form of the family X_λ near the saddle point. There exists a sequence $(\alpha_i(\lambda))_i$ of analytic germs at $0 \in \mathbb{R}^\ell$, such that for any $k \in \mathbb{N}$, one has an analytic chart near $0 \in \mathbb{R}^2$, where the family X_λ, up to an analytic equivalence can be written:

$$X_\lambda \begin{cases} \dot{x} = +x \\ \dot{y} = -y\Big[1 - \sum_{i=0}^{k} \alpha_{i+1}\ (\lambda)(xy)^i\ + (xy)^{k+1}\ F(x,y,\lambda)\Big]. \end{cases} \tag{29}$$

The analytic germs α_{i+1} which appear in this normal form are related to the holonomy map H_λ:

Lemma 7. *The ideal generated by the germs $(\alpha_i, 0)$ in the ring $\mathcal{O}(\mathbb{C}^\ell)$ is equal to the ideal of holonomy $\mathcal{I}_H$. More precisely, if the holonomy map is given by:*

$$H_\lambda(y) = y + \sum_{i \geq 0} \beta_{i+1}(\lambda)\ y^{i+1},$$

for any $k \geq 0$, the ideal generated by $\alpha_1, \cdots, \alpha_{k+1}$ is equal to the ideal generated by $\beta_1, \cdots, \beta_{k+1}$.

Proof. We want to compute the holonomy map for the vector field X_λ considered as a complex vector field defined in a neighborhood of $\mathbb{C}^2$.

So, let us suppose that x, y are now complex variables.

In the normal form (29), the unstable manifold is given by $\{y = 0\}$ and the transversal section on which we want to compute the holonomy map can be chosen to be $\tau = \{x = 1\}$, parametrized by y near 0. One considers the holonomy map from τ to τ, above the loop $x(\theta) = e^{i\theta}$, $\theta \in [0, 2\pi]$, on the unstable manifold. The complex flow of X_λ, through the point $(y, 1) \in \tau$ is given by $x(t) = e^t$ and $y(t)$. We will write:

$$y(t) = ye^{r(\lambda)t}\Phi(t, y, \lambda) \tag{30}$$

where $r(\lambda) = -1 + \alpha_1$ is the stable eigenvalue.

One has $\Phi(0, y, \lambda) \equiv 1$ so that one can expand Φ in series in y, with a constant term equal to 1:

$$\Phi(t, y, \lambda) = 1 + \sum_{j=1}^{k} \Phi_{j+1}(t, \lambda)y^j\ + \mathcal{O}(y^{k+1}). \tag{31}$$

If we bring this expansion in the second equation (29) one obtains, for $\dot{\Phi} = \dfrac{\partial \Phi}{\partial t}\ (t, \lambda)$:

$$\dot{\Phi} = \Phi\Big(\sum_{j=1}^{k} \alpha_{j+1}\ e^{j\alpha_1 t}\ y^j\ \Phi^j\ + \mathcal{O}(y^{k+1})\Big). \tag{32}$$

The equation (32) gives for the analytic functions $\Phi_j(t, \lambda)$ a recurrent system:

$$\dot{\Phi}_2 = \alpha_2\ e^{\alpha_1 t}$$
$$\dot{\Phi}_3 = \alpha_3\ e^{2\alpha_1 t}\ + 2\Phi_2\ \alpha_2\ e^{\alpha_1 t}$$

and more generally, for any j such that $2 \leq j \leq k+1$:

$$\dot{\Phi}_j = \alpha_j\ e^{(j-1)\alpha_1 t}\ + P_j(\alpha_1, \dots, \alpha_{j-1},\ \Phi_1, \dots, \Phi_{j-1}, e^{\alpha_1 t}) \tag{33}$$

In the expression (33), P_j is a polynomial in $\Phi_1, \dots, \Phi_{j-1}$ and $e^{\alpha_1 t}$, with linear coefficients in $\alpha_1, \dots, \alpha_{j-1}$. It follows that Φ_j has the form:

$$\Phi_j(t, \lambda) = \sum_{\ell=1}^{j}\ \alpha_l \nu_{\ell j}(t, \lambda). \tag{34}$$

for some analytic functions $\nu_{\ell j}$. Then, the holonomy map:

$$\begin{aligned} H_\lambda(y) &= y(2\pi i) = ye^{2\pi i r(\lambda)}\ \Phi(2\pi i,\ y,\ \lambda) \\ &= ye^{2\pi i \alpha_1}\ \Big(1 + \sum_{j=1}^{k}\ \Phi_j(2\pi i) y^k + \mathcal{O}(y^{k+1})\Big) \end{aligned} \tag{35}$$

has the required property. □

Let us return now to the map $\Delta_\lambda(x)$. Using the Dulac normal form (29), it was proved in [$\mathbf{R}_1$] that, for any $k \geq 1$, and any analytic parametrizations of transversal segments σ, τ, the map $\Delta_\lambda(x)$ has (x, ω)-expansion of order k:

$$\Delta_\lambda(x) = \sum_{i=0}^{k}\ \beta_i(\lambda)[x^i + \cdots] + \sum_{j=1}^{k}\ \alpha_j(\lambda)[x^j \omega + \cdots] + \psi_k(x, \lambda) \tag{36}$$

has it was already explained in the introduction.

The coefficients β_i, α_j are analytic and the α_j are the coefficients of the Dulac normal form.

Definition. *Let be $\mathcal{I}_0$ the ideal generated in $\mathcal{O}_0(\mathbb{C}^\ell)$ by the coefficients of all the (x,ω)-expansion at any order.*

It is clear that this ideal is independent of the choice of the transversal segments σ, τ and of their analytic parametrization. Moreover, it follows from the formula (36) and the lemma 7 that:

Lemma 8. $\mathcal{I}_H \subset \mathcal{I}_0$.

Proof. Using formula (36) we see that any coefficient α_j of the Dulac normal form appears as a coefficient of a (ω, x)-expansion, if the order k is chosen large enough. Now the lemma 7 says that the ideal of holonomy is generated by the coefficients of the normal forms. □

In $[\mathbf{R}_1]$, it was defined a third ideal, the Bautin ideal $\mathcal{I}$ generated by the germs $\lambda \longrightarrow \Delta_\lambda(x)$ for $x \in \sigma^+ - \{0\}$. Here we will notice by $\mathcal{I}$ the ideal generated in $\mathcal{O}_0(\mathbb{C}^\ell)$ (called in $[\mathbf{R}_1]$ the complexification of the real ideal $\mathcal{I}$). To prove the theorem 3, we now consider the complex extension of D_λ, R_λ that we will denote by the same notations. Moreover, σ, τ will be now small disks centered at $0 \in \mathbb{C}$ and σ^+ a sector in σ with angle less than 2π. The theorem 2 gives us an analytic parametrization z of σ such that $D_\lambda(z) - z$ has coefficients in $\mathcal{I}_H$. We suppose chosen such a parametrization. Let $\mathcal{I}_R$ be the ideal generated by the coefficients of the map $R_\lambda(z) - z$:

$$R_\lambda(z) - z = \sum_{i=0}^{\infty} \gamma_i(\lambda) z^i. \tag{37}$$

First we prove that $\mathcal{I}_R \subset \mathcal{I}_0$. The ideal $\mathcal{I}_R$ is generated by the coefficients $\gamma_i(\lambda)$ of the series (37). Let be any $i \in \mathbb{N}$ and a (x,ω)-expansion of $\Delta_\lambda(x)$ of order $k > i$.

Consider the coefficient $\beta_i(\lambda)$ in this expansion:

$$\beta_i(\lambda) = \widetilde{\beta}_i(\lambda) - \gamma_i(\lambda) \tag{38}$$

where $\widetilde{\beta}_i(\lambda)$ is the coefficient from $D_\lambda(z) - z$ and γ_i is the coefficient from $R_\lambda(z) - z$.

The germ of $\widetilde{\beta}_i$ belongs to the ideal $\mathcal{I}_H \subset \mathcal{I}_0$ and because the germ β_i belongs to the ideal $\mathcal{I}_0$, one has also that $\gamma_i \in \mathcal{I}_0$.

Consider now any $z \in \sigma^+ - \{0\}$. Writing again $\Delta_\lambda(z) = (D_\lambda(z) - z) - (R_\lambda(z) - z)$, we obtain that the germ of $\lambda \longrightarrow \Delta_\lambda(z)$ is a difference of a germ in $\mathcal{I}_H$ and a germ in $\mathcal{I}_R$. Then, it belongs to $\mathcal{I}_0$. Because the germs of maps $\lambda \longrightarrow \Delta_\lambda(z)$, for $z \in \sigma^+ - \{0\}$ generate the Bautin ideal, we have proved that $\mathcal{I} \subset \mathcal{I}_0$.

The inverse inclusion was proved in [**R**$_1$], and finally we have proved the theorem 3: $\mathcal{I}_0 = \mathcal{I}$.

Consider now any (x, ω)-expansion of $\Delta_\lambda(x)$ (36). We now know that the germ of $\lambda \longrightarrow \Delta_\lambda(x)$ belongs to $\mathcal{I}_0$. It is the same, by definition for the principal part of (36). Then it results that, for any $x \in \sigma^+ - \{0\}$, the map germ of the remainder: $\lambda \longrightarrow \psi_l(x, \lambda)$ belongs also at $\mathcal{I}_0$, for any l. It was proved in [**R**$_1$] that if the remainder $\lambda \longrightarrow \psi_{2k+1}(x, \lambda)$ belongs to some ideal $\mathcal{I}$ for all $x \in \sigma^+ - \{0\}$, then this function $\psi_{2k+1}(x, \lambda)$ can be divided in the ideal, in class $\mathcal{C}^k$. This implies that if $\{\varphi_1, \cdots, \varphi_\ell\}$ are a set of generators for $\mathcal{I}$, then one can write, for x, λ small enough:

$$\psi_{2k+1}(x, \lambda) = \sum_{s=1}^{\ell} \varphi_s(\lambda) \nu_s(x, \lambda) \tag{39}$$

with $\nu_s(x, \lambda)$ of class $\mathcal{C}^k$, analytic in λ, and $(2k+1)$-flat at $x = 0$, as the function ψ_{2k+1}.

It follows from this, that if the ideal $\mathcal{I}_0$ is generated by the $\ell = 2k$ first coefficients β_0, α_1, β_1, $\alpha_2, \cdots, \beta_{k-1}$, α_k, one can write :

$$\Delta_\lambda(x) = \beta_0[x + \cdots] + \alpha_1[x\omega + \cdots] + \cdots + \alpha_k[x^k\omega + \cdots]$$

where each bracket has a principal part which is a polynomial in x, ω, and a remainder term which have a finite but arbitrarily large class of differentiability and flatness $K >> k$. One can write a similar formula if $\mathcal{I}_0$ is generated by the $\ell = 2k+1$ first coefficients.

It is well known that such a linear combination has at most ℓ zeros in $x \in \sigma^+$, for σ^+ and the parameter space sufficiently small. ([**J**], [**M**], [**R**$_2$]). This proves the theorem 4.

References

[**A**] L. V. Ahlfors, *"Lectures on quasi conformal mappings"*, The Wadsworth and

Brooks/Cole Mathematics Series (1987).

[H] M. Hervé, *"Several Complex Variables"*, Oxford University Press (1963).

[I] Yu. Il'Yashenko, *"Limit cycles of polynomial vector fields with non-degenerate singular points on the real plane"*, Funk. Anal. Ego. Pri., **18(3):** (1984), 32-34, Func. Ana. and Appl., **18(3):** (1985), 199-209.

[J] P. Joyal, *"The Generalized Homoclinic Bifurcation"*, J.D.E., **107:** (1994), 1-45.

[L] O. Lehto, *"Univalent Functions and Teichmüller Spaces"*, Graduate Texts in Mathematics, **109:** (1987), Springer-Verlag World Publishing Corp.

[M] P. Mardesic, *"Le déploiement versel du cusp d'ordre n"*, thèse Université de Bourgogne (1992). *"Chebychev systems and the versal unfolding of the cusp of order n"*. Travaux en Cours, **57**, (1998), 1-153.

[P-Y] R. Pérez-Marco, J.-C. Yoccoz, *"Germes de feuilletages holomorphes à holonomie prescrite"*, in: Complex Analytic Methods in Dynamical Systems, IMPA january 1992, Astérisque **222:** (1994), 345-371.

[R$_1$] R. Roussarie, *"Cyclicité finie des lacets et des points cuspidaux"*, Nonlinearity, fasc. **2:** (1989), 73-117.

[R$_2$] R. Roussarie, *"Bifurcations of Planar Vector Fields and Hilbert's sixteenth problem"*. Progress in Mathematics, Birkhaüser Ed. **164**, (1998), 1-204.

Robert Roussarie
Université de Bourgogne
Laboratoire de Topologie-U.M.R. 5584 du C.N.R.S.
U.F.R. des Sciences et Techniques
9, avenue Alain Savary B.P. 400
21011 Dijon Cedex
e-mail: roussari@u-bourgogne.fr