Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol. 30, N. 1, 31-52

Conformal measures and Hausdorff dimension for infinitely renormalizable quadratic polynomials

Eduardo A. Prado

Abstract. We show in this paper that if f is a quadratic infinitely many times renormalizable polynomial of sufficient high combinatorial type, then: $\mathrm{HD}(J(f)) = \inf\{\delta : \exists\ \delta - conformal\ measure\ for\ f\}$ We use Lyubich's construction of the principal nest ([**Lyu97**]) in order to prove this result.

Keywords: conformal measures, Hausdorff dimension, polynomial dynamics.

Introduction

Let $f: C \to C$ be a quadratic polynomial. Sullivan showed in [**Sul80**] that it is possible to construct a conformal measure for f with support on $J(f)$, the Julia set of f, for at least one positive exponent δ. By a conformal measure (or δ-conformal measure, to be more precise) we understand a Borel probability measure μ satisfying the following condition:

$$\mu(f(A)) = \int_A |Df(z)|^\delta d\mu(z),$$

whenever f restricted to the set A is one to one.

Conformal measures are one of the tools in the study of the Hausdorff dimension of Julia sets. From the work of Bowen, Sullivan, and Walters ([**Bow75**], [**Sul80**] and [**Wal78**]) we know that if f is expanding on $J(f)$, then the Hausdorff measure (which is finite and non-zero) is the only conformal measure on $J(f)$. In other words, there exists only one exponent δ for which a δ-conformal measure for f exists. This δ is

Received 12 November 1997.
Partially supported by CNPq-Brazil grant # 300534/96-5

the Hausdorff dimension of $J(f)$ and this δ-conformal measure is equivalent to the Hausdorff measure of $J(f)$. Denker and Urbański showed in [**DU91**] (with a technical problem solved in [**Prz93**]) that the hyperbolic dimension of the Julia set of any rational function f is equal to $\inf\{\delta > 0 : \exists \quad \delta - conformal\ measure\ for\ f\}$. Let us call this last quantity δ_{inf}.

Urbański showed in [**Urb**] that if f is a rational function with no recurrent critical point then the Hausdorff dimension of the Julia set of f is equal to δ_{inf}. In this case critical points are allowed to be inside $J(f)$. In [**Prz96**] Przytycki showed this same result if f is a non-renormalizable quadratic polynomial or a Collet-Eckmann map. The goal of this work is to extend Przytycki's result to some infinitely renormalizable quadratic polynomials: the *polynomials of high combinatorial type* (see Section 2.2 for the definition of such polynomials). We use Przytycki's techniques together with Lyubich's construction of the *Principal nest* ([**Lyu97**]) in order to do that. In other words we show the following:

Theorem 1. *For any polynomial of high combinatorial type f we have the following equality:*

$$\inf\{\delta : \exists\ \delta - conformal\ measure\ for\ f\} = \mathrm{HD}(J(f)),$$

where $\mathrm{HD}(J(f))$ *stands for the Hausdorff dimension of the Julia set of f.*

2. Renormalization and combinatorics

2.1 Yoccoz polynomials

We will briefly describe how to construct the Yoccoz puzzle pieces for a quadratic polynomial. See [**Hub**] and [**Mil91**] for a complete exposition of such construction.

In this section we will just consider quadratic polynomials f with repelling periodic points.

We say that $g: U \to U'$ is a $quadratic - like\ map$ if it is a double branched covering and U and U' are open topological discs with U properly contained in U'. In addition to that we require the filled in Julia set of g to be connected. By *filled in Julia set* of g we understand the set $\{z \in U : g^n(z)$ is defined for all natural numbers $n\}$. There are two fixed points of g inside its filled in Julia set. One of them, the dividing fixed point, disconnects the filled in Julia set of g in more than one connected component. The other does not. Usually the dividing fixed point is denoted by α.

Remember that f, a quadratic-like map or a polynomial, is *renormalizable* if there exist open topological discs $U \subset U'$ with $0 \in U$ with $R(f): U \to U'$ being a quadratic-like map, $R(f) = f^k|_U$, with k the smallest natural number bigger than 1 satisfying this statement (we call k the *period of renormalization*). Here $R(f)$ stands for this renormalization of f. We can ask whether $R(f)$ is renormalizable or not and then define renormalizations of f of higher orders. So, each renormalization of f defines a quadratic polynomial-like map.

Let f be a degree two non-renormalizable polynomial and let G be the Green function of the filled in Julia set of f. There are q external rays landing at the dividing fixed point of f, where $q \geq 2$. The q *Yoccoz puzzle pieces of depth zero* are the components of the topological disc defined by $G(z) < G_0$, where G_0 is any fixed positive constant, cut along the q external rays landing at the dividing fixed points. We denote $Y^0(x)$ the puzzle piece of depth zero containing x. We define the *puzzle pieces of depth* n as being the connected components of the pre-images of any puzzle piece of depth zero under f^n. Again, if x is an element of a given puzzle piece of depth n we denote such puzzle piece by $Y^n(x)$.

Suppose now that f is at most finitely renormalizable without indifferent periodic points. Let α be the dividing fixed point of the last renormalization of f. Let G be the Green function of the filled in Julia set of f. In that case we define the puzzle pieces of depth zero as being the components of the topological disc $G(z) < G_0$, G_0 a positive constant, cut along the rays landing at all points of the f-periodic orbit of α. As before we define the puzzle pieces of depth n as being the connected components of the pre-images under f^n of the puzzle pieces of depth zero. The puzzle piece at depth n containing x is denoted by $Y^n(x)$.

We will consider the Yoccoz puzzle pieces as open topological discs.

Under this consideration the Yoccoz partition will be well defined over the Julia set of the polynomial f minus the set of pre-images of the dividing fixed point of the last renormalization of f (which is f itself in the non-renormalizable case).

A quadratic polynomial is a *Yoccoz polynomial* if it is at most finitely renormalizable without indifferent periodic points. We will need the following result:

Theorem 2.1. (**Yoccoz.**)*If f is a Yoccoz polynomial then*

$$\bigcap_{n\geq 0} Y^n(x) = \{x\}$$

for any x where the Yoccoz partition is defined.

2.2 Polynomials of high combinatorial type

Let us pass to the main class of polynomials that we will be considering, namely the polynomials of high combinatorial type. See [**Lyu97**] for a detailed exposition on this matter. We will need some technical definitions.

Let us start with a quadratic polynomial f with a recurrent critical point, say zero, and without indifferent periodic points. Given a Yoccoz puzzle piece Y_i^n of f and a point x such that $f^j(x)$ belongs to Y_i^n. We define the *pull back of Y_i^n along the orbit of x* as being the only connected component of $f^{-j}(Y_i^n)$ containing x. If moreover x belongs to Y_i^n and j is minimal and non-zero, then we say that j is *the first return time of x to Y_i^n*. A puzzle piece is said to be critical if it contains the critical point. Notice that if we pull back a critical puzzle piece $Y^n(0)$ along the first return of the critical point to $Y^n(0)$ we get a new critical puzzle piece.

We say that a polynomial (or a polynomial-like map) is Douady-Hubbard immediately renormalizable if it is renormalizable and the critical orbit never escapes the puzzle pieces of level one whose closures contain the dividing fixed point of f (see [**Lyu97**]). If f is not Douady-Hubbard immediately renormalizable it is possible to find a first time t_0 when the critical orbit escapes the union of the puzzle pieces of level one whose closures contain the dividing fixed point of f. The pull back of $Y^1(f^{t_0}(0))$ to 0 is denoted by $V^{0,0}$. This is the first critical puzzle piece such that the closure of its pull back along the first return of the critical point to itself is properly contained in itself. We say that this is our puzzle piece of level zero. The pull back of $V^{0,0}$ along the first return of the critical point to $V^{0,0}$ will be denoted $V^{0,1}$. We keep repeating this procedure: define $V^{0,t+1}$, the puzzle piece of level $t+1$, as being the pull back of $V^{0,t}$, the puzzle piece of level t, along the first return of the critical point to $V^{0,t}$. This procedure stops if the critical point does not return to a certain critical puzzle piece. If we assume that the critical point is combinatorially recurrent (i. e., the critical orbit enters every critical puzzle piece), then we can repeat this procedure forever, so let us assume that (as the opposite case is a well understood case). The collection $V^{0,t}$ for t being a natural number is the *principal nest of the first renormalization level.*

Now we have a sequence of first return maps $f^{l(t)} : V^{0,t+1} \to V^{0,t}$. By definition $V^{0,0}$ properly contains $V^{0,1}$. This implies that each $V^{0,t}$ properly contains $V^{0,t+1}$. It is also easy to see that each $f^{l(t)} : V^{0,t+1} \to V^{0,t}$ is a quadratic-like map.

We say that $f^{l(t)} : V^{0,t+1} \to V^{0,t}$ is a *central*: *return* or that $t+1$ is a *central*: *return*: *level* if $f^{l(t)}(0)$ belongs to $V^{0,t+1}$ (this implies that $l(t+1) = l(t)$). A *cascade*: *of*: *central*: *returns* is a set of subsequent central return levels. More precisely, a cascade of central returns is a collection of central return levels $t = t_0 + 1, ..., t_0 + N$ followed by a non-central return at level $t_0 + N + 1$. In this case we say that the above cascade of central returns has length N. We could also have an infinite cascade of central returns. Notice that with the above terminology a non-central return level is a cascade of central return of length zero. We also say that $V^{0,t}$ is on top of the cascade of central returns.

It is possible to show that the principal nest of the first renormalization level ends with an infinite cascade of central returns if and only if f is renormalizable. In that case, denote the first level of this infinite cascade of central returns by $t(0)+1$. Then we define the first renormalization $R(f)$ of f as being the quadratic-like map $f^{l(t(0))} : V^{0,t(0)+1} \to$

$V^{0,t(0)}$. The filled in Julia set of $R(f)$ is connected (it is also possible to show that $\bigcap V^{0,n} = J(R(f))$). Again we can find the dividing fixed point of the Julia set of $R(f)$, some external rays landing at it and define new puzzle pieces over the Julia set of $R(f)$. The rays landing at the new dividing fixed point are not canonically defined (remember that $R(f)$ is a polynomial-like map, and not a polynomial). We are not taking the external rays of the original polynomial. Instead we need to make a proper selection of those rays to be able to state the Theorem at the end of this Subsection (see [**Lyu97**]). As before we can construct the principal nest for $R(f)$, provided that $R(f)$ is not Douady-Hubbard immediately renormalizable. The elements of this new principal nest are denoted by $V^{1,0}, V^{1,1}, ..., V^{1,t}, ...$ and the nest is called the *principal nest of the second renormalization level.* If this new principal nest also ends in an infinite cascade of central returns, we repeat the procedure just described and construct a third principal nest. We repeat this process as many times as we can.

Now we define the *principal nest* of the polynomial f as being the set of critical puzzle pieces

$$V^{0,0} \supset V^{0,1} \supset \cdots \supset V^{0,t(0)} \supset V^{0,t(0)+1} \supset V^{1,0} \supset V^{1,1} \supset \cdots , V^{1,t(1)} \supset$$
$$\supset V^{1,t(1)+1} \supset \cdots \supset V^{m,0} \supset V^{m,1} \supset \cdots \supset V^{m,t(m)} \supset V^{m,t(m)+1} \supset \cdots$$

In order to go ahead with the definition of the class of polynomials we are interested in, we need the notion of a truncated secondary limb. A *limb* in the Mandelbrot set M is the connected component of $M \setminus \{c_0\}$ not containing 0, where c_0 is a bifurcation point on the main cardioid. If we remove from the limb a neighborhood of its root c_0, we get a *truncated limb.* A similar object corresponding to the second bifurcation from the main cardioid is a *truncated secondary limb.*

A polynomial of high combinatorial type is defined as being an infinitely many times renormalizable polynomial satisfying the two following properties:

(i) First select in the Mandelbrot set a finite number of truncated secondary limbs. We require all the quadratic-like renormalizations to be in these limbs.

(ii) We also require that in between two quadratic-like renormalization levels we have a sufficiently high number of non-central returns (called height) which depends on the a priori selection of the limbs.

Such class of polynomials was originally introduced and studied in [**Lyu93**].

Before we go to the main Theorem about polynomials of high combinatorial type, we will need one last notation. If $V^{i,k}$ is an element of the principal nest then we denote by $n(k)$ the number of central cascades in between $V^{i,0}$ and $V^{i,k}$. Remember that a non-central return is viewed as a central cascade of length zero.

The following result in [**Lyu93**] (see [**Lyu97**]) will allow us to create the "Koebe space" for Yoccoz polynomials and polynomials of high combinatorial type:

Theorem 2.2. (**Lyubich.**) *The principal modulus* $\operatorname{mod}(V^{i,k} \setminus V^{i,k+1})$ *grows linearly with n(k) for any polynomial of high combinatorial type, if* $V^{i,k}$ *is on top of a cascade of central returns. Moreover,* $\operatorname{mod}(V^{i,0} \setminus V^{i,1}) \geq c > 0$.

2.3 Combinatorics of Yoccoz pieces

In this section we will state some basic properties of the combinatorics of the Yoccoz puzzle defined before. Those properties will be used in the next sections of the paper.

We first notice the following: if $Y^n(z)$ and $Y^m(w)$ are two Yoccoz pieces and $m \geq n$, then either $\operatorname{int}(Y^n(z)) \cap \operatorname{int}(Y^m(w)) = \varnothing$ or $Y^m(w) \subset Y^n(z)$. We call this property *the Markov property of the puzzle pieces.*

For the next Lemma, consider a critical puzzle piece $Y^n(0)$. Let m be the time of first return of the critical point to $Y^n(0)$. Let $Y^{n+m}(0)$ be the pull back of $Y^n(0)$ along the first return of the critical point to $Y^n(0)$.

Lemma 2.3. *Let z be an element of $J(f)$ and let t be the smallest time that $f^t(z) = Y^{n+m}(0)$. Then we can univalently pull $Y^n(0)$ back along the orbit $z, \ldots, f^t(z)$.*

Proof. If not, $f^{-s}(Y^n(0))$ would contain the critical point, for some s

less than t (here f^{-s} means the branch of f^{-s} along the orbit of z). That would mean that s is greater or equal to m, the first return time of 0 to $Y^n(0)$. That would imply $f^{-s}(Y^n(0)) \subset Y^{n+m}(0)$ by the Markov property of puzzle pieces. In other words, z would hit $Y^n(0)$ on a time strictly less than t, contradicting the definition of t. $\square$

We finish this section introducing the concept of *generalized renormalization* ([**Lyu91**]). If $f^{l(m,t(m))} : V^{m,t(m)+1} \to V^{m,t(m)}$ is a renormalization level, and if we denote the closure of the critical orbit by $\overline{O}$ then the set $\overline{O} \cap V^{m,t(m)}$ is contained in $V^{m,t(m)+1}$. If $f^{l(m,i)} : V^{m,i+1} \to V^{m,i}$ is not a renormalization level, then we need extra puzzle pieces to cover $\overline{O} \cap V^{m,i}$. We can find finitely many puzzle pieces $V_j^{m,i+1}$, $0 \leq j \leq k(m, i+1)$ contained in $V^{m,i}$ such that $V_j^{m,i+1} = V^{m,i+1}$ when $j = 0$ and each one of those $V_j^{m,i+1}$ is the pull back of $V^{m,i}$ along the first return of some point in $\overline{O} \cap V^{m,i}$ back to $V^{m,i}$. We also assume that $\cup V_j^{m,i+1}$ covers $\overline{O} \cap V^{m,i}$. The map $g : \cup V_j^{m,i+1} \to V^{m,i}$ is called the i^{th} generalized renormalization of the m^{th} renormalization level of f. The map g is defined as being f^t in each $V_j^{m,i+1}$, where t is the return time of $V_j^{m,i+1}$ to $V^{m,i}$. We say that the puzzle pieces are of level $i+1$ (inside the renormalization level m). We also say that the puzzle pieces $V_j^{m,i+1}$ are non-critical if $j \neq 0$, and critical otherwise.

3. Modified principal nest

The goal of this section is to construct *the modified principal nest*, starting from the principal nest constructed in [**Lyu97**] and described in Section 2.2. The elements of the modified principal nest will be related to each other via maps which are compositions of a quadratic map and an isomorphism with bounded distortion (depending just on the map f). To simplify notation we will denote the elements of the principal nest of the first renormalization level (see section 2.2) by $V^0, V^1, ..., V^n, ...$. Remember that we divide the principal nest into disjoint unions of cascades of central returns. Remember also that a non-central return level is a cascade of central returns of length zero or a trivial cascade. Let $n(k)$ be the number of cascades of central returns before the level k. We use the following notation: $\mu_n = \mathrm{mod}(V^n \setminus V^{n+1})$.

Beginning of the construction of the modified principal nest: The elements of the modified principal nest will be denoted by W^i. Let $n+1$ be the first level of the first non-trivial cascade of central returns. We define $W^i = V^i$ for all $i = 1, \ldots, n+2$. Suppose that this first non-trivial cascade has its last element at level $n+k$, i. e., the first non-central return appears on $f^{l(n+k)} : V^{n+k} \to V^{n+k-1}$. We will construct the next element W^{n+3} of our modified principal nest as being a puzzle piece satisfying:

(i) $V^{n+k+1} \subset W^{n+3} \subset V^{n+k}$ and $0 \in W^{n+3}$;

(ii) the puzzle piece W^{n+3} is mapped as a branched covering of degree two onto $V^{n+2} = W^{n+2}$;

(iii) The above map is the composition of a pure quadratic map and an isomorphism with bounded distortion;

(iv) $\mathrm{mod}(V^{n+k} \setminus W^{n+3}) \geq \frac{1}{2}\mu_n = \mu_{n+1}$.

Let us construct W^{n+3} (see Figure 1). As we are in a cascade of central returns there exists a number p such that $f^p = f^{l(i)} : V^i \to V^{i-1}$ for all $i = n+1, \ldots, n+k$. It is also true that $f^{ip}(0) \in V^{n+k-i} \setminus V^{n+k-i+1}$ for $i = 1, \ldots, k$ (where 0 is the critical point). In particular $f^{(k-1)p}(0) \in V^{n+1} \setminus V^{n+2}$.

We know that the orbit of $f^{(k-1)p}(0)$ should enter V^{n+2}. That is because the orbit of V^{n+k+1} should enter V^{n+2} in order to return to V^{n+k}. Let us call S^{k-1} the pull back of V^{n+2} along the orbit of $f^{(k-1)p}(0)$. To be more precise, let $t > p > 0$ be the first time that $f^t(f^{(k-1)p}(0)) \in V^{n+2}$. Then $S^{k-1} = f^{-t}(V^{n+2})$. Here we are considering the branch of f^{-t} that takes $f^t(f^{(k-1)p}(0))$ to $f^{(k-1)p}(0)$. Now we define $S^i = f^{-(k-i-1)p}(S^{k-1})$, where we consider $f^{-(k-i-1)p}$ as being the branch taking $f^{(k-1)p}(0)$ to $f^{ip}(0)$, for $i = 1, \ldots, k-1$. In particular $f^p(0) \in S^1$. We finally define $W^{n+3} = f^{-p}(S^1)$, where we understand f^{-p} as the branch taking $f^p(0)$ to 0.

Having this definition of W^{n+3}, the first property is obvious by construction and from the Markov property of puzzle pieces. The next property follows from the fact that each one of the maps $f^p : S^i \to$

$S^{i+1}, i = 1, ..., k-2$ is an isomorphism. That is true because we are inside a cascade of central returns.

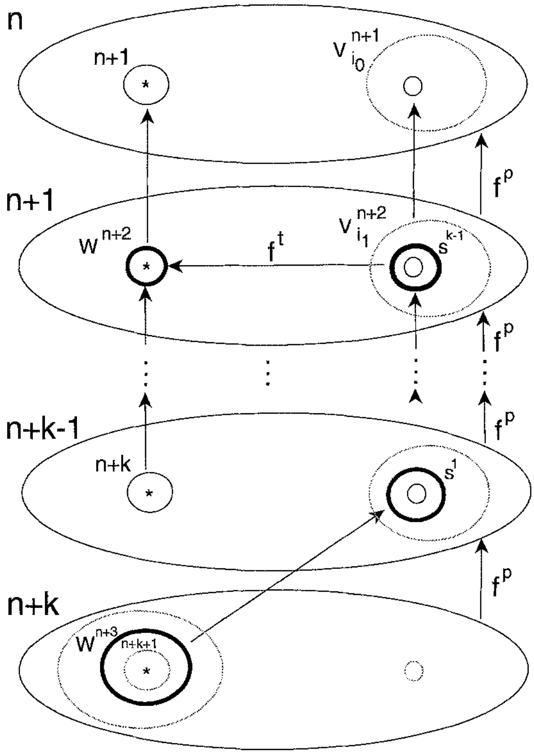

Figure 1: Construction of W^{n+3}.

Let us prove the third property. There exists a puzzle piece $V_{i_1}^{n+2}$ of level $n+2$ not containing the critical point such that $S^{k-1} \subset V_{i_1}^{n+2} \subset V^{n+1}$. The puzzle piece $V_{i_1}^{n+2}$ is defined as being the pull back of V^{n+1} along the first return of $f^{(k-1)p}(0)$ to V^{n+1}. Notice that the return time of $f^{(k-1)p}$ to V^{n+1} is smaller or equal to t (the smallest positive time so that $f^t(f^{(k-1)p}(0)) \in V^{n+2}$). This together with the Markov property of the puzzle pieces imply the inclusion $S^{k-1} \subset V_{i_1}^{n+2}$. The critical point does not belong to $V_{i_1}^{n+2}$ because otherwise we would find a non-central return at some level between $n+1$ and $n+k-1$. In a similar fashion we also show that exists a puzzle piece $V_{i_0}^{n+1}$ of level $n+1$ such that $f^p(V_{i_1}^{n+2}) \subset V_{i_0}^{n+1}$, $V_{i_0}^{n+1}$ not critical. The puzzle piece $V_{i_0}^{n+1}$ is obtained as the pull back of V^n along the first return of $f^{kp}(0)$ back to V^n.

We have $\mathrm{mod}(V_{i_0}^{n+1} \setminus f^p(V_{i_1}^{n+2})) \geq \mu_n$. If the orbit of $f^p(V_{i_1}^{n+2})$ hits V^{n+1} at the same time as the orbit of $V_{i_0}^{n+1}$ returns to V^n, then we obviously have $\mathrm{mod}(V_{i_0}^{n+1} \setminus f^p(V_{i_1}^{n+2})) = \mu_n$. If the orbit of $f^p(V_{i_1}^{n+2})$ hits V^{n+1} latter than the orbit of $V_{i_0}^{n+1}$ returns to V^n, then there is a puzzle piece V such that $f^p(V_{i_1}^{n+2}) \subset V \subset V_{i_0}^{n+1}$ and the orbit of $f^p(V_{i_1}^{n+2})$ hits V^{n+1} at the same time as the orbit of V covers univalently V^n. So we get $\mathrm{mod}(V_{i_0}^{n+1} \setminus f^p(V_{i_1}^{n+2})) \geq \mathrm{mod}(V \setminus f^p(V_{i_1}^{n+2})) = \mu_n$.

The map $f^{(k-1)p} : V^{n+k-1} \to V^n$ has all its critical points inside V^{n+k} (because we are inside a cascade of central returns). As $V_{i_0}^{n+1} \neq V^{n+1}$, we conclude that we can isomorphically pull $V_{i_0}^{n+1}$ back along the orbit $S^1, S^2, ..., S^{k-1}, f^p(S^{k-1})$. That means that $\mathrm{mod}(f^{-(k-1)p}(V_{i_0}^{n+1}) \setminus S^1) \geq \mu_n$. If we make one extra pull back we will get $\mathrm{mod}(f^{-kp}(V_{i_0}^{n+1}) \setminus W^{n+3}) \geq \frac{1}{2}\mu_n$.

Remember that f is a quadratic polynomial. Putting this fact together with the information from the last paragraph we conclude that we can decompose $f^{pk} : W^{n+3} \to S^{k-1}$ into a pure quadratic map followed by an analytic isomorphism of bounded distortion (by Koebe's Theorem). The distortion depends only on the principal modulus μ_n, which is definite.

The inverse of the map $f^t : S^{k-1} \to V^{n+2}$ can be extended to V^{n+1} (see Lemma 2.3. That means, by Koebe's Theorem, that the distortion of f^t is bounded (depending just on $\mu_{n+1} = \frac{1}{2}\mu_n$) when restricted to S^{k-1}.

Putting the information of the last paragraphs together we can show the third property. The last property also follows from the previous argument.

We will now define the element W^{n+4} following W^{n+3} in our modified principal nest. This new element W^{n+4} will be defined as a certain pull-back of W^{n+3}. We start to define this pull back now. Immediately after the first non-trivial cascade of central returns we will find either another non-trivial cascade of central returns or a trivial cascade of central return. We need to consider both cases.

Continuation of the modified principal nest through a non-trivial cascade: Suppose that we have another non-trivial cascade of central returns. In that case, we would find central returns on all levels from $n+k+1$ to $n+k+m-1$ (so this new cascade of central returns has length m). We can find W^{n+4} such that:

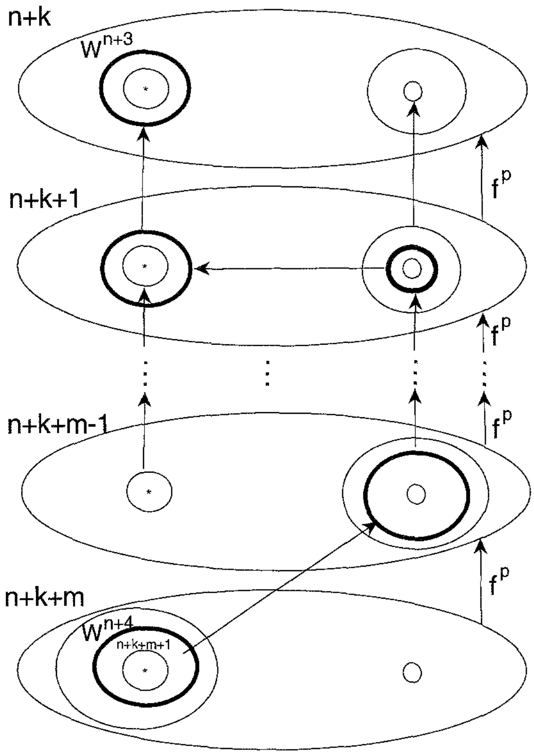

Figure 2: First construction of W^{n+4}.

(i) $V^{n+k+m+1} \subset W^{n+4} \subset V^{n+k+m}$ and $0 \in W^{n+4}$;

(ii) W^{n+4} is mapped as a branched covering of degree four onto W^{n+3};

(iii) The above map is the composition of a pure quadratic map followed by an isomorphism with bounded distortion, an other pure quadratic map and an isomorphism with bounded distortion;

(iv) $\text{mod}(V^{n+k+m} \setminus W^{n+4}) \geq \frac{1}{2}\mu_{n+k}$.

The above properties and their justifications are similar to the ones for W^{n+3} stated before (see Figure 2). This finishes the construction of

W^{n+4} in the case that we have a non-trivial cascade of central returns following the first non-trivial cascade of central returns. We would like to point out that the value of $\text{mod}(V^{n+k+m} \setminus W^{n+4})$ does not depend on the value of $\text{mod}(V^{n+k} \setminus W^{n+3})$.

Continuation of the modified principal nest through a trivial cascade: In this case we suppose that we have a trivial cascade following the first non-trivial cascade of central returns. In other words the first return map $f^{l(n+k+1)} : V^{n+k+1} \to V^{n+k}$ is a non-central return. In that case we can find W^{n+4} such that:

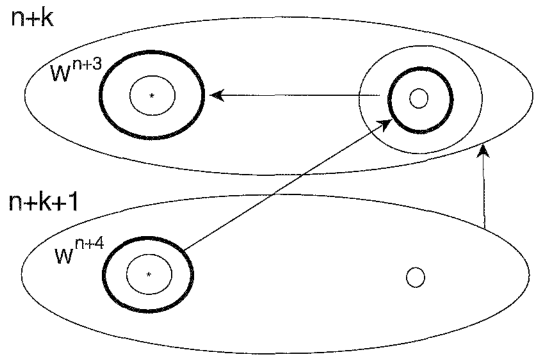

Figure 3: Second construction of W^{n+4}.

(i) $V^{n+k+2} \subset W^{n+4} \subset V^{n+k+1}$ and $0 \in W^{n+4}$;

(ii) W^{n+4} is mapped as a branched covering of degree two onto W^{n+3};

(iii) The above map is the composition of a pure quadratic map and an isomorphism with bounded distortion;

(iv) $\text{mod}(V^{n+k+1} \setminus W^{n+4}) \geq \frac{1}{2}\text{mod}(V^{n+k} \setminus W^{n+3}) \geq \frac{1}{4}\mu_n$.

Again with the same type of argument we used to show properties of W^{n+3} we show the above properties (see Figure 3).

The definition of the modified principal nest: It follows now by induction. We assume that we have the definition of one of its elements, say W^i. Such element is inside the last level of a given cascade of central returns. We can construct the next one, W^{i+1} following the constructions just as we described in the previous two cases.

The modified principal nest as constructed has the following properties. Given any cascade of central returns with last element V^{n+1}:

(i) There exist W in the modified principal nest such that $V^{n+1} \subset W \subset V^n$ with $\text{mod}(V^n \setminus W)$ growing linearly with the number of cascades of central returns;

(ii) If W' is the element following W in the modified principal nest, then $\text{mod}(W \setminus W')$ is growing linearly (with the number of cascades of central returns);

(iii) W' is mapped as a branched covering either of degree two or degree four onto W;

(iv) The map in the previous item is the composition of a pure quadratic map and an isomorphism with bounded distortion in the degree two case. In the degree four case, this map is a composition of two maps as in the degree two case.

The last three properties of the modified principal nest are true by construction (the third one is a consequence of the first). The first property is obvious except perhaps in one case, namely when we have more than one trivial cascade of central returns together. Notice that in our construction of the modified principal nest through a trivial cascade (see Figure 3) we got the following estimate :

$$\text{mod}(V^{n+k+1} \setminus W^{n+4}) \geq \frac{1}{2}\text{mod}(V^{n+k} \setminus W^{n+3}).$$

One can ask whether we will keep dividing by two the bound for the modulus comparing the principal and the modified nests at a certain level, if we have several trivial cascades, one following the other. If that would be the case we would spoil our estimates concerning the modified principal nest.

Let us analyze what happens when we have two consecutive trivial cascades. On Figure 4, let V_i^{n+2} be a non-critical puzzle piece of level $n+2$ containing the orbit of W^{i+2}, contained inside V^{n+1}. Let us assume that the return time of V_i^{n+2} to V^{n+1} is t and that the return time of V^{n+1} to V^n is s. In general $t \geq s$. Let us analyze two situations: one when $t > s$ and the other when $t = s$. If $t > s$, then $f^s(V_i^{n+2})$

is contained in V^n. So there is a puzzle piece of level $n+1$ V_j^{n+1} containing $f^s(V_i^{n+2})$. Due to the fact that V_j^{n+1} will be mapped to V^n and V_i^{n+2} will be mapped to V^{n+1} we can prove that $\mathrm{mod}(V_j^{n+1} \setminus f^s(V_i^{n+2})) \geq \mu_n$. This is enough to show that $\mathrm{mod}(V^{n+2} \setminus W^{i+2}) \geq \frac{1}{4}\mu_n$. If $t = s$, then $\mathrm{mod}(V^{n+1} \setminus V_i^{n+2}) \geq \frac{1}{2}\mu_n$. Again, this is enough to show that $\mathrm{mod}(V^{n+2} \setminus W^{i+2}) \geq \frac{1}{4}\mu_n$. In both cases we are showing that $\mathrm{mod}(V^{n+2} \setminus W^{i+2})$ is greater then $\frac{1}{4}\mu_n$, which is definite number because it is the principal modulus at the top of a (trivial) cascade of central returns.

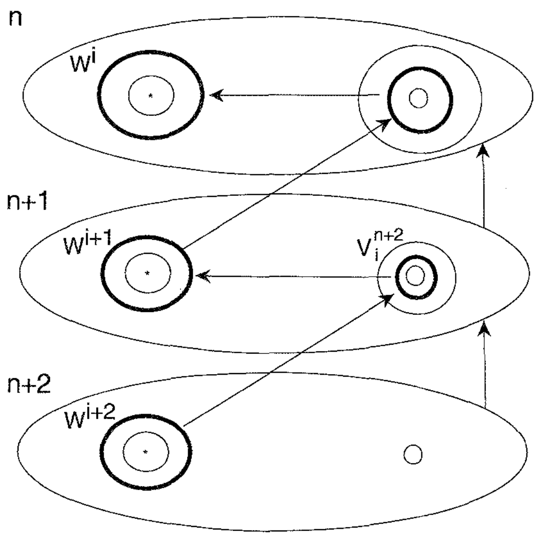

Figure 4: Two consecutive trivial cascades of central returns.

From our considerations we see that for each cascade of central return we have one element of the modified principal nest. Therefore we can enumerate the elements of the modified principal nest counting the cascades of central returns. So W^n is the n^{th} element of the modified principal nest, i.e. , the element at the end of the n^{th} cascade of central return.

Suppose that the first principal nest of f ends in an infinite cascade of central returns. Then the modified principal nest also ends in an infinite cascade of central returns. The construction of the modified principal nest through the renormalization level is the same as the construction of the principal nest defined in [**Lyu97**] through the renormalization level. Now we complete the modified principal nest repeating the construction that we have just described.

4. Proof of the Theorem

Before we start the proof of Theorem 1 we state a fundamental result due to Denker and Urbański that we use. This result is possible due to a technical problem solved in [**Prz93**].

Definition 4.1([**DU91**], [**Shi91**].) We define the hyperbolic dimension of f as:

$$\mathrm{hypdim}(f) = \sup\{\mathrm{HD}(X) : X \subset J(f) \text{ is hyperbolic for } f\}$$

Theorem 4.2. ([**DU91**].)*If f any rational map, then*

$$\delta_{inf} = \inf\{\delta : \exists\colon \delta - conformal\colon measure\colon for\colon f\} = \mathrm{hypdim}(f).$$

We use the following notation: if a_n and b_n are two sequences of positive real numbers, then we write $a_n \asymp b_n$ if $K^{-1} \leq \frac{a_n}{b_n} \leq K$, for some constant K.

Let f be a polynomial of high combinatorial type and μ a δ-conformal measure for f. Take W and W' being two consecutive elements of the modified principal nest and $f^a : W' \to W$ the first return map of the critical point to W.

Lemma 4.3. *Let f be any polynomial of high combinatorial type and assume that W and W' are two consecutive elements of the modified principal nest belonging to the same renormalization level. Then*

$$\frac{\mu(W)}{(\mathrm{diam}(W))^\delta} \leq K \frac{\mu(W')}{(\mathrm{diam}(W'))^\delta}$$

where K is a constant depending on the selection of the secondary limbs.

Proof. Suppose first that $f^a : W' \to W$ is a degree two branched covering. Then the map $f^{a-1} : f(W') \to W$ has bounded distortion (by construction of the modified principal nest). That implies that

$$\frac{\mu(f(W'))}{(\mathrm{diam}(f(W')))^\delta} \asymp \frac{\mu(W)}{(\mathrm{diam}(W))^\delta}$$

(because of the definition of conformal measure). So we conclude that there is a constant K_1 such that:

$$\frac{(\text{diam}(f(W')))^\delta}{(\text{diam}(W))^\delta} \leq K_1 \frac{\mu(f(W'))}{\mu(W)}. \tag{1}$$

As $f : W' \to f(W')$ is pure quadratic we have the following inequality: $\text{diam}(W') \leq L|Df(x)|^{-1}\text{diam}(f(W'))$, where L is a constant and x is any element in W'. As μ is conformal we find z_0 in W' such that $\mu(f(W')) = \frac{1}{2}|Df(z_0)|^\delta \mu(W')$.

Putting last two observations together we get:

$$\frac{(\text{diam} W')^\delta}{(\text{diam}(f(W')))^\delta} \leq L^\delta \frac{1}{2} \frac{\mu(W')}{\mu(f(W'))}. \tag{2}$$

Multiplying equations (1) and (2) we get the Lemma. Remember that we are assuming that $f : W \to W'$ has degree two.

Suppose now that $f^a : W' \to W$ is a degree four branched covering. Then it is a composition of two maps which are themselves the composition of a pure quadratic map followed by some map with bounded distortion. Then we repeat the previous argument twice to get the same result. □

For the next Lemma we will consider two consecutive elements of the modified principal nest W and W' such that $f^a : W' \to W$ (the first return map of the critical point to W') has connected Julia set. In other words, we will be considering the level of the modified principal nest corresponding to a change of renormalization level. Let W'' be the first element of the modified principal nest following W'.

Lemma 4.4. *Suppose that $f^a : W' \to W$ has a connected Julia set. Then*

$$\frac{\mu(W')}{(\text{diam}(W'))^\delta} \leq K \frac{\mu(W'')}{(\text{diam}(W''))^\delta}$$

where W'' is the first element of the modified principal nest following W' and K depends just on the selection of the secondary limbs.

Proof. Suppose that the α fixed point of $g = f^a : W' \to W$ is the landing point of p external rays. For each $i \neq 0$ and for each puzzle piece of level zero Y_i^0 for g we define $S_i = Y_i^0 \bigcap J(g)$. Let us define Y^1

as being the intersection of $J(g)$ with the critical puzzle piece of level zero for g. We also define $W_i = -S_i$.

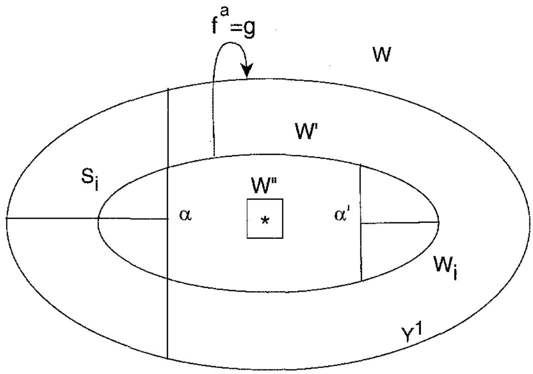

Figure 5: Renormalization level.

The pieces S_i can be enumerated following the orbit of the critical point. It follows from bounded geometry of the puzzle pieces in Figure 5, as shown in [**Lyu97**], that $g : S_i \to S_{i+1}$ has bounded distortion for $i = 1, ..., p-1$ (depending only on the choice of the secondary limbs). The same happens to the map $g : S_{p-1} \to Y^1$. This implies that for $i = 1, ..., p-2$ we have:

$$\frac{\mu(S_i)}{\operatorname{diam}(S_i)^\delta} \asymp \frac{\mu(S_{i+1})}{\operatorname{diam}(S_{i+1})^\delta}$$

and

$$\frac{\mu(S_{p-1})}{\operatorname{diam}(S_{p-1})^\delta} \asymp \frac{\mu(Y^1)}{\operatorname{diam}(Y^1)^\delta}.$$

By the hypothesis of polynomials of high combinatorial type (it is not Douady-Hubbard immediately renormalizable) we know that there is a time j (this being minimal) such that $g^j(0) \in W_i$, for some i. According to [**Lyu97**], W'' is the pull back of W_i along the critical orbit, back to the critical point. As this map is the composition of a pure quadratic polynomial with an isomorphism with bounded distortion (again by bounded geometry), we can repeat the proof of Lemma 4.3 to get:

$$\frac{\mu(W'')}{\operatorname{diam}(W'')^\delta} \geq K_0 \frac{\mu(W_i)}{\operatorname{diam}(W_i)^\delta} = \frac{\mu(S_i)}{\operatorname{diam}(S_i)^\delta}.$$

Last equality follows from the fact that $S_i = -W_i$. Notice that K_0 depends just on the selection of secondary limbs. There exist an integer k such that $g^k : W_i \to Y^1$. This map has bounded distortion (again because of bounded geometry). Then we get:

$$\frac{\mu(W_i)}{\operatorname{diam}(W_i)^\delta} \asymp \frac{\mu(Y^1)}{\operatorname{diam}(Y^1)^\delta}.$$

Putting all the previous estimates together we get the following (remember that p is the number of external rays landing at α, which depends just on the selection of the secondary limbs):

$$\begin{aligned} p\frac{\mu(W'')}{\operatorname{diam}(W'')^\delta} &\geq K\frac{\mu(Y^1)}{\operatorname{diam}(Y^1)^\delta} + K\sum_{i=1}^{p-1}\frac{\mu(S_i)}{\operatorname{diam}(S_i)^\delta} \geq \\ &\geq K\frac{\mu(Y^1)+\sum_i \mu(S_i)}{\operatorname{diam}(W')^\delta} \\ &= K\frac{\mu(W')}{\operatorname{diam}(W')^\delta}. \end{aligned}$$

So we get the Lemma. □

Lemma 4.5. *The diameter of the puzzle-pieces of the modified principal nest decreases super exponentially fast, i.e.,*

$$\operatorname{diam}(W^n) < e^{-Ln}$$

for any L, if f is a polynomial of high combinatorial type.

Proof. According to our construction of the modified principal nest, the principal modulus $A_n = W^n \setminus W^{n+1}$ grows linearly with n in between to renormalization levels. That means that

$$\operatorname{mod}(W^{m,0} \setminus W^{m,n}) > C \cdot \sum_n L^n$$

(remember that n counts the number of central cascades in a given renormalization level). The constant L is uniform according to [**Lyu97**]. □

Now if we put Lemmas 4.3, 4.4 and 4.5 together we conclude that for any element of the modified principal nest W^n we have (here we are enumerating the whole modified principal nest with just one index):

$$\frac{\mu(W^n)}{(\text{diam}(W^n))^\delta} \geq K(C)^n \frac{\mu(W^0)}{(\text{diam}(W^0))^\delta}.$$

In other words,

$$\frac{\mu(W^n)}{(\text{diam}(W^n))^\delta} \geq aK^n.$$

If we take $\delta' > \delta$ then

$$\frac{\mu(W^n)}{(\text{diam}(W^n))^{\delta'}} = \frac{\mu(W^n)}{(\text{diam}(W^n))^\delta} \frac{1}{(\text{diam}(W^n))^{\delta'-\delta}}.$$

As $\text{diam}(W^n)$ goes to zero super exponentially fast, we have:

$$\frac{\mu(W^n)}{(\text{diam}(W^n))^{\delta'}} > C \geq 0. \tag{3}$$

Let us prove Theorem 1. For each n, let O_n be a family of open sets such that:.

1. For each n, O_n is an open cover of the set $X_n = \{z \in J(f) : \exists j, f^j(z) \in W^n\}$;
2. For $U_n(z) \in O_n$, $\text{diam}(U_n(z))$ goes to zero as n goes to infinity;
3. $\dfrac{\mu(U_n(x))}{\text{diam}(U_n(x))^{\delta'}} > C \geq 0$, for $\delta' > \delta \geq \delta_{inf}$.

The open cover O_n is simply the union of the sets $U_n(z)$ defined as the pull back of W^n along the orbit of $z \in X_n$ from time $t = 0$ to the first time the orbit of z enters W^n.

The first property described above follows from the definition of O_n. The second property follows from the proof of the local connectivity of $J(f)$ for a polynomial high combinatorial type (see [**Lyu97**]). The last property follows from the bounds we have for $\text{mod}(W^n \setminus W^{n-1})$, Lemma 2.3 and Koebe distortion Lemma.

The second and the third property of the covers O_n imply that $\text{HD}(X_n) \leq \delta$. As we are considering δ as an exponent of some conformal measure for f, we actually showed:

$$\text{HD}(X_n) \leq \delta_{inf}. \tag{4}$$

Let $Y_n = J(f) \setminus X_n$. Each set Y_n is f invariant and the dynamics $f : Y_n \to Y_n$ is hyperbolic (one can see this by means of a standard hyperbolic metric argument, using the fact that the dynamics restricted to the closure of the critical orbit is minimal). So, due to Theorem 4.2 we have:

$$\mathrm{HD}(Y_n) \leq \mathrm{hypdym}(J(f)) = \delta_{inf}. \tag{5}$$

Putting all the information together we get:

$$\delta_{inf} = \mathrm{hypdim}(J(f)) \leq \mathrm{HD}(J(f)) \leq \delta_{inf},$$

and then we prove Theorem 1.

Acknowledgments. I would like to thank Misha Lyubich for the many hours of mathematical conversations and for continuous encouragement. I also thank Brian Yarrington and LeRoy Wenstrom for various valuable conversations and suggestions. I am grateful to CNPq-Brazil, Universidade de São Paulo and the State University of New York at Stony Brook for financial support.

References

[**Bow75**] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Mathematics*. Springer-Verlag, 1975.

[**DU91**] M. Denker and M. Urbański. On Sullivan's conformal measures for rational maps of the Riemann sphere. *Nonlinearity*, 4, 1991.

[**Hub**] J. Hubbard. Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. In *Topological methods in modern mathematics, A symposium in honor of John Milnor*. Publish or Perish.

[**Lyu91**] M. Lyubich. On the Lebesgue measure of the Julia set of a quadratic polynomial. *IMS@Stony Brook preprint series*, (1991/10), 1991.

[**Lyu93**] M. Lyubich. Geometry of quadratic polynomials: moduli, rigidity and local connectivity. *IMS@Stony Brook preprint series*, (1993/9), 1993.

[**Lyu97**] M. Lyubich. Dynamics of quadratic polynomials, I-II. *Acta Math.*, 178, (1997).

[**Mil91**] J. Milnor. Local connectivity of Julia sets: expository lectures. *IMS@Stony Brook preprint series*, (1991/10), 1991.

[**Prz93**] F. Przyticki. Lyapunov characteristic exponents are non-negative. *Proc A. M. S.*, 119(1), 1993.

[**Prz96**] F. Przyticki. Iterations of holomorphic Collet-Eckmann maps: conformal and invariant measures. Preprint.

[**Shi91**] M. Shishikura. Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets. *IMS@Stony Brook preprint series*, (1991/7), 1991.

[Sul80] D. Sullivan. *Conformal dynamics*, volume 1007 of *Lecture Notes in Mathematics*. Springer-Verlag, 1980.

[Urb] M. Urbański. Rational functions with no recurrent critical points.

[Wal78] P. Walters. Invariant measures and equilibrium states for some mappings which expand distance. *Trans. of the A. M. S.*, 1978.

Eduardo A. Prado
Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Postal 66281 CEP 05315-970
São Paulo, Brazil

E-mail: eprado@tecepe.com.br