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Complete rotation hypersurfaces with H_k constant in space forms

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Abstract. In this paper we classify all complete rotation hypersurfaces with H_k constant in $\mathbb{R}^{n+1}$ and H^{n+1}, where H_k is the normalized k-th symmetric function of the principal curvatures. Partial results are also given for S^{n+1}.
Keywords: Rotation hypersurfaces, Space forms.

Introduction

Minimal surfaces are among the most studied objects in differential geometry. They are characterized by $H = 0$, where H is the mean curvature of the surface. In recent years, some of their properties have been generalized to constant mean curvature hypersurfaces, and also, to hypersurfaces with H_k constant, where H_k is the normalized k-th symmetric function of the principal curvatures of the hypersurface.

Until now, there have been few examples of this second class of hypersurfaces. In [**1**], do Carmo and Dajczer studied the rotation hypersurfaces with constant mean curvature, and, some years later, Leite and Mori ([**3**], [**4**]) classified the complete rotation hypersurfaces (c.r.h., for short) with constant scalar curvature in space forms.

In this paper we follow the techniques on the papers above to classify c.r.h. with H_k constant in $\mathbb{R}^{n+1}$, $\mathbf{H}^{n+1}$ and $\mathbf{S}^{n+1}$. In the case of $\mathbf{H}^{n+1}$, we will describe all three types of rotational hypersurfaces, as defined in [**1**]. We mainly use Leite's methods, introduced to us by professor M. P. do Carmo, to whom we are indebted for encouragement and constant

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guidance. We had also fruitful discussions with M. L. Leite; some of the results here stated are contained in her joint work with Hounie [**2**], obtained independently from us.

1. Spherical rotation hypersurfaces in space forms

1.1. Notation and basic facts

Let $\bar{M}^{n+1}(c)$ be a complete, simply-connected riemannian manifold with constant curvature $c, c = 0, -1, 1$. Our models for $\bar{M}^{n+1}$ will be the euclidean space $\mathbb{R}^{n+1}$, for $c = 0$; the upper semispace

$$\mathbf{H}^{n+1} = \{x \in \mathbb{R}^{n+1}; x_{n+1} > 0\},$$

for $c = -1$; and the unit sphere $\mathbf{S}^{n+1} \subseteq \mathbb{R}^{n+2}$, for $c = 1$, with the usual metrics.

Definition 1. A (spherical) *rotation hypersurface* $M^n \subseteq \bar{M}^{n+1}(c)$ is an $O(n)$-invariant hypersurface, where $O(n)$ is considered as a subgroup of isometries of $\bar{M}^{n+1}(c)$.

Remark. Strictly speaking, we should add the word *spherical* to our definition in the case $c = -1$, because in this case do Carmo and Dacjzer [**1**] defined another types of rotations (giving rise to the so-called parabolic and hyperbolic hypersurfaces, which we will analyze in the second part of this paper). As in this first part of the paper we will consider only $O(n)$-invariant hypersurfaces, we will drop the word *spherical* for the moment.

$O(n)$ fixes a geodesic γ (the *revolution axis*) and rotates a curve α, called the *profile curve*. We choose γ as $\{x \in \bar{M}^{n+1}(c); x_1 = \cdots = x_n = 0\}$ and α contained in $\{x \in \bar{M}^{n+1}(c); x_2 = \cdots = x_n = 0, x_1 \geq 0\}$.

The orbit of every point in α is an $(n-1)$-dimensional sphere. We choose as parameters of our rotation hypersurface (s, Θ), where s is the arc length of α and $\Theta = (\theta_1, \ldots, \theta_{n-1})$ parameterizes the $(n-1)$-dimensional sphere given by the orbit of $\alpha(s)$. We will also use the following notation: $r(s)$ will denote the (Riemannian) distance from $\alpha(s)$ to γ, realized by a point $P(s)$ in γ, and $h(s)$ will be the (Riemannian) height of $P(s)$ in γ, with respect to a fixed point in γ. Then (see [**3**] or

[5]) the first fundamental form of M is given by

$$I = f^2(r(s)) \sum g_{ij}(\Theta) d\theta_i \otimes d\theta_j + ds \otimes ds$$

where g_{ij} is the metric of constant sectional curvature 1 in an $(n-1)$-dimensional sphere, and $f(r) = r, \sinh r$, or $\sin r$, for $c = 0, -1, 1$, respectively.

Also, the fact that the profile curve α is parameterized by arc length imposes the following restriction over f and h:

$$\dot{r}^2 + \left(\frac{df}{dr}\right)^2 \dot{h}^2 = 1. \tag{1}$$

Theorem 1. **(do Carmo, Dajczer [1].)** *The principal curvatures κ_i of M are*

$$\kappa_i = \frac{\sqrt{1 - cf^2 - \dot{f}^2}}{f}$$

for $i = 1, ..., n-1$, and

$$\kappa_n = -\frac{\ddot{f} + cf}{\sqrt{1 - cf^2 - \dot{f}^2}}$$

where the dot denotes the derivative with respect to s.

The formulas in the theorem above are valid only when $\dot{f}^2 \leq 1 - cf^2$. The set of pairs $(f, \dot{f})$ satisfying this constraint and $f \geq 0$ will be called the *relevant region*.

Let H_k be the normalized k-th symmetric function of the principal curvatures of an hypersurface:

$$\binom{n}{k} H_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_k} \tag{2}$$

Proposition 1. *The rotation hypersurface M^n has the prescribed curvature H_k, $k \leq n$, if and only if f satisfies the following differential equation:*

$$nH_k f^k = (n-k)(1 - cf^2 - \dot{f}^2)^{\frac{k}{2}} - k(1 - cf^2 - \dot{f}^2)^{\frac{k-2}{2}} (\ddot{f} + cf) f \tag{3}$$

for $k \leq n$.

Proof. It follows from (2) and theorem 1. □

From now on, we will suppose that H_k is constant.

Proposition 2. *Equation (3) is equivalent to its first integral*

$$G_k(f,\dot{f}) = f^{n-k}((1-cf^2-\dot{f}^2)^{\frac{k}{2}} - H_k f^k) = A = const. \tag{4}$$

for $k \leq n$.

Proof. We obtain (4) multiplying (3) by f^{n-k-1} and integrating. □

For later reference, we write also the formula for the gradient of G_k:

$$\nabla G_k(f,\dot{f}) = f^{n-k-1}((1-cf^2-\dot{f}^2)^{\frac{k-2}{2}}((n-k)(1-\dot{f}^2)-cnf^2) - nH_k f^k, \\ -kf\dot{f}(1-cf^2-\dot{f}^2)^{\frac{k-2}{2}})$$

for $k < n$, and

$$\nabla G_n(f,\dot{f}) = (-ncf(1-cf^2-\dot{f}^2)^{\frac{n-2}{2}} - nH_n f^{n-1}, -n\dot{f}(1-cf^2-\dot{f}^2)^{\frac{n-2}{2}})$$

for $k = n$.

Following Leite [**3**], we will obtain our results studying the level curves of G_k. The cases $k = 1, 2$ were studied in [**1**], [**3**] and [**4**] and some of the results here stated in the case $k > 2$ were obtained independently by Hounie and Leite in [**2**].

Equation (4) tells us that a local solution f of (3), paired with its first derivative, denoted $(f,\dot{f})$, is a level curve of the function

$$G_k(u,v) = u^{n-k}((1-cu^2-v^2)^{\frac{k}{2}} - H_k u^k) \tag{5}$$

with $u > 0$ and $1 - cu^2 - v^2 \geq 0$.

Lemma 1. *The sets $(f,\dot{f})$, where f is a solution of (3), are the connected components of the level curves of G_k contained in the relevant region.*

Proof. The theory of ODE implies that any local solution of (3) can be extended through values for which $(f,\dot{f})$ is interior to the relevant region. □

Definition 2. A solution of (3) is *complete* if either f is defined for all s or if the pair $(f,\dot{f})$ only admits $(0,\pm 1)$ as limit values.

Geometrically, complete solutions of (3) give rise to a complete rotation hypersurface. When $(f,\dot{f})$ has $(0,1)$ or $(0,-1)$ as limit value, we claim that the profile curve meets orthogonally the axis of rotation, because $\dot{f}^2 = 1$ implies $(\frac{df}{dr}\frac{dr}{ds})^2 = 1$; but $\frac{df}{dr}(0) = 1$, so that $(\frac{dr}{ds})^2 = 1$; substituting this into (1) we have $\frac{dh}{ds} = 0$, so $\frac{dh}{dr} = 0$; this last equation proves our claim.

Before concluding this section, let us say that an hypersurface corresponding to a constant solution of (3) is called a *cylinder*. Also, we say that a rotation hypersurface $M^n \subset \bar{M}^{n+1}(c)$ with axis γ is *cylindrically bounded* if there exist a complete cylinder with same axis γ such that M is contained in the closure of the component of $\bar{M} - C$ containing γ.

1.2. Complete rotation hypersurfaces in $\mathbb{R}^{n+1}$

1.2.1. The case $k < n$

Equations (3) and (4) read in this case

$$nH_k f^k = (n-k)(1-\dot{f}^2)^{\frac{k}{2}} - k(1-\dot{f}^2)^{\frac{k-2}{2}}\ddot{f}f$$
$$G_k(f,\dot{f}) = f^{n-k}((1-\dot{f}^2)^{\frac{k}{2}} - H_k f^k) = A.$$

We first look for the cylinders in $\mathbb{R}^{n+1}$ with H_k constant; they must satisfy the condition

$$nH_k f^k = n - k \tag{6}$$

Proposition 3. (Complete cylinders with H_k constant in $\mathbb{R}^{n+1}$, $k < n$.)

(i) *There are no complete cylinders in $\mathbb{R}^{n+1}$ with $H_k < 0$, k even.*

(ii) *There are no complete cylinders in $\mathbb{R}^{n+1}$ with $H_k = 0$.*

(iii) *For every $H_k > 0$, there is a complete cylinder in $\mathbb{R}^{n+1}$ given by*

$$f^k = \frac{n-k}{nH_k}$$

Proof. It follows directly from equation (6). □

We note also, in case (iii) of the above proposition, that the corresponding value of $A = G_k(f,\dot{f})$, which we denote by A_0, is given by

$$A_0 = \frac{k}{n}\left(\frac{n-k}{nH_k}\right)^{\frac{n-k}{k}}.$$

Theorem 2. (Classification of c.r.h. with H_k constant in $\mathbb{R}^{n+1}$, $k < n$)

(i) *There are no c.r.h. in $\mathbb{R}^{n+1}$ with $H_k < 0$ for k even.*

(ii) *Up to isometries, there is only one monoparametric family of embedded c.r.h. with $H_k = 0$, which converges to a hyperplane. If $2(n-k)/k = 1$, the profile curve is a parabola, if $(n-k)/k = 1$, it is a catenary and if $(n-k)/k > 1$, it asymptotizes two horizontal lines.*

(iii) *Up to isometries, there is only one monoparametric family of embedded c.r.h. with H_k constant for any $H_k > 0$; these hypersurfaces are periodic and cylindrically bounded, and they converge, on one side, to a sequence of spheres, pairwise and vertically tangent; and on the other, to the cylinder given in case (iii) of Proposition 2.*

Proof. Every level curve (see figure 1) can be seen as the smooth union of two graphs

$$(\pm\dot{f})^2 = 1 - (H_k f^k + \frac{A}{f^{n-k}})^{2/k}$$

Figure 1(a) corresponds to the case $H_k < 0$. For every $A > 0$, the corresponding level curve leaves the relevant region when $H_k f^n + A = 0$, so there are no complete hypersurfaces in this case.

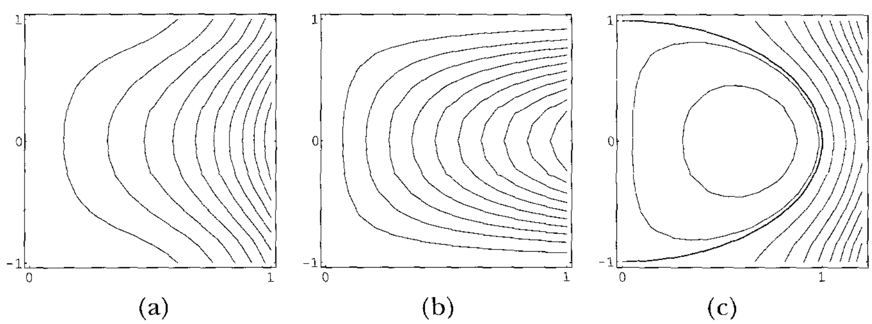


Figure 1: Level curves of G_k, $k < n$, for $\mathbb{R}^{n+1}$. (a) $H_k < 0$; (b) $H_k = 0$; and (c) $H_k > 0$.

Now, let us consider $H_k = 0$; G_k has the form

$$G_k(f, \dot{f}) = f^{n-k}(1 - \dot{f}^2)^{\frac{k}{2}} = A \tag{7}$$

From this formula and the restrictions over f and $\dot{f}$, we have that the set of admissible values for A is $[0, \infty)$. $A = 0$ gives $\dot{f}^2 = 1$, so that

$f(s) = r(s) = \pm s$ and $h(s) = 0$, equations corresponding to a hyperplane.

If $A \neq 0$, we solve (7) for $\dot{f}^2$ to obtain

$$\dot{f}^2 = 1 - \left(\frac{A}{f^{n-k}}\right)^{\frac{2}{k}}.$$

This expression shows that, for every such A, f can assume arbitrarily large values, so $r = f$ has no upper bound and every corresponding hypersurface is not cylindrically bounded. Also, $f = r$ attains a minimum $r_1 > 0$ and this last expression let us set $A = r_1^{n-k}$. We use (1) to write

$$\dot{h}^2 = \left(\frac{A}{f^{n-k}}\right)^{\frac{2}{k}}.$$

Away from r_1, we divide $\dot{h}^2$ by $\dot{f}^2 = \dot{r}^2$ to get

$$\left(\frac{dh}{dr}\right)^2 = \frac{r_1^{2(n-k)/k}}{r^{2(n-k)/k} - r_1^{2(n-k)/k}}$$

This implies that h is given by the following integrals:

$$h = \pm r_1^{(n-k)/k} \int \frac{1}{\sqrt{r^{2(n-k)/k} - r_1^{2(n-k)/k}}}$$

The analysis of the convergence of these integrals for $2(n-k)/k = 1$, $(n-k)/k = 1$ and $(n-k)/k > 1$ was done in [**3**] (p. 294) and we shall omit it. We must mention that Hounie and Leite [**2**] made a more detailed analysis of the convergence of this integrals, so we remit the interested reader to their paper.

When $H_k > 0$, the level curves [see figure 1(c)] corresponding to complete hypersurfaces are given by $A \in [0, A_0]$, where A_0 is the value obtained after Proposition 2; the value $A = 0$ gives, for example, the portion of the ellipse

$$H_k^{2/k} f^2 + \dot{f}^2 = 1$$

contained in the relevant region; this curve joins $(0,1)$ to $(0,-1)$, and its corresponding hypersurface is a sphere parameterized by

$$r(s) = \frac{1}{H_k^{1/k}} \sin\left(H_k^{1/k} s\right), h(s) = \frac{1}{H_k^{1/k}} \cos\left(H_k^{1/k} s\right)$$

The translations of this sphere along the revolution axis give the sequence of spheres pairwise and vertically tangent.

As we said before, the value $A = A_0$ gives a cylinder; all level curves given by $A \in (0, A_0)$ correspond to complete, periodic and cylindrically bounded hypersurfaces (the proof of this fact is entirely similar to that of the case $k = 2$, which can be seen in [**3**], p. 295). □

1.2.2. The case $k = n$

In this case, equation (3) takes the form

$$H_n f^n = -(1 - \dot{f}^2)^{\frac{n-2}{2}} \ddot{f} f \tag{8}$$

Again, first we look for constant solutions of (8), but in this case, the condition $\dot{f} = 0$ implies $H_n = 0$; conversely, $H_n = 0$ implies that every constant $f = c$ is solution of (8).

Proposition 4. *There are complete cylinders with H_n constant in $\mathbb{R}^{n+1}$ if and only if $H_n = 0$.*

Now we study the general case.

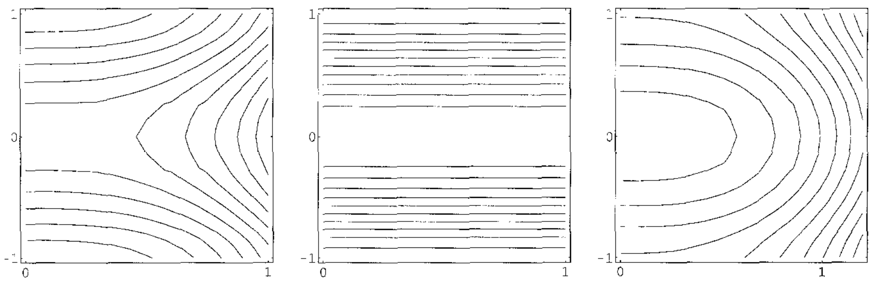
Theorem 3. (Classification of c.r.h. with H_n constant in $\mathbb{R}^{n+1}$) *The only c.r.h. with H_n constant in $\mathbb{R}^{n+1}$ are the hyperplanes, the cylinders and the spheres.*

Proof. Figure 2(a) shows the level curves of G_n for $H_n < 0$; again, it is easy to show that every level curve leaves the relevant region. Also, figure 2(c) shows the level curves for $H_n > 0$; in this case, the only complete solution of (8), according to our definition, corresponds to the value $A = 0$, which gives, for example, the sphere parameterized by

$$r(s) = \frac{1}{H_n^{1/n}} \sin\left(H_n^{1/n} s\right),$$

$$h(s) = \frac{1}{H_n^{1/n}} \cos\left(H_n^{1/n} s\right)$$

This implies that the only c.r.h. in $\mathbb{R}^{n+1}$ with $H_n \neq 0$ are the spheres.



(a) (b) (c)

Figure 2: Level curves of G_n, for $\mathbb{R}^{n+1}$. (a) $H_n < 0$; (b) $H_n = 0$; and (c) $H_n > 0$.

For $H_n = 0$, the level curves of

$$G_n(f, \dot{f}) = (1 - \dot{f}^2)^{n/2}$$

are horizontal lines [see figure 2(b)]; those corresponding to complete hypersurfaces are given by $\dot{f}^2 = 1$ (so, $f(s) = r(s) = \pm s$ and $h(s) = 0$, an hyperplane) and $\dot{f} = 0$, which gives a cylinder. □

We note that, if the level curve has $(0, a)$ as limit value, where $a \in (-1, 1)$, then the corresponding hypersurface meets the rotation axis with a non-right angle, so this hypersurface is not complete.

1.3. Complete rotation hypersurfaces in $\mathbf{H}^{n+1}$

1.3.1. The case $k < n$

Let us write down the formulas (3) and (4) for this case:

$$nH_k f^k = (n-k)(1 + f^2 - \dot{f}^2)^{\frac{k}{2}} - k(1 + f^2 - \dot{f}^2)^{\frac{k-2}{2}}(\ddot{f} - f)f,$$
$$G_k(f, \dot{f}) = f^{n-k}((1 + f^2 - \dot{f}^2)^{\frac{k}{2}} - H_k f^k) = A. \tag{9}$$

We will use the *hyperbolic expression* of $\dot{h}^2$, obtained from (1):

$$\dot{h}^2 = \frac{1 - \dot{r}^2}{\cosh^2 s} = \frac{1 + f^2 - \dot{f}^2}{(1 + f^2)^2}. \tag{10}$$

Before stating our next theorem, we recall that we are dealing with *spherical* hypersurfaces in $\mathbf{H}^{n+1}$.

Theorem 4. (Classification of c.r.h. with H_k constant in $\mathbf{H}^{n+1}$, $k < n$)

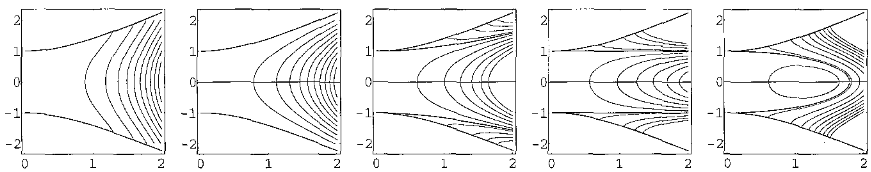
(i) *There are no c.r.h. with* $H_k < 0$ *for* k *even.*

(ii) *Up to isometries, there is only one monoparametric family of embedded c.r.h. with* $H_k \in [0, 1)$. *These hypersurfaces are not cylindrically bounded, and for* $H_k = 0$, *they converge to a totally geodesic hyperbolic space* $\mathbf{H}^n$. *The profile curves are asymptotic to two geodesics.*

(iii) *Up to isometries, there is only one monoparametric family of embedded c.r.h. with* $H_k = 1$. *These hypersurfaces are not cylindrically bounded and they converge to a horosphere.*

(iv) *For any* $H_k > 1$, *there is a one-parameter family of embedded c.r.h. with* H_k *constant, periodic and cylindrically bounded, which converges to a sequence of geodesic spheres.*

Proof. Again, we will study the level curves of $G_k(f, \dot{f})$ with the restrictions $1 + f^2 - \dot{f}^2 \geq 0$ and $f \geq 0$. The proofs are entirely similar to those in the euclidean case and we shall only point out some details. Figure 3 shows these level curves.



(a) (b) (c) (d) (e)

Figure 3: Level curves of G_k, $k < n$, for $\mathbf{H}^{n+1}$ (spherical case). (a) $H_k < 0$; (b) $H_k = 0$; (c) $H_k \in (0, 1)$; (d) $H_k = 1$; and (e) $H_k > 1$.

(i) Figure 3(a) shows the case $H_k < 0$. Every level curve leaves the relevant region in a finite time, so we have no complete hypersurfaces in this case.

(ii) Figure 3(b) shows the case $H_k = 0$; now, G_k can be written as

$$G_k(f, \dot{f}) = f^{n-k} \left(1 + f^2 - \dot{f}^2\right)^{k/2} = A \tag{11}$$

If $A = 0$ and $f(0) = 0$, then $f(s) = \sinh(s)$, so that $r(s) = s$, $h(s) = 0$ and M is an n-dimensional hyperbolic subspace of $\mathbf{H}^{n+1}$.

Figure 3(c) shows the case $H_k \in (0,1)$; if $A = 0$ and $f(0) = 0$, f is given by

$$f(s) = \frac{\sinh\left(\sqrt{1 - H_k^{2/k}}\, s\right)}{\sqrt{1 - H_k^{2/k}}}$$

For $A \neq 0$, the function f, and therefore r, has no upper bound, so no hypersurface is cylindrically bounded. Also, for every such A, f attains a unique minimum f_1. Using (9), we have

$$\dot{f}^2 = 1 + f^2 - \left(\frac{A + H_k f^n}{f^{n-k}}\right)^{2/k}$$

Away from f_1, we may divide $\dot{h}^2$ given in (10) by $\dot{f}^2$ to obtain

$$\left(\frac{dh}{df}\right)^2 = \frac{1}{(1+f^2)^2}\frac{(A + H_k f^n)^{2/k}}{(1+f^2)f^{2(n-k)/k} - (A + H_k f^n)^{2/k}}$$

but the second factor converges when $f \to \infty$; this implies that $h(f)$ is uniformly bounded, which means that the profile curves asymptotizes two geodesics.

(iii) [See figure 3(d)]; when $A = 0$ and $H_k = 1$ in (9), we obtain $1 - \dot{f}^2 = 0$; if $f(0) = 0$, then $f(s) = \pm s$. From (10),

$$\dot{h}^2 = \left(\frac{s}{1+s^2}\right)^2$$

If $h(0) = 0$, then $\pm h(s) = \log\sqrt{1+s^2}$. Using polar coordinates (r,ϕ) and recalling, from hyperbolic geometry, that $\tan\phi = \sinh r(s) = f(s) = \pm s$ and $e^h = \rho$ (where ρ and $\pi/2 - \phi$ are the standard polar coordinates in the plane), we have $\rho = (\sec\phi)^{\pm 1}$. The level curve corresponding to $\rho = \sec\phi$, or $\rho\cos\phi = 1$, is a horizontal line; the curve corresponding to $\rho = \cos\phi$ is the inverse of this horizontal line with respect to the unit circle. The associated hypersurfaces are horospheres.

(iv) Finally, figure 3(e) shows the level curves of G_k for $H_k > 1$. All facts asserted in (iv) can be proved as in the euclidean case for $H_k > 0$.

□

1.3.2. The case $k = n$

Theorem 5. (Classification of c.r.h. with H_n constant in $\mathbf{H}^{n+1}$)

(i) *There are no c.r.h. with $H_n < 0$ for k even.*

(ii) *Up to isometries, there is only one monoparametric family of embedded c.r.h. with $H_n \in [0,1]$, not cylindrically bounded.*

(iii) *Up to isometries, there is only one monoparametric family of compact embedded c.r.h. with $H_n > 1$, cylindrically bounded, convergent on one side to a cylinder.*

Proof. The level curves corresponding to this case appear in figure 4. Formulas (3) and (4) read

$$H_n f^n = -(1 + f^2 - \dot{f}^2)^{\frac{n-2}{2}}(\ddot{f} - f)f$$

$$G_n(f, \dot{f}) = (1 + f^2 - \dot{f}^2)^{\frac{n}{2}} - H_n f^n = A$$

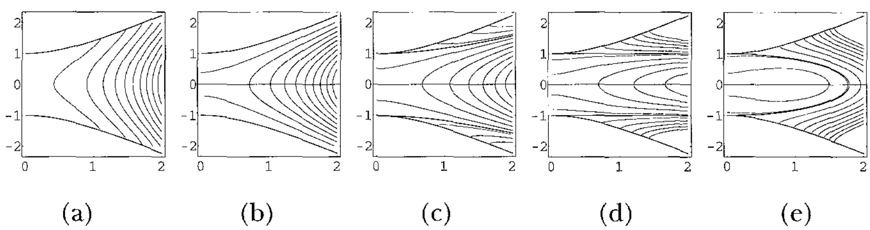


Figure 4: Level curves of G_n, for $\mathbf{H}^{n+1}$ (spherical case). (a) $H_n < 0$; (b) $H_n = 0$; (c) $H_n \in (0,1)$; (d) $H_n = 1$; and (e) $H_n > 1$.

As before, when $H_n < 0$, the level curves of G_n leave the relevant region $\dot{f}^2 - f^2 \leq 1$ [see figure 4(a)].

When $H_n = 0$ [see figure 4(b)], the level curves of $G_n = A$ are hyperbolas (possibly degenerate). We obtain the following expressions for f:

$$f(s) = \begin{cases} \sinh s, & A = 0 \\ \sqrt{1 - A^{2/n}} \sinh s, & A \in (0,1) \\ e^{\pm s}, & A = 1 \\ \sqrt{A^{2/n} - 1} \cosh s, & A > 1 \end{cases}$$

The more interesting case is $A = 0$, which gives, as in the previous case, an n-dimensional hyperbolic space.

The analysis of the remaining cases is similar to that of the case $k < n$; we show in Figures 4(c), (d) and (e) the behaviour of the level curves for $H_n \in (0,1)$, $H_n = 1$ and $H_n > 1$, respectively. $\square$

1.4. Complete rotation hypersurfaces in $\mathbb{S}^{n+1}$

We only have partial results in this case; in particular, the problem of embeddedness is not as clear as in $\mathbb{R}^{n+1}$ or $\mathbf{H}^{n+1}$. The level curves of G_k are similar to the ones obtained in [**3**].

1.4.1. The case $k < n$

Formulas (3) and (4) read:

$$nH_k f^k = (n-k)(1-f^2-\dot{f}^2)^{\frac{k}{2}} - k(1-f^2-\dot{f}^2)^{\frac{k-2}{2}}(\ddot{f}-f)f,$$
$$G_k(f,\dot{f}) = f^{n-k}((1-f^2-\dot{f}^2)^{\frac{k}{2}} - H_k f^k) = A.$$

First we study the critical points of G_k; calculating ∇G_k we see that these critical points must satisfy $1-f^2-\dot{f}^2 = 0$, in which case $H_k = 0$, or $\dot{f} = 0$, which gives the following condition on f:

$$\left(1-f^2\right)^{\frac{k-2}{2}}\left((n-k)-nf^2\right) - nH_k f^k = 0.$$

There is a special value of H_k, denoted by H_k^0, which has only one critical point; it is given by

$$H_k^0 = -\frac{2}{n}\left(\frac{k-2}{n-k}\right)^{\frac{k-2}{2}}.$$

For this value of H_k, the corresponding value of f satisfy

$$f^2 = \frac{n-k}{n-2}.$$

Now we can state our theorem; until now, we have only the following (partial) result.

Theorem 6. (c.r.h. with H_k constant in $\mathbf{S}^{n+1}$, $k < n$)

(i) *There are no c.r.h. with $H_k < H_k^0$ for k even.*

(ii) *Up to isometries, there is only one c.r.h. (in fact, an embedded cylinder) with $H_k = H_k^0$.*

Proof. (i) Figure 5(a) shows the level curves for $H_k < H_k^0$. Every level curve leaves this region in a finite time, so we have no complete hypersurfaces in this case.

(ii) The situation for this case is almost the same as in case (i), but now a critical point appears suddenly, giving rise to a cylinder [See figures 5(b) and (c)]. □

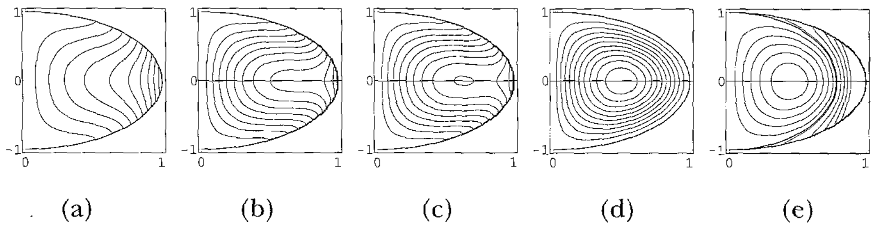


Figure 5: Level curves of G_k, $k < n$, for $\mathbf{S}^{n+1}$. (a) and (b) $H_k < H_k^0$; (c) $H_k^0 < H_k < 0$; (d) $H_k = 0$; and (e) $H_k > 0$.

We will make some remarks on the cases $H_k > H_k^0$ in the final section of this paper.

1.4.2. The case $k = n$

The function G_n is given by

$$G_n(f, \dot{f}) = \left(1 - f^2 - \dot{f}^2\right)^{\frac{n}{2}} - H_n f^n = A$$

Figure 6 shows the level curves of G_n in this case.

If $H_n < 0$ [Figures 6(a) and (b)], the only complete hypersurface corresponds to the unique critical point of G_n, which satisfies

$$nf(1 - f^2)^{(n-2)/2} + nH_n f^{n-1} = 0,$$

or

$$1 = (1 + (-H_n)^{2/(n-2)}) f^2.$$

All other level curves leave the relevant region.

If $H_n \geq 0$ [Figures 6(c) and (d)], G_n has no critical points and the only complete solution according to our definition is obtained from $G_n(f, \dot{f}) = 0$, so that

$$1 = (H_n^{2/n} + 1) f^2 + \dot{f}^2.$$

We call the corresponding hypersurface a *parallel*, in analogy with the situation in $\mathbf{S}^2$. We have then the following:

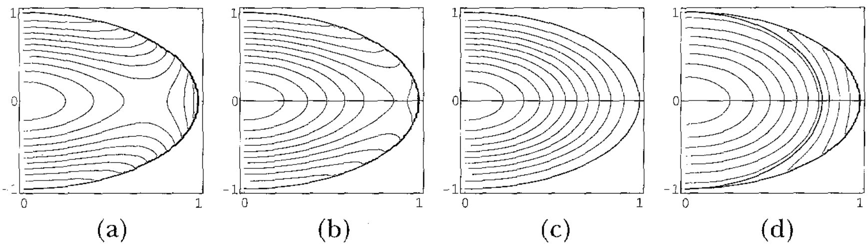


Figure 6: Level curves of G_n, for $\mathbf{S}^{n+1}$. (a) and (b) $H_n < 0$; (c) $H_n = 0$; and (d) $H_n > 0$.

Theorem 7. (c.r.h. with H_n constant in $\mathbf{S}^{n+1}$) *The only c.r.h. with H_n constant in $\mathbf{S}^{n+1}$ are the cylinders and the parallels.*

2. Rotation hypersurfaces in hyperbolic space

2.1. Basic facts

In this section we define the parabolic and hyperbolic rotation hypersurfaces, as given in [**1**], using the hyperboloid model for the hyperbolic space; for completeness, we have included in the definitions the spherical case just analyzed.

Let

$$L^{n+2} = \{x = (x_1, \dots, x_{n+2}), x_i \in \mathbb{R}\}$$

with the Lorentzian metric

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + \cdots + x_{n+2} y_{n+2},$$

where $y = (y_1, \dots, y_{n+2})$. The $(n+1)$*-dimensional hyperbolic space* is given by

$$\mathbf{H}^{n+1} = \{x \in L^{n+2}; \langle x, x \rangle = -1\}$$

An *orthogonal transformation* of L^{n+2} is a linear map preserving $\langle\ ,\ \rangle$, and the orthogonal transformations define, by restriction, all isometries of $\mathbf{H}^{n+1}$. P^k will denote a k-dimensional linear subspace of L^{n+2}, and

$O(P^k)$ will be the set of orthogonal transformations of L^{n+2} with positive determinant which leave P^k pointwise fixed.

Definition. Let $P^2 \subset P^3$ and C be a regular C^2 curve in $P^3 \cap \mathbf{H}^{n+1}$ which does not meet P^2. The orbit of C under the action of $O(P^2)$ is a *spherical* (resp. *parabolic, hyperbolic*) rotation hypersurface if $\langle\ ,\ \rangle|_{P^2}$ is a Lorentzian metric (resp. Riemannian metric, degenerate quadratic form).

In [**1**], do Carmo and Dajczer obtained explicit parameterizations for these hypersurfaces, as follows:

Let $e_1, \ldots, e_{n+2}$ be a basis of L^{n+2} with the following conditions:

1. P^2 is generated by e_{n+1} and e_{n+2};
2. (a) $\langle e_{n+2}, e_{n+2}\rangle = -1$ (spherical case)
   (b) $\langle e_1, e_1\rangle = \langle e_{n+1}, e_{n+1}\rangle = 0$, $\langle e_1, e_{n+1}\rangle = 1$ (parabolic case)
   (c) $\langle e_1, e_1\rangle = -1$ (hyperbolic case)
   (d) $\langle e_i, e_j\rangle = \delta_{ij}$ for all i, j not specified above.

If $x = \sum x_i e_i$ and $y = \sum y_i e_i$, then $\langle x, y\rangle$ is given by

$$\begin{aligned}
&x_1y_1 + \cdots + x_{n+1}y_{n+1} - x_{n+2}y_{n+2} && \text{(spherical case)}\\
&x_1y_{n+1} + x_2y_2 + \cdots + x_ny_n + x_{n+1}y_1 + x_{n+2}y_{n+2} && \text{(parabolic case)}\\
&-x_1y_1 + x_2y_2 + \cdots + x_{n+2}y_{n+2} && \text{(hyperbolic case)}
\end{aligned}$$

Let P^3 be the 3-plane generated by e_1, e_{n+1}, e_{n+2} and the curve C given by $x_1 = x(s)$, $x_{n+1} = x_{n+1}(s)$, $x_{n+2} = x_{n+2}(s)$, $s \in J$, where s is the arc length of C and J is an open interval.

Proposition 5. (**do Carmo, Dajczer [1].**) *With respect to the basis $e_1, \ldots, e_{n+2}$, the following are local parameterizations for the rotation hypersurfaces in $\mathbf{H}^{n+1}$:*

1. *Spherical case:*

$$f(s, \theta_1, \ldots, \theta_{n-1}) = (x\phi_1, \ldots, x\phi_n, x_{n+1}, x_{n+2}),$$

where $\phi = (\phi_1, \ldots, \phi_n)$ is an orthogonal parameterization of the unit sphere in the space generated by $e_1, \ldots, e_n$.

2. *Parabolic case:*

$$f(s,\theta_1,\ldots,\theta_{n-1}) =$$
$$= \left(x, x\theta_1,\ldots,x\theta_i,\ldots,x\theta_{n-1}, -\frac{1+x_{n+2}^2+x^2\sum\theta_i^2}{2x}, x_{n+2}\right)$$

3. *Hyperbolic case:*

$$f(s,\theta_1,\ldots,\theta_{n-1}) = (x\phi_1,\ldots,x\phi_n,x_{n+1},x_{n+2}),$$

where $\phi = (\phi_1,\ldots,\phi_n)$ *is an orthogonal parameterization of the unit hyperbolic space of* $e_1,\ldots,e_n$.

It can be shown (see [4]) that f is an immersion if and only if $x > 0$ in the spherical and parabolic cases, and $x \geq 1$ in the hyperbolic case. These conditions will hold from now on.

We will use the notation M_δ, $\delta = 1, 0$ or -1, for a rotation hypersurface in $\mathbf{H}^{n+1}$, where $\delta = 1$ (resp. $\delta = 0, -1$) means that M_δ is spherical (resp. parabolic, hyperbolic). From now on, we also assume that $\delta + x^2 - \dot{x}^2 \geq 0$ on J, where the dot denotes derivative with respect to s.

Theorem 8. **(do Carmo, Dajczer [1].)** *Let M_δ be a rotation hypersurface in $\mathbf{H}^{n+1}$ defined by the immersion f. Then the directions corresponding to the parameters $\theta_1,\ldots,\theta_{n-1}$ are principal directions; the principal curvatures κ_i along the coordinate curves corresponding to θ_i are all equal and given by*

$$\kappa_i = \frac{\sqrt{\delta + x^2 - \dot{x}^2}}{x}$$

$i = 1,\ldots,n-1$*; the principal curvature along the coordinate curve corresponding to the parameter s is given by*

$$\kappa_n = -\frac{\ddot{x} - x}{\sqrt{\delta + x^2 - \dot{x}^2}}.$$

2.2. Rotation hypersurfaces with H_k constant

Using the definition of H_k given in (2) and proposition 5, we can conclude:

Proposition 6. *The rotation hypersurface M^n_δ has the prescribed curvature H_k, $k \leq n$, if and only if x satisfies the following differential equation:*

$$nH_k x^k = (n-k)(\delta + x^2 - \dot{x}^2)^{\frac{k}{2}} - k(\delta + x^2 - \dot{x}^2)^{\frac{k-2}{2}}(\ddot{x} - x)x \qquad (12)$$

for $k \leq n$.

From now on, we will suppose that H_k is constant. Also, we will analyze only the parabolic and hyperbolic cases.

Proposition 7. *For H_k constant, equation* (12) *has the following first integral:*

$$G_k(x, \dot{x}) = x^{n-k}((\delta + x^2 - \dot{x}^2)^{\frac{k}{2}} - H_k x^k) = A \qquad (13)$$

for $k \leq n$; here A is a constant. In the parabolic case ($\delta = 0$), there exist constant solutions of (12) *if and only if $H_k = 1$; moreover, in this case, every constant function $x = c$ is a solution of* (12)*, and the corresponding value of A in* (13) *is* 0. *In the hyperbolic case ($\delta = -1$), there exist constant solutions of* (12) *if and only if $H_k \in [0, 1)$.*

Proof. The fact that G_k is a first integral of (12) is a straightforward calculation which we shall omit; so, we will analyze the existence of constant solutions.

If we substitute $\delta = 0$ and $\dot{x} = 0$ in (12), it follows that $nH_k x^k = nx^k$, which in turn implies $H_k = 1$ (recall that $x > 0$). Conversely, if $H_k = 1$, then every constant function $x = c$ is a solution of (12).

If $\delta = -1$ and $\dot{x} = 0$ in (12), we solve the equation obtained for H_k to get

$$H_k = \frac{1}{x^k}\left(x^2 - 1\right)^{\frac{k-2}{2}}\left(x^2 - \frac{n-k}{n}\right)$$

The left side of this equation (defined for $x \geq 1$) is an injective function with range equal to the interval $[0, 1)$; this means that for every $H_k \in [0, 1)$ there exists only one constant solution $x = c$ of (12) corresponding to this H_k. $\square$

We will call the rotation hypersurfaces corresponding to constant solutions of (12) *cylinders*. In view of Proposition 7, we have:

Corollary 1. *Every constant function $x = c$ gives rise to a parabolic cylinder with $H_k = 1$. If $H_k \in [0,1)$, there exist only one hyperbolic cylinder with H_k constant.*

Definition 4. A solution $x = x(s)$ of (12) is *complete* if and only if x is defined for all $s \in R$ and $\delta + x^2 - \dot{x}^2 \geq 0$ for all s.

The reason for this definition is that such a solution gives rise to a *complete* rotation hypersurface. As in the first part of this paper, we will investigate the completeness of x by means of the level curves of G_k.

2.3. Parabolic rotation hypersurfaces

Figure 7 shows the level curves of G_k.

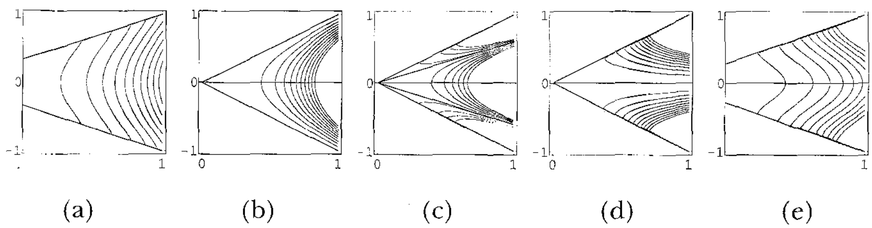


Figure 7: Level curves of G_k, $k \leq n$, for $\mathbf{H}^{n+1}$ (parabolic case). (a) $H_k < 0$; (b) $H_k = 0$; (c) $H_k \in (0,1)$; (d) $H_k = 1$; and (e) $H_k > 1$.

Figures 7(a) and (e) show the level curves of G_k for $H_k < 0$ and $H_k > 1$, respectively; all these level curves leave the relevant region and we have no complete hypersurfaces in these cases. For $H_k = 1$, the only level curves which do not leave this region corresponds to $A = 0$. As we have seen before, these level curve contains all constant solutions of (12).

So, the remaining case is $H_k \in [0,1)$. We recall that G_k has the form

$$G_k(x,\dot{x}) = x^{n-k}((x^2 - \dot{x}^2)^{\frac{k}{2}} - H_k x^k) = A \tag{14}$$

If $A = 0$ in (14), we obtain

$$\left(1 - H_k^{2/k}\right) x^2 - \dot{x}^2 = 0,$$

or

$$\dot{x} = \pm\sqrt{1 - H_k^{2/k}}\, x,$$

and if we impose the initial condition $x(0) = 1$, then

$$x(s) = e^{\pm\sqrt{1-H_k^{2/k}}\,s}.$$

Now, we claim that any two parabolic hypersurfaces with the same H_k and $A \neq 0$ are the same (see [**1**]). For that purpose, let us rewrite (14) in the form

$$\left(x^{(n-k)/k}\dot{x}\right)^2 = x^{2n/k} - (H_k x^n + A)^{2/k}$$

Let $z = x^{n/k}$, so we can write the former equation as

$$\dot{z}^2 = \frac{n^2}{k^2}\left(z^2 - (H_k z^k + A)^{2/k}\right)$$

Reordering and integrating, we have

$$s = \frac{k}{n}\int \frac{dz}{\sqrt{z^2 - (H_k z^k + A)^{2/k}}}$$

Now, take $z = A^{1/k}w$ to obtain

$$s = \frac{k}{n}\int \frac{dw}{\sqrt{w^2 - (H_k w^k + 1)^{2/k}}}$$

This last expression does not depend on A. As in [**1**], we can see that the principal curvatures κ_i also do not depend on A, as well as the first and second fundamental forms. This proves our claim.

We collect all these facts in the following result.

Theorem 9. (Classification of complete parabolic hypersurfaces, $k \leq n$)

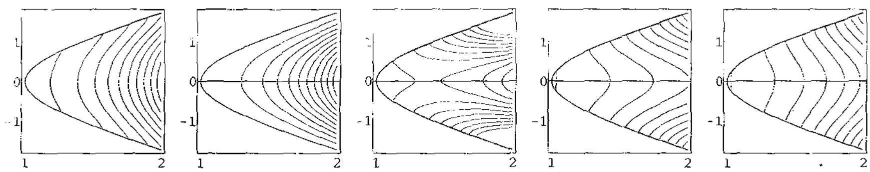
1. *There are no complete parabolic hypersurfaces with* $|H_k| > 1$.
2. *There are no complete parabolic hypersurfaces with* $H_k \in [-1, 0)$, *for* k *even.*
3. *For each* $H_k \in [0, 1)$, *there are only two complete parabolic hypersurfaces with such* H_k *(up to isometries).*

2.4. Hyperbolic rotation hypersurfaces

The analysis of G_k is very similar to that in the parabolic case. In figure 8 we have depicted some level curves of this function in a $(u, \dot{x})$-plane. Our result is as follows.

Theorem 10. (Classification of complete hyperbolic hypersurfaces, $k \leq n$) *If H_k is the (constant) k-th curvature of a hypersurface, then*

1. *There are no complete hyperbolic hypersurfaces with $|H_k| > 1$.*
   T here are no complete hyperbolic hypersurfaces with $H_k \in [-1, 0)$, for k even.
2. *For each $H_k \in [0, 1)$, there exist a one-parameter family of complete hyperbolic hypersurfaces with such H_k.*



(a) (b) (c) (d) (e)

Figure 8: Level curves of G_k, $k \leq n$, for $\mathbf{H}^{n+1}$ (hyperbolic case). (a) $H_k < 0$; (b) $H_k = 0$; (c) $H_k \in (0, 1)$; (d) $H_k = 1$; and (e) $H_k > 1$.

3. Open questions

In hyperbolic space, we have not been able to determine explicitly many of the hypersurfaces which "bound" the families here described.

In the case of $\mathbf{S}^{n+1}$, we have included the figures of the cases which we have not studied in detail. Figures 5(a) and (b) show the level curves corresponding to two different values of $H_k < H_k^0$. Figure 5(c) shows that, when $H_k^0 < H_k < 0$, G_k has two critical points, one of which corresponds to a cylinder.

Let us call A_1 the value of G_k at this critical point, and A_0 the value of G_k at the other critical point. Then, every level curve corresponding to $A \in (A_0, A_1)$ is a closed curve, which may or not correspond to a hypersurface in $\mathbf{S}^{n+1}$, this fact depending on the period of the profile curve.

We may calculate this period as follows: solving the expression of G_k for $\dot{f}^2$, and substituting the result in the *spherical expression of* $\dot{h}^2$ [obtained from (1)], we get

$$\dot{h}^2 = \frac{(H_k f^n + A)^{2/k}}{f^{2(n-k)/k}\left(1 - f^2\right)^2}.$$

For every $A \in (A_0, A_1)$, f attains a minimum, so r attains a minimum r_1. Away from r_1, we divide this last formula by $\dot{f}^2$ to get

$$\left(\frac{dh}{df}\right)^2 = \frac{(H_k f^n + A)^{2/k}}{f^{2(n-k)/k}\left(1 - f^2\right)^2 - (H_k f^n + A)^{2/k}} \cdot \frac{1}{(1-f^2)^2}.$$

Then, thinking on h as a function of f, the period P of the profile curve is

$$P = 2\int_{f_0}^{f_1} \frac{dh}{df} = 2\int_{f_0}^{f_1} \sqrt{\frac{(H_k f^n + A)^{2/k}}{f^{2(n-k)/k}\left(1 - f^2\right)^2 - (H_k f^n + A)^{2/k}} \cdot \frac{1}{1-f^2}},$$

where the limits f_0, f_1 of the above integral are solutions of $G_k(f, 0) = A$. It is clear that the profile curve gives rise to an immersed hypersurface if and only if the period is a rational multiple of 2π, and to an embedded hypersurface if and only if the period is precisely 2π.

In [**3**], Leite analyzed the above integrals for $k = 2$, showing that, for $H_2 > H_k^0$ (our notation differs slightly), there exists a countable family of c.r.h. with such H_2. In fact, H. Mori [**4**] asserted that it is a monoparametric family, but we are not sure of this fact. We expect that Leite's result can be generalized to H_k.

Finally, we did not draw the profile curves, because they are completely analogous to the case $k = 2$, studied in detail by Leite. We refer the interested reader to [**3**].

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