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Irreducibility of moduli space of harmonic 2-spheres in S^4

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Abstract. In this paper we prove that for $d \geq 3$, the moduli spaces of degree d branched superminimal immersions of the 2-sphere into S^4 has 2 irreducible components. Consequently, the moduli space of degree d harmonic 2-spheres in S^4 has 3 irreducible components.

Keywords: harmonic maps, superminimal surfaces, moduli space.

0. Introduction

Recall that in the Calabi construction (see [**C**]) the space of branched minimal immersions (or, equivalently, the space of harmonic maps) of S^2 into S^{2n} decomposes into a union of moduli spaces H_k labelled by the twistor degree $k \geq 1$. If f is a linearly full (that is, its image is not contained in a totally geodesic proper subsphere) branched minimal immersion of S^2 to S^4, Calabi showed that either f or $-f = A \circ f$ (where A is the antipodal map on S^{2n}) has a holomorphic horizontal lift to the twistor space of S^{2n}. A holomorphic curve in the twistor space can be classified by its homology (or twistor) degree. Barbosa [**Ba**] later showed that if the degree of the lifted curve is $k \in \mathbb{Z}^+$, then $\mathrm{Area}(f(S^2)) = \mathrm{Energy}(f) = 4\pi k$. When $n = 2$, the twistor space of S^4 is $\mathbb{CP}^3$. The study of the space of harmonic maps of S^2 into S^4 (modulo the antipodal map on S^4) thus reduces to the study of holomorphic horizontal rational curves in $\mathbb{CP}^3$.

In [**Lo1**] the study of holomorphic horizontal rational curves of degree k in $\mathbb{CP}^3$ was reduced to the study of the moduli space $\mathcal{M}_k$ of pairs of

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meromorphic functions of degree k with the same ramification divisor. (This has been further refined and generalized in [**Lo2**].)

The moduli space of harmonic spheres in S^4 was also studied by Verdier in [**V1**] and [**V2**]. In [**V1**] he asserted that when $d \geq 3$, the moduli space of degree d harmonic maps of S^2 to S^4 has three irreducible components. However, he merely provided a sketch of the proof of this assertion. In this paper, we shall furnish the details of the proof. Many of the spaces described in [**V1**] are dual to the corresponding ones described in this paper. The way the moduli spaces are described in this paper enables one to describe the moduli of higher genus branched superminimal surfaces in S^4 as well the moduli of quaternionic branched superminimal surfaces in $\mathbb{HP}^n$. (See [**Lo2**] and [**KL**].)

1. The moduli space

Let us first recall the twistor fibration $\mathbb{CP}^1 \to \mathbb{CP}^3 \xrightarrow{\pi} S^4$. Consider $\mathbb{H}^2$ as a quaternion module with right scalar multiplication. The quaternionic projective line, $\mathbb{HP}^1$, is just the quotient of $\mathbb{H}^2 \setminus \{0\}$ by the action of right scalar multiplication by nonzero quaternions. We identify $\mathbb{C}^4$ with $\mathbb{H}^2$ via $(z_0, z_1, z_2, z_3) \mapsto (z_0 + jz_1, z_2 + jz_3)$, and hence obtain the formula for the twistor fibration over $S^4 \cong \mathbb{HP}^1$:

$$\mathbb{CP}^3 \ni [z_0, z_1, z_2, z_3] \stackrel{\pi}{\longmapsto} [z_0 + jz_1, z_2 + jz_3] \in \mathbb{HP}^1.$$

Using the Fubini-Study metric we obtain a splitting of the tangent space, $T\mathbb{CP}^3 = \mathcal{V} \oplus \mathcal{H}$, into vertical and horizontal components. The horizontal distribution, $\mathcal{H}$, has complex codimension 1 and is a holomorphic subbundle of $T\mathbb{CP}^3$. This distribution is a holomorphic contact structure: it is the kernel of a contact 1-form θ on $\mathbb{CP}^3$ with values in a holomorphic line bundle $\mathcal{L}$, and it is encoded in the following exact sequence:

$$0 \to \mathcal{H} \to T\mathbb{CP}^3 \xrightarrow{\theta} \mathcal{L} \to 0.$$

This contact form has a lifting to $\mathbb{C}^4$ given by

$$\Omega = z_0\, dz_1 - z_1\, dz_0 + z_2\, dz_3 - z_3\, dz_2. \tag{1.1}$$

Surfaces which arise as twistor projections of holomorphic curves tangent to the horizontal distribution in $\mathbb{CP}^3$ are known to be harmonic, and in fact branched minimal (see [**Br**], [**La**], [**Lo1**], [**ES**] and [**BR**]). They are called *branched superminimal surfaces.* In the genus zero case, if f is branched minimal, then either f or $-f$ is branched superminimal.

Remark 1.2. It is well known that if $f: S^2 \to S^4$ is harmonic, and $A: S^4 \to S^4$ is the antipodal map, then either f or $-f := A \circ f$ has a holomorphic horizontal lift to $\mathbb{CP}^3$. (If f is totally geodesic, then both f and $-f$ have holomorphic horizontal lifts.) Also, if $g: S^2 \to S^4$ is branched superminimal, then both g and $A \circ g$ are harmonic. (cf. [**V1**]).

Let $\gamma: S^2 \to \mathbb{CP}^3$ be a holomorphic map of degree d. This map can be expressed using homogeneous coordinates in $\mathbb{CP}^3$ as follows: $\gamma(z) = [s_0(z), s_1(z), s_2(z), s_3(z)]$, where $s_0, s_1, s_2, s_3 \in H^0(S^2, \mathcal{O}(d))$. (Note that if we choose homogeneous coordinates z_0, z_1 for $S^2 = \mathbb{CP}^1$, the s_i's are just homogeneous polynomials of degree d in the variables z_0, z_1.) Since $\gamma(S^2) \subset \mathbb{CP}^3$ is a curve of degree d, the linear system $\langle s_0, s_1, s_2, s_3\rangle$ has no base points. It follows from (1.1) that $\gamma = [s_0, s_1, s_2, s_3]$ is a holomorphic horizontal curve of degree d if

$$\begin{aligned} &s_0, s_1, s_2, s_3 \in H^0(S^2, \mathcal{O}(d)), \\ &\langle s_0, s_1, s_2, s_3\rangle \text{ is base-point-free and} \\ &s_0 s_1' - s_1 s_0' + s_2 s_3' - s_3 s_2' = 0. \end{aligned} \tag{1.3}$$

Notation. In this paper we shall let $W_d = H^0(S^2, \mathcal{O}(d))$ and W_d^k denote the k-fold product of W_d. We shall let K denote the canonical line bundle over S^2, and $W_{2d-2} = H^0(S^2, \mathcal{O}(d)^2 \otimes K) = H^0(S^2, \mathcal{O}(2d-2))$. Also, given a vector space V, we shall let $\mathbb{P}V$ denote the projective space of lines through the origin in V. Now let S be a subset (not necessarily a subspace) of V. Suppose that S is closed under scalar multiplication. Define an equivalence relation on $S \setminus \{0\}$ as follows: $s \sim \lambda s$ for any $\lambda \in \mathbb{C} \setminus \{0\}$. We shall let $\mathbb{P}S$ denote the set of equivalence classes $(S \setminus \{0\})/\!\sim$.

Consider the following maps:

$$\begin{aligned} \text{Ram:} \quad & W_d^2 \longrightarrow W_{2d-2} \\ & (s, t) \longmapsto s\, dt - t\, ds, \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} \psi\colon\quad W_d^4 &\longrightarrow W_{2d-2} \\ (s_0, s_1, s_2, s_3) &\longmapsto s_0\, ds_1 - s_1\, ds_0 + s_2\, ds_3 - s_3\, ds_2. \end{aligned} \tag{1.5}$$

Observe that $\psi(s_0, s_1, s_2, s_3) = \mathrm{Ram}(s_0, s_1) + \mathrm{Ram}(s_2, s_3)$. We are interested in the space $\psi^{-1}(0)$. (Note: $\psi^{-1}(0)$ is closed under scalar multiplication.) In particular, we shall study the projective variety

$$\mathcal{M}_d := \mathbb{P}(\psi^{-1}(0)) \subset \mathbb{P}W_d^4 \cong \mathbb{CP}^{4d+3}.$$

Lemma 1.6. *Each irreducible component of $\mathcal{M}_d$ has dimension greater than or equal to $2d+4$.*

Proof. From (1.5) and the fact that $\dim W_{2d-2} = 2d-1$, we obtain $2d-1$ relations for $\psi^{-1}(0)$. The lemma follows since $4d+3-(2d-1) = 2d+4$. $\square$

Let $\mathcal{C}_d := \{[s_0, s_1, s_2, s_3] \in \mathbb{P}W_d^4 \mid \langle s_0, s_1, s_2, s_3\rangle \text{ has base-points}\}$. Observe that elements of $\mathcal{C}_d$ correspond to parametrized rational curves in $\mathbb{CP}^3$ of degree strictly less than d. It follows from (1.3) that branched superminimal immersions of S^2 into S^4 of area $4\pi d$ are parametrized by the quasi-projective variety

$$\mathcal{S}_d := \mathcal{M}_d \setminus \mathcal{C}_d. \tag{1.7}$$

In order to understand the structure of $\mathcal{S}_d$, we need to understand the structure of $\mathcal{M}_d$. To do this, we need to study the map Ram in greater detail.

Remark. Since $\mathrm{Ram}(s,t) = -\mathrm{Ram}(t,s)$, Ram is well defined on simple bivectors: $s \wedge t \mapsto s\,dt - t\,ds$ and thus extends linearly to a map

$$\begin{aligned} \Lambda : {\textstyle\bigwedge^2} W_d &\longrightarrow W_{2d-2} \\ \sum_{i<j} a_{ij} s_i \wedge s_j &\longmapsto \sum_{i<j} a_{ij}\{s_i\, ds_j - s_j\, ds_i\}. \end{aligned} \tag{1.8}$$

This map is well-known and it is called the *Gaussian map* (see [**Wa1**], [**Wa2**]).

Consider a diagram

$$\begin{tikzcd} & X \times X \arrow[d, "\phi\times\psi"] \\ Y \arrow[r, "\Delta"] & Y \times Y, \end{tikzcd}$$

where Δ is the diagonal inclusion. The fibre product $X \times_Y X$ is just the restriction to the diagonal of the product $\phi \times \psi : X \times X \to Y \times Y$, that is, $X \times_Y X = \Delta^*(X \times X)$.

From the observation following (1.5), the horizontality condition can be expressed as follows:

$$\mathrm{Ram}(s_0, s_1) = -\mathrm{Ram}(s_2, s_3).$$

From the diagram:

$$\begin{tikzcd} & W_d^2 \times W_d^2 = W_d^4 \arrow[d, "\mathrm{Ram}\times\{-\mathrm{Ram}\}"] \\ W_{2d-2} \arrow[r, "\Delta"] & W_{2d-2} \times W_{2d-2}, \end{tikzcd} \tag{1.9}$$

where Δ is the diagonal inclusion, consider the fibre product

$$\mathcal{F}_d := W_d^2 \times_{W_{2d-2}} W_d^2 \subset W_d^4.$$

Note that $\mathcal{F}_d$ is not a subspace of W_d^4; nevertheless, it is closed under scalar multiplication since $\mathrm{Ram}(\lambda s, \lambda t) = \lambda^2 \mathrm{Ram}(s,t)$ for any $\lambda \in \mathbb{C}$ and $(s,t) \in W_d^2$. The projectivized fibre product $\mathbb{P}\mathcal{F}_d$ is defined as $(\mathcal{F}_d \setminus \{0\})/\sim$, the set of equivalence classes under scalar multiplication. Observe that we have a natural identification:

$$\mathcal{M}_d \cong \mathbb{P}\mathcal{F}_d \subset \mathbb{P}W_d^4. \tag{1.10}$$

Let us first examine the fibre above $0 \in W_{2d-2}$ in the fibre product $\mathcal{F}_d$. Let $\mathcal{B}_d := \{(s,t) \in W_d^2 \mid \mathrm{Ram}(s,t) = 0\}$. Observe that $(s,t) \in \mathcal{B}_d$ if and only if $\mu s = \lambda t$ for some $(\mu, \lambda) \in \mathbb{C}^2 \setminus \{(0,0)\}$ if and only if $s \wedge t = 0$. Thus,

$$\mathcal{B}_d = \{(a\sigma, b\sigma) \in W_d^2 \mid \sigma \in W_d,\ [a,b] \in \mathbb{CP}^1\}. \tag{1.11}$$

It follows from (1.9) and (1.11) that the fibre above $0 \in W_{2d-2}$ in the fibre product $\mathcal{F}_d$ is just the set

$$\mathcal{B}_d \times \mathcal{B}_d = \{(a\sigma, b\sigma, c\tau, d\tau) \mid (\sigma,\tau) \in W_d^2, \ \ [a,b],[c,d] \in \mathbb{CP}^1\}. \quad (1.12)$$

Let N and S denote the pair of antipodal points $[1,0],[0,1] \in \mathbb{HP}^1 \cong S^4$ respectively, and consider the following pair of projective lines in $\mathbb{CP}^3$ corresponding to the twistor fibres above N and S:

$$L_0 := \{[z_0, z_1, 0, 0] \mid (z_0, z_1) \in \mathbb{C}^2 \setminus \{(0,0)\}\} = \pi^{-1}([1,0]),$$

and

$$L_1 := \{[0, 0, z_2, z_3] \mid (z_2, z_3) \in \mathbb{C}^2 \setminus \{(0,0)\}\} = \pi^{-1}([0,1]).$$

Fix $([a,b],[c,d]) \in \mathbb{CP}^1 \times \mathbb{CP}^1$, let ℓ denote the line in $\mathbb{CP}^3$ spanned by the points $[a,b,0,0] \in L_0$ and $[0,0,c,d] \in L_1$ and let $S_\ell = \pi(\ell) \subset S^4$. It follows from [**Lo1**] that ℓ is horizontal and that S_ℓ is a totally geodesic 2-sphere passing through N and S in S^4. Now choose $(\sigma,\tau) \in W_d^2$ such that the pencil $\langle\sigma,\tau\rangle$ is base-point-free and set $\phi = [(a\sigma, b\sigma, c\tau, d\tau)] \in \mathbb{P}(\mathcal{B}_d \times \mathcal{B}_d)$. Then $\phi = [\sigma(a,b,0,0) + \tau(0,0,c,d)]$ and hence $\phi(S^2) \subset \ell$, that is, $\phi : S^2 \to \ell$ is a branched covering map. Fixing a line $\ell \subset \mathbb{CP}^3$ which intersects the twistor lines L_0 and L_1 amounts to fixing a totally geodesic 2-sphere $S_\ell \subset S^4$ passing through $N, S \in S^4$, and the composition $\pi \circ \phi$ is just a branched covering map $S^2 \xrightarrow{\pi\circ\phi} S_\ell$. Thus, the space $\mathbb{P}(\mathcal{B}_d \times \mathcal{B}_d) \backslash \mathcal{C}_d$ parametrizes a family of horizontal lines in $\mathbb{CP}^3$ which project to totally geodesic 2-spheres passing through N and S in S^4.

Since $\dim W_d = d+1$, (1.11) implies that $\dim \mathcal{B}_d = d+2$, and thus from (1.12)

$$\dim(\mathbb{P}(\mathcal{B}_d \times \mathcal{B}_d)) = 2d+3. \quad (1.13)$$

Remark. Since elements of $\mathbb{P}(\mathcal{B}_L \times \mathcal{B}_L)$ are curves in $\mathbb{CP}^3$ which intersect $L_0 \cup L_1$, they are elements in the *boundary* of the moduli space described in [**Lo1**].

We now examine the case when $\mathrm{Ram}(s,t) \neq 0$ or, equivalently, $s \wedge t \neq 0$. Since $[s,t] \in \mathbb{P}(W_d^2 \backslash \mathcal{B}_d)$ implies that $s \wedge t \neq 0$, we may consider $[s,t]$ as a projectivized ordered 2-frame in W_d. We can thus identify $\mathbb{P}V(2, W_d)$, the *projectivized Stiefel manifold* of ordered 2-frames in W_d, with the set $\mathbb{P}(W_d^2 \setminus \mathcal{B}_d)$. Now, consider the action

$$\begin{aligned} \mathrm{PGL}(2,\mathbb{C})\times\mathbb{P}V(2,W_d) &\longrightarrow \mathbb{P}V(2,W_d) \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, [s,t]\right) &\longmapsto [as+bt, cs+dt]. \end{aligned}$$

Let $[s \wedge t]$ denote the 2-plane in W_d spanned by s and t. Observe that the principal $\mathrm{PGL}(2,\mathbb{C})$-bundle over the Grassmannian $G(2,W_d)$ is given by

$$\begin{aligned} \mathbb{P}V(2,W_d) &\xrightarrow{p} G(2,W_d) \\ [s,t] &\longmapsto [s\wedge t]. \end{aligned}$$

Given a line bundle $L \to S^2$ and a holomorphic section $\alpha \in H^0(S^2, L)$, we shall let (α) denote the divisor of α. We have an induced map

$$(\mathrm{Ram})\colon \mathbb{P}V(2,W_d) = \mathbb{P}(W_d^2 \setminus \mathcal{B}_d) \longrightarrow \mathbb{P}W_{2d-2} \tag{1.14}$$

defined by sending a point $[s,t]$ to the divisor $(s\,dt - t\,ds)$. A simple computation shows that the ramification map (1.14) factors through the Grassmannian, that is, the following diagram commutes:

$$\begin{tikzcd} \mathbb{P}V(2,W_d) \arrow[r, equal] \arrow[d, "p"'] & \mathbb{P}V(2,W_d) \arrow[d, "(\mathrm{Ram})"] \\ G(2,W_d) \arrow[r, "{[\mathrm{Ram}]}"] & \mathbb{P}W_{2d-2}, \end{tikzcd} \tag{1.15}$$

where the map [Ram] is given by $[s\wedge t] \mapsto (s\,dt - t\,ds)$. (This is independent of the choice of spanning vectors of the 2-plane.) Consider the Plücker embedding $G(2,W_d) \hookrightarrow \mathbb{P}(\bigwedge^2 W_d)$ and let $\mathcal{G}$ denote the image of the Grassmannian in $\mathbb{P}(\bigwedge^2 W_d)$. ($\mathcal{G}$ can be identified with the set of projectivized simple bivectors in W_d.) Recall the map:

$$\Lambda\colon \bigwedge^2 W_d \to W_{2d-2}. \tag{1.8}$$

Projectivizing (1.8), we obtain a rational map:

$$\bar{\Lambda}\colon \mathbb{P}\bigwedge^2 W_d \dashrightarrow \mathbb{P}W_{2d-2}. \tag{1.16}$$

Since $\Lambda(v \wedge w) = 0$ implies that $v \wedge w = 0$, the base locus of the map $\bar{\Lambda}$ does not intersect $\mathcal{G}$.

Remark 1.17. The map $\bar{\Lambda}|_{\mathcal{G}}: \mathcal{G} \to \mathbb{P}W_{2d-2}$ is precisely the map [Ram]: $G(2, W_d) \to \mathbb{P}W_{2d-2}$. It was shown in [**Lo1**] that $\bar{\Lambda}|_{\mathcal{G}}$ is a finite map of degree $(2d-2)!/d!(d-1)!$ onto $\mathbb{P}W_{2d-2}$. (This is precisely the degree of $\mathcal{G}$ in $\mathbb{P}\bigwedge^2 W_d$.)

Recall from [**Lo1**] that a holomorphic horizontal map from S^2 to $\mathbb{CP}^3$ of degree $d \leq 2$ is just a (branched) covering map to a horizontal projective line in $\mathbb{CP}^3$ with twistor image lying in a totally geodesic 2-sphere in S^4. Henceforth, we shall assume that $d \geq 3$.

Consider the diagram

$$\begin{array}{ccc} & & G(2, W_d) \times G(2, W_d) \\ & & \Big\downarrow {\scriptstyle [\mathrm{Ram}] \times [\mathrm{Ram}]} \\ \mathbb{P}W_{2d-2} & \xrightarrow{\Delta} & \mathbb{P}W_{2d-2} \times \mathbb{P}W_{2d-2}, \end{array}$$

where Δ is the diagonal inclusion. Since [Ram] : $G(2, W_d) \to \mathbb{P}W_{2d-2}$ is a finite map by Remark 1.17, we see that

$$\dim(G(2, W_d) \times_{\mathbb{P}W_{2d-2}} G(2, W_d)) = \dim(G(2, W_d)) = 2(d-1). \tag{1.18}$$

Similarly, we obtain the fibre product $\mathbb{P}V(2, W_d) \times_{\mathbb{P}W_{2d-2}} \mathbb{P}V(2, W_d)$ from the diagram:

$$\begin{array}{ccc} & & \mathbb{P}V(2, W_d) \times \mathbb{P}V(2, W_d) \\ & & \Big\downarrow {\scriptstyle [\mathrm{Ram}] \times [\mathrm{Ram}]} \\ \mathbb{P}W_{2d-2} & \xrightarrow{\Delta} & \mathbb{P}W_{2d-2} \times \mathbb{P}W_{2d-2}, \end{array}$$

where Δ is the diagonal inclusion.

Remark. The total space of the $(\mathrm{PGL}(2, \mathbb{C}) \times \mathrm{PGL}(2, \mathbb{C}))$-bundle over the fibre product $G(2, W_d) \times_{\mathbb{P}W_{2d-2}} G(2, W_d)$ is $\mathbb{P}V(2, W_d) \times_{\mathbb{P}W_{2d-2}} \mathbb{P}V(2, W_d)$.

Recall the following lemma:

Lemma 1.19. *Let L be a line bundle over S^2 and let $s_0, s_1 \in H^0(S^2, L)$ be two nonzero sections. If $(s_0) = (s_1) \in \mathbb{P}H^0(S^2, L)$, then there is a unique $\alpha \in \mathbb{C} \setminus \{0\}$ such that $s_0 = \alpha s_1$.*

Let $([s,t],[u,v]) \in \mathbb{P}V(2,W_d) \times_{\mathbb{P}W_{2d-2}} \mathbb{P}V(2,W_d)$. Since

$$[\mathrm{Ram}]([s \wedge t]) = [\mathrm{Ram}]([u \wedge v]) \in \mathbb{P}W_{2d-2},$$

by the above lemma there exists a *unique* $-\lambda^2 \in \mathbb{C} \setminus 0$ such that

$$\mathrm{Ram}(s,t) = -\lambda^2 \,\mathrm{Ram}(u,v) = -\,\mathrm{Ram}(\pm\lambda u, \pm\lambda v),$$

thereby giving us a pair $[s,t,\pm\lambda u,\pm\lambda v] \in \mathcal{M}_d$. This gives us a 2:1 correspondence

$$\begin{aligned} \mathcal{M}_d \setminus \mathbb{P}(\mathcal{B}_d \times \mathcal{B}_d) &\longrightarrow \mathbb{P}V(2,W_d) \times_{\mathbb{P}W_{2d-2}} \mathbb{P}V(2,W_d) \\ [s,t,\pm\lambda u,\pm\lambda v] &\longmapsto ([s,t],[u,v]). \end{aligned}$$

This 2:1 correspondence is related to the contact involution discussed in [**Lo1**].

Remark 1.20. Let $G = (\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))/\pm 1$, where $1 = (I,I)$.
(1) G acts on $\mathcal{M}_d$ and leaves the set $\mathbb{P}(\mathcal{B}_d \times \mathcal{B}_d)$ invariant.
(2) G acts freely on $\mathcal{M}_d \setminus \mathbb{P}(\mathcal{B}_d \times \mathcal{B}_d)$. In fact

$$\begin{aligned} \mathcal{M}_d \setminus \mathbb{P}(\mathcal{B}_d \times \mathcal{B}_d) &\longrightarrow G(2,W_d) \times_{\mathbb{P}W_{2d-2}} G(2,W_d) \\ [s_0,s_1,s_2,s_3] &\longmapsto ([s_0 \wedge s_1],[s_2 \wedge s_3]) \end{aligned}$$

is a principal G-bundle.
(3) $G \longrightarrow \mathrm{PGL}(2,\mathbb{C}) \times \mathrm{PGL}(2,\mathbb{C})$ is a 2:1 covering map and thus so is

$$\mathcal{M}_d \setminus \mathbb{P}(\mathcal{B}_d \times \mathcal{B}_d) \longrightarrow \mathbb{P}V(2,W_d) \times_{\mathbb{P}W_{2d-2}} \mathbb{P}V(2,W_d).$$

Observe from (1.13) that $\dim \mathbb{P}(\mathcal{B}_d \times \mathcal{B}_d) = 2d+3$ and hence, by Lemma 1.6, $\mathbb{P}(\mathcal{B}_d \times \mathcal{B}_d)$ cannot be a component of $\mathcal{M}_d$. The fact that $\dim G = 6$, together with Lemma 1.6, Equation 1.18 and Remark 1.20 (2), imply that each irreducible component of $\mathcal{M}_d$ is of dimension $2(d-1)+6 = 2d+4$. Thus:

Lemma 1.21. *Each irreducible component of $\mathcal{M}_d$, and hence of $\mathcal{S}_d$, is of pure dimension $2d+4$.*

2. Monodromy and irreducibility

Given a finite covering map $A \to B$ with fibre F, the monodromy group is the image of $\pi_1(B)$, the fundamental group of the base, in $\mathfrak{S}\rho(F)$, the symmetric group of the fibre. Recall that [Ram] : $G(2, W_d) \to \mathbb{P}W_{2d-2}$ is a branched covering map. Let $\mathfrak{R}$ and $\mathfrak{B}$ denote the ramification and branching loci of [Ram] respectively. Then

$$[\mathrm{Ram}] : G(2, W_d) \setminus \mathfrak{R} \longrightarrow \mathbb{P}W_{2d-2} \setminus \mathfrak{B}$$

is a covering map with fibre F consisting of $\rho = (2d-2)!/d!(d-1)!$ points.

Monodromy Theorem (Eisenbud-Harris). *The monodromy of*

$$[\mathrm{Ram}] : G(2, W_d) \setminus \mathfrak{R} \longrightarrow \mathbb{P}W_{2d-2} \setminus \mathfrak{B}$$

is the full symmetric group $\mathfrak{S}_\rho(F)$.

This theorem was proved by Eisenbud and Harris (cf. [**EH**]) in their study of limit linear series. It is used in the proof of the following:

Theorem 1. *The moduli space* $\mathcal{S}_d$ *has two irreducible components when* $d \geq 3$.

This theorem was first mentioned by Verdier (cf. [**V1**]), who claimed that it followed immediately from the Monodromy Theorem. However, since Verdier did not provide a complete proof of the irreducibility result, we shall furnish one here. We begin with some notation and lemmas on topology.

Let (Y, y_0) be a connected manifold with base point y_0, let Z be a manifold which is not necessarily connected and let $p : (Z, z_0) \to (Y, y_0)$ be a finite covering map. Let $F := p^{-1}(y_0)$ denote the fibre above the base point (so $z_0 \in F$), and let $\mathcal{G}$ denote the monodromy group. Finally let $\widetilde{Y}$ denote the universal cover of Y.

Consider the normal subgroup

$$H := \bigcap_{\zeta \in F} p_* \pi_1(Z, \zeta) \lhd \pi_1(Y, y_0).$$

Observe that $\alpha \in \pi_1(Y, y_0)$ induces the identity transformation of F if and only if $\alpha \in H$. Consider the factor group $G = \pi_1(Y, y_0)/H$. Note that G is an effective transformation group of F. Let $\chi : \pi_1(Y, y_0) \to G$ be the natural homomorphism. The bundle structure theorem (cf. [**S**], pp. 68–71) says that $Z \xrightarrow{p} Y$ admits a bundle structure with fibre F, group G and characteristic class χ. Note that in the case of the universal cover $\widetilde{Y} \to Y$, the structure group is the whole fundamental group $\pi_1(Y, y_0)$. Since the fibre F can be identified with the left coset space $\pi_1(Y, y_0)/H$ (which in turn is identified with the group G), by the associated bundle construction we obtain the identification

$$Z = \widetilde{Y} \times_{\pi_1(Y,y_0)} F.$$

Lemma 2.1. *Let (Y, y_0) be a connected manifold with base point y_0. Suppose $\pi_1(Y, y_0)$ acts on some finite set F. Consider the fibre space*

$$p : Z = \widetilde{Y} \times_{\pi_1(Y,y_0)} F \longrightarrow Y.$$

If $\pi_1(Y, y_0)$ acts transitively on F, then Z is connected.

Proof. First, note that p is a covering map (cf. [**S, p. 69**]). Suppose instead that Z is not connected. Then it has at least 2 connected components. Let U_1 and U_2 be two of them. Let F_1 and F_2 denote the intersection of $p^{-1}(y_0)$ with U_1 and U_2 respectively. Observe that neither F_1 nor F_2 is empty since Y is connected. Let $x_1 \in F_1$ and $x_2 \in F_2$. Since any based loop in Y lifts to a curve in a connected component of Z, there is no element of $\pi_1(Y, y_0)$ sending x_1 to x_2. This contradicts transitivity. □

Lemma 2.2. *Let $p : (Z, z_0) \to (Y, y_0)$ be a finite covering and let $F := p^{-1}(y_0)$. Suppose the monodromy $\mathcal{G}$ acts doubly transitively on the fibre F, then the fibre product $Z \times_Y Z$ has two connected components.*

Proof. Observe that via the diagonal action of $\mathcal{G}$ on $F \times F$, we can identify the fibre product $Z \times_Y Z$ with the fibre space $\widetilde{Y} \times_{\mathcal{G}} (F \times F)$. Furthermore, the diagonal action of $\mathcal{G}$ on $F \times F$ preserves the diagonal $\Delta_{(F\times F)}$ and its complement $C\Delta_{(F\times F)}$. For simplicity, we will let S_1 and S_2 denote the diagonal and its complement respectively. Double transitivity of the action of $\mathcal{G}$ on F is nothing other than transitivity of the diagonal action on both S_1 and S_2. The lemma follows since

$$\begin{aligned} Z \times_Y Z &= \widetilde{Y} \times_{\mathcal{G}} (F \times F) \\ &= \widetilde{Y} \times_{\mathcal{G}} (S_1 \amalg S_2) \\ &= (\widetilde{Y} \times_{\mathcal{G}} S_1) \amalg (\widetilde{Y} \times_{\mathcal{G}} S_2), \end{aligned}$$

and by Lemma 2.1, $(\widetilde{Y} \times_G S_i)$ is connected for $i = 1, 2$. □

We next prove some lemmas on algebraic curves in algebraic varieties. In particular, we will prove some "covering curve" lemmas which will be used in the proof of Theorem 1.

Lemma 2.3. *Let V be an irreducible algebraic variety and let U be a proper subvariety. Then for any point $p \in U$, there is an irreducible algebraic curve γ in V which intersects U only at p. Furthermore if p is a smooth point of V, then γ can be chosen to be smooth at p.*

Proof. Pick a general point $q \in V \setminus U$. Then there is an irreducible algebraic curve C in V which contains p and q and which intersects U at only a finite number of points, say $\{x_1, \ldots, x_n\}$. The curve C can be chosen to be smooth at q since q is a general point, and furthermore if p is a smooth point of V, C can be chosen to be smooth at p. (See [**We**].) Take $\gamma = \{p\} \cup (C \setminus \{x_1, \ldots, x_n\})$. □

Remark 2.4. In fact, the curve γ above can be chosen to be smooth by removing the set of singular points in C.

Lemma 2.5. *Let V be an irreducible proper subvariety of $G(2, W_d)$ and let $U := [\mathrm{Ram}](V)$ denote its image in $\mathbb{P}W_{2d-2}$. Let $p \in U$. Then there exists a smooth algebraic curve $\gamma \subset \mathbb{P}W_{2d-2}$ emanating from p such that $\gamma \cap U = \{p\}$. Furthermore, if $q \in V \cap [\mathrm{Ram}]^{-1}(p)$, then there is an algebraic curve $\tilde{\gamma}$ emanating from q which covers γ.*

Proof. Observe that $[\mathrm{Ram}] : G(2, W_d) \to \mathbb{P}W_{2d-2}$ is proper and finite and hence U is an irreducible, proper subvariety of $\mathbb{P}W_{2d-2}$. Since $\mathbb{P}W_{2d-2}$ is smooth, by Lemma 2.3 and Remark 2.4 there is a smooth algebraic curve $C \subset \mathbb{P}W_{2d-2}$ emanating from p such that $C \cap U = \{p\}$. If $\gamma : C \to \mathbb{P}W_{2d-2}$ denotes the inclusion map, we have the curve required in the first part of the lemma.

Now consider

$$\begin{array}{ccc} \widetilde{G} & \longrightarrow & G(2, W_d) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle [\mathrm{Ram}]} \\ C & \xrightarrow{\gamma} & \mathbb{P}W_{2d-2}, \end{array}$$

where $\widetilde{G}$ is the fibre product

$$\begin{aligned} C \times_{\mathbb{P}W_{2d-2}} G(2, W_d) &= \gamma^*(G(2, W_d)) = [\mathrm{Ram}]^*(C) \\ &= \{(x, y) \in C \times G(2, W_d) \mid [\mathrm{Ram}](y) = \gamma(x)\}, \end{aligned}$$

and π_1 is the first factor projection map. By [**H**, 9.3a, p. 266] [Ram] is flat (it is also proper). Thus π_1 is flat by [**H**, Proposition 9.2b, p. 254] and proper by [**H**, Corollary 4.8c, p. 102], and hence each irreducible component of $\widetilde{G}$ maps onto C (cf. [**H**, Corollary 9.6, p. 257]). Take an irreducible component Γ through $(p, q) \in \widetilde{G}$. If necessary, normalize Γ to obtain a smooth algebraic curve $\widetilde{C}$ (we let $\widetilde{C} = \Gamma$ if Γ is smooth). We thus have the commutative diagram:

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{\tilde{\gamma}} & G(2, W_d) \\ {\scriptstyle \tilde{\pi}_1}\downarrow & & \downarrow{\scriptstyle [\mathrm{Ram}]} \\ C & \xrightarrow{\gamma} & \mathbb{P}W_{2d-2}, \end{array}$$

where $\tilde{\gamma}(\widetilde{C})$ is the desired curve emanating from q which covers γ. $\square$

Now consider the map $[\mathrm{Ram}] : G(2, W_d) \times_{\mathbb{P}W_{2d-2}} G(2, W_d) \to \mathbb{P}W_{2d-2}$.

Lemma 2.6. *Let $p \in \mathfrak{B} \subset \mathbb{P}W_{2d-2}$ and let $q = (q_1, q_2)$ be a point in the inverse image $[\mathrm{Ram}]^{-1}(p) \subset G(2, W_d) \times_{\mathbb{P}W_{2d-2}} G(2, W_d)$. Then there exists an algebraic curve $\tilde{\gamma} \subset G(2, W_d) \times_{\mathbb{P}W_{2d-2}} G(2, W_d)$ emanating from q such that*

$$\tilde{\gamma} \setminus \{q\} \subset G(2, W_d) \times_{\mathbb{P}W_{2d-2}} G(2, W_d) \setminus [\mathrm{Ram}]^{-1}(\mathfrak{B}).$$

Proof. Lemma 2.5 gives us a smooth curve $\gamma : C \to \mathbb{P}W_{2d-2}$ emanating from p with $\gamma \cap \mathfrak{B} = \{p\}$. It also furnishes us with the curves $\tilde{\gamma}_i : C_i \to G(2, W_d)$ emanating from q_i, $i = 1, 2$, which covers γ. Let $f_i : C_i \to C$, $i = 1, 2$, denote the projection maps. Observe that the fibre product $C_1 \times_C C_2$ is just the restriction of $C_1 \times C_2$ to the diagonal $C \subset C \times C$, *i.e.*

$$\begin{array}{ccc} C_1 \times_C C_2 & \longrightarrow & C_1 \times C_2 \\ \downarrow & & \downarrow{\scriptstyle f_1 \times f_2} \\ C & \longrightarrow & C \times C. \end{array}$$

Since C is smooth, the product $C \times C$ is smooth and hence the diagonal $C \subset C \times C$ is described by one algebraic equation (which is locally $x = y$). Consequently, the fibre product $C_1 \times_C C_2$ is described by one algebraic equation (which is locally $f_1 = f_2$). This implies that every component of the fibre product has codimension 1, *i.e.* every component has dimension 1, implying that $q = (q_1, q_2)$ is not an isolated point. Note that the curves C_i were chosen such that $C_i - \{q_i\}$ are mapped onto $C - \{p\}$, for $i = 1, 2$, and thus the inverse image of p in the fibre product $C_1 \times_C C_2$ contains *only* the point q. Let S be an irreducible component of $C_1 \times_C C_2$ containing the point $q = (q_1, q_2)$. From the commutative diagram:

$$\begin{array}{ccc} C_1 \times_C C_2 & \xrightarrow{\tilde{\gamma}} & G(2, W_d) \times_{\mathbb{P}W_{2d-2}} G(2, W_d) \\ \downarrow & & \downarrow{\scriptstyle [\mathrm{Ram}]} \\ C & \xrightarrow{\gamma} & \mathbb{P}W_{2d-2}, \end{array}$$

we see that $\tilde{\gamma}$ restricted to $S \subset C_1 \times_C C_2$ is the desired curve emanating from $q = (q_1, q_2)$. $\square$

Remark 2.7. If $q_1 = q_2$, consider the diagonal curve

$$\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_1) : C_1 \to G(2, W_d) \times_{\mathbb{P}W_{2d-2}} G(2, W_d)$$

defined by sending $c \in C_1$ to the point $(\tilde{\gamma}_1(c), \tilde{\gamma}_1(c))$ which obviously lies in the fibre product $G(2, W_d) \times_{\mathbb{P}W_{2d-2}} G(2, W_d)$. The curve $\tilde{\gamma}$ emanates from the diagonal point (q_1, q_1) and covers γ.

Proposition 2.8. *The fibre product* $G(2, W_d) \times_{\mathbb{P}W_{2d-2}} G(2, W_d)$ *has two irreducible components when* $d \geq 3$.

Proof. First, observe that if the monodromy acts as the full symmetric group on the fibre, then it acts doubly transitively. By the Monodromy Theorem and Lemma 2.2, the fibre product $(G(2, W_d) \setminus \mathfrak{R}) \times_{\mathbb{P}W_{2d-2}} (G(2, W_d) \setminus \mathfrak{R})$ has two connected components, say V_1 and V_2, corresponding to the diagonal and the complement of the diagonal respectively. It follows from Lemma 2.6 that the Zariski closure of $(G(2, W_d) \setminus \mathfrak{R}) \times_{\mathbb{P}W_{2d-2}} (G(2, W_d) \setminus \mathfrak{R})$ is $G(2, W_d) \times_{\mathbb{P}W_{2d-2}} G(2, W_d)$.

Note that for smooth algebraic varieties, irreducibility is equivalent to connectedness. Since

$$(G(2,W_d)\setminus\mathfrak{R})\times_{\mathbb{P}W_{2d-2}}(G(2,W_d)\setminus\mathfrak{R})$$

is smooth, V_1 and V_2 are irreducible. Recall that the Zariski closure of an irreducible set is irreducible. Thus, $\overline{V}_1$ and $\overline{V}_2$ are irreducible. Since $\overline{V}_1\neq\overline{V}_2$, in order to show that $G(2,W_d)\times_{\mathbb{P}W_{2d-2}}G(2,W_d)$ has two irreducible components, it suffices to prove that $G(2,W_d)\times_{\mathbb{P}W_{2d-2}}G(2,W_d)=\overline{V}_1\cup\overline{V}_2$. It is clear from Lemmas 2.5 and 2.6 that

$$\overline{V}_1\cup\overline{V}_2\subset G(2,W_d)\times_{\mathbb{P}W_{2d-2}}G(2,W_d).$$

We now need to show that $G(2,W_d)\times_{\mathbb{P}W_{2d-2}}G(2,W_d)\subset\overline{V}_1\cup\overline{V}_2$.

Suppose x lies in the diagonal, *i.e.* $x=(q,q)\in G(2,W_d)\times G(2,W_d)$. By Lemma 2.6 and Remark 2.7, there is a diagonal algebraic curve $\tilde{\gamma}$ emanating from x such that

$$\tilde{\gamma}\setminus\{x\}\subset(G(2,W_d)\setminus\mathfrak{R})\times_{\mathbb{P}W_{2d-2}}(G(2,W_d)\setminus\mathfrak{R}).$$

This shows that $x\in\overline{V}_1$.

Now suppose $x=(q_1,q_2)\in G(2,W_d)\times_{\mathbb{P}W_{2d-2}}G(2,W_d)$ is in the complement of the diagonal. By Lemma 2.6, there is an algebraic curve $\tilde{\gamma}$ in $G(2,W_d)\times_{\mathbb{P}W_{2d-2}}G(2,W_d)$ emanating from x such that

$$\tilde{\gamma}\setminus\{x\}\subset(G(2,W_d)\setminus\mathfrak{R})\times_{\mathbb{P}W_{2d-2}}(G(2,W_d)\setminus\mathfrak{R}).$$

Note that $\tilde{\gamma}$ intersects the diagonal in a finite number of points. Let σ denote the curve obtained from $\tilde{\gamma}$ by removing those points. Then $\sigma\setminus\{x\}$ is a *non-diagonal* curve in V_2, thus showing that $x\in\overline{V}_2$.

We have shown that $G(2,W_d)\times_{\mathbb{P}W_{2d-2}}G(2,W_d)\subset\overline{V}_1\cup\overline{V}_2$, which implies that $G(2,W_d)\times_{\mathbb{P}W_{2d-2}}G(2,W_d)=\overline{V}_1\cup\overline{V}_2$. Hence, $G(2,W_d)\times_{\mathbb{P}W_{2d-2}}G(2,W_d)$ has two irreducible components. □

Remark 2.9.

(1) When $d=1$: $G(2,W_1)$ is just a point and thus $\mathcal{M}_1$ is isomorphic to $G=(\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C}))/\pm 1$, which is irreducible. Thus $\mathcal{M}_1$ is an irreducible projective variety of dimension 6. Let

$$A=\{[a\sigma,b\sigma,c\sigma,d\sigma]\mid[a,b,c,d]\in\mathbb{CP}^3,\sigma\in W_1\setminus\{0\}\}.$$

Since $\dim A = 4$ we see that $\mathcal{S}_1 = \mathcal{M}_1 \setminus A$ is an irreducible quasi-projective variety of dimension 6.

(2) When $d = 2$: the map [Ram] : $G(2, W_2) \to \mathbb{P}W_2 \cong \mathbb{CP}^2$ is a biholomorphism (cf. [**Lo1**], Proposition 2.4). So

$$G(2, W_2) \times_{\mathbb{P}W_2} G(2, W_2) \cong G(2, W_2)$$

which is irreducible. Since $\mathcal{M}_2$ is a G-bundle over the fibre product, we see that $\mathcal{M}_2$ is an irreducible projective variety of dimensional 8. The subset of $\mathcal{M}_2$ consisting of quadruples of sections of W_2 with base points is isomorphic to $\mathbb{CP}^1 \times \mathcal{M}_1$, which is of dimension 7. Thus $\mathcal{S}_2$ is an irreducible quasi-projective variety of dimensional 8.

Proof of Theorem 1. For $d \geq 3$, let X_1 and X_2 denote the two irreducible components of $G(2, W_d) \times_{\mathbb{P}W_{2d-2}} G(2, W_d)$. $\mathcal{M}_d$ is the total space of the G-bundle over $X_1 \cup X_2$, where

$$G = (\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}))/ \pm 1.$$

Since G is irreducible, we claim that the total space restricted to each component X_i, $i = 1, 2$, is also irreducible. Otherwise, pick a component of the total space and project it to X_i. If this is not surjective then we have split off a component of X_i, a contradiction; if it is surjective, then by restricting to a general fibre we see that the general fibre has a component splitting off, again a contradiction. Thus, $\mathcal{M}_d$ has two irreducible components and each component is of dimension 2d+4. The subset of $\mathcal{M}_d$ consisting of quadruples of sections of W_d with base points is isomorphic to $\mathbb{CP}^1 \times \mathcal{M}_{d-1}$, which is of dimension 2d+3 and is hence not a component of $\mathcal{M}_d$. Thus, $\mathcal{S}_d$ has two irreducible components, each of dimension 2d+4. $\square$

Remark 2.10. Proposition 2.8 together with Theorem 1 tells us that for $d \geq 3$, $\mathcal{S}_d$ has 2 irreducible components: the "diagonal component", $\mathcal{S}_d^0$, and the "non-diagonal component", $\mathcal{S}_d^+$. Recall that $\mathcal{S}_d$ parametrizes the space of holomorphic horizontal curves of degree d in $\mathbb{CP}^3$. By post-composition with the twistor projection $\pi : \mathbb{CP}^3 \to S^4$ we obtain branched superminimal immersions of S^2 into S^4 of area (or energy)

$4\pi d$. Let

$$\begin{aligned}\mathcal{H}_d^0 &= \{\pi \circ f \mid f \in \mathcal{S}_d^0\},\\ \mathcal{H}_d^+ &= \{\pi \circ f \mid f \in \mathcal{S}_d^+\},\\ \mathcal{H}_d^- &= \{A \circ \pi \circ f \mid f \in \mathcal{S}_d^+\},\end{aligned}$$

where $A : S^4 \to S^4$ is the antipodal map. Elements of $\mathcal{H}_d^0$ correspond to totally geodesic maps from S^2 to S^4, that is, branched coverings of S^2 to a totally geodesic 2-sphere in S^4. Elements of H_d^+ and H_d^- are "linearly full" harmonic maps from S^2 to S^4, that is, their images are not contained in any strict linear subspace of $\mathbb{R}^5 \supset S^4$. The action of the antipodal map A preserves $\mathcal{H}_d^0$ and interchanges $\mathcal{H}_d^+$ and $\mathcal{H}_d^-$. Every map $f \in \mathcal{H}_d^-$ is harmonic but has no holomorphic horizontal lift to $\mathbb{CP}^3$. (cf. Remark 1.2 and [**V1**].)

By Remarks 1.2 and 2.10, we obtain:

Theorem 2. *For $d \geq 3$, the moduli space $\mathcal{H}_d$ of harmonic maps from S^2 to S^4 of energy $4\pi d$ has three irreducible irreducible components: $\mathcal{H}_d^0$, $\mathcal{H}_d^+$ and $\mathcal{H}_d^-$.*

3. Final Remarks

Since $\mathcal{S}_d := \mathcal{M}_d \setminus \mathcal{C}_d$ is not compact, one may ask: "What would be a natural compactification of $\mathcal{S}_d$? Can we compactify $\mathcal{S}_d$ by considering the whole of $\mathcal{M}_d$?" Unfortunately, elements of $\mathcal{C}_d$ correspond to parametrized rational curves of degree strictly less than d. Consider a curve C emanating from a point $p \in \mathcal{M}_d \setminus \mathcal{S}_d$. Then the punctured curve $C \setminus \{p\}$ is the parameter space for a flat family of parametrized rational curves in $\mathbb{CP}^3$ of degree d. To each point x in $C \setminus \{p\}$, we can associate the graph Γ_x of the corresponding parametrized rational curve, that is, a curve of bidegree $(1, d)$ in $\mathbb{CP}^1 \times \mathbb{CP}^3$. Thus $C \setminus \{p\}$ parametrizes a flat family of graphs of bidegree $(1, d)$ in $\mathbb{CP}^1 \times \mathbb{CP}^3$. There is a notion of taking limits to p in the algebro-geometric sense since flatness extends across a puncture. Thus, we can associate to the point p a limit graph of bidegree $(1, d)$. (This is discussed using some examples in [**Lo2**].) Taking all possible limits, one should obtain a natural compactification of $\mathcal{S}_d$.

The question is, how can one do this globally? This turns out to be a rather difficult problem.

In [**LV1**], a simpler compactification problem was studied: the variety parametrizing graphs of holomorphic maps from $\mathbb{CP}^1$ to $\mathbb{CP}^1 \times \mathbb{CP}^1$. (The graph of such a map is a complete intersection.) Here a global approach was taken: obtain the compactification via an explicit sequence of blow-ups along smooth centres. The bidegree (1,1) case is described in [**LV1**], and the bidegree (2,2) case in [**LV2**]. It is hoped that an understanding of this simpler compactification problem will be of use in the compactification problem for the moduli space $\mathcal{S}_d$.

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