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Foliations with a Kupka component on algebraic manifolds

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Abstract. We consider codimension one holomorphic foliations in complex projective manifolds of dimension at least 3, having a compact Kupka component and represented by integrable holomorphic sections ω of the bundle $TM^* \otimes L$, where L denotes a very ample holomorphic line bundle. We will show that, if the transversal type is not the radial vector field and $H^1(M, \mathbb{C}) = 0$, then the foliation has a meromorphic first integral.

Keywords: Integrating factor, transversal affine structure, positive vector bundle, very ample line bundle.

0. Introduction.

Recall that an "open book foliation" on a real manifold M of dimension $n \geq 3$, consists of a codimension two submanifold $K \subset M$, called the *binding* of the book, and a codimension one foliation $\mathcal{F}$ on $M - K$, arising from the fibers of a fibration $\pi : M - K \to \mathbb{S}^1 = \mathbb{P}^1_{\mathbb{R}}$.

In this note, we are going to consider the complex analogue: Let M be a compact complex manifold of dimension ≥ 3. By a *complex open book foliation* on M, we mean a codimension one holomorphic foliation with singularities $\mathcal{F}$ on M satisfying the following conditions:

(1) The foliation $\mathcal{F}$ is represented by an integrable section ω of the bundle $T^*M \otimes L$, where L is a *very ample* holomorphic line bundle on M.

(2) The singular set of the foliation has a compact connected component K of the Kupka singular set.

The Kupka singular set of the foliation represented by the section ω,

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is defined by

$$K_\omega = \{p \in M | \, \omega(p) = 0 \quad d\omega(p) \neq 0\},$$

and it is a codimension two smooth submanifold of M.

A complex open book foliation on the projective space $\mathbb{P}^n$ $n \geq 3$, is given by the fibers of a generic rational map if, and only if, the Kupka component K is a complete intersection [**C**–**L**], and then, the foliation looks like a 'fibration' $\pi : \mathbb{P}^n - K \to \mathbb{P}^1_{\mathbb{C}}$, moreover, the binding of the book is precisely the Kupka set, which correspond to the base points of the rational map.

A codimension one holomorphic foliation on M, admits an *affine transversal structure* if there exists an open cover $\{U_\alpha\}$, and a family of submersions $f_\alpha : U_\alpha \to \mathbb{C}$ which defines the foliation on U_α, such that

$$f_\alpha = c_{\alpha\beta} \cdot f_\beta + d_{\alpha\beta} \quad c_{\alpha\beta} \in \mathbb{C}^*, \quad d_{\alpha\beta} \in \mathbb{C}.$$

In this note, we are going to prove the following theorem:

Theorem 3.4. *Let $\mathcal{F}$ be a complex open book foliation of M, with Kupka component K, and represented by a section $\omega \in \Gamma(M; T^*M \otimes L)$, where L is a very ample line bundle. If the transversal type of K is not the radial vector field, then the foliation has at least one compact leaf, moreover, in the complement of this leaf, the foliation admits an affine transversal structure.*

In particular, when $H^1(M;\mathbb{C}) = 0$, we can say more:

Theorem 3.5. *Let $\mathcal{F}$ be a complex open book foliation of M, with Kupka component K, and represented by a section $\omega \in \Gamma(M; T^*M \otimes L)$, where L is a very ample line bundle. If the transversal type of K is not the radial vector field, and $H^1(M;\mathbb{C}) = 0$, then the foliation has a meromorphic first integral.*

And the foliation looks like the fibers of a 'fibration' $\pi : M - K \to \mathbb{P}^1$.

1. Positive Vector Bundles.

In this section, we will discuss the notion of positivity of holomorphic vector bundles.

Let M be a compact, complex manifold and let $E \to M$ be a holomorphic vector bundle; we denote by $H^0(M, \mathcal{O}(E))$ or by $\Gamma(E)$, the (finite dimensional) complex vector space of holomorphic sections of the vector bundle E.

Definition 1.1. *A holomorphic vector bundle $E \to M$ is called* **positive** *if there exists an hermitian metric in E whose curvature tensor $\Theta = (\Theta^{\rho}_{\sigma ij})$ has the property that the hermitian quadratic form*

$$\Theta(\zeta, \eta) = \sum_{\rho,\sigma,i,j} \Theta^{\rho}_{\sigma ij} \zeta^{\sigma} \overline{\zeta}^{\rho} \eta^{i} \overline{\eta}^{j}$$

is positive definite in the variables ζ, η.

Recall that the positivity of a holomorphic line bundle is a topological property, that is, a holomorphic line bundle L is positive if and only if its Chern class may be represented by a positive $(1,1)$–form in $H^2_{DR}(M)$ [**G–H**].

A holomorphic line bundle L over the complex manifold M is called *very ample*, if the holomorphic sections of L gives an embedding into projective spaces; in other words, there is an embedding $M \subset \mathbb{P}^N$, such that $L = [\mathbb{H} \cap M]$, the line bundle associated to the hyperplane section. If a line bundle L is positive, then there exists $n_0 \in \mathbb{N}$ such that $L^{\otimes n}$ is very ample for $n \geq n_0$.

A complex manifold X of complex dimension n is called is k*–convex*, if there exists a $\mathcal{C}^2$ function $\varphi : X \to \mathbb{R}$ such that its Levy form

$$L(\varphi) = \left(\frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j} \right),$$

has at least $n-k+1$ positive eigenvalues, and φ is an *exhustion function*, that is

$$X = \bigcup_{a \in \mathbb{R}} \varphi^{-1}(a, \infty).$$

Observe that if X is an n–dimensional manifold 1–convex, then the function φ is *plurisubharmonic*, and then, it is a Stein manifold.

The following theorem, relates the geometric properties of a smooth submanifold $X \subset M$ and the convexity properties of the complement $M - X$, the proof may be found in [**S**].

Theorem 1.2. *Let M be a compact, complex manifold and let $X \subset M$ be a compact, $1 \leq k$-codimensional smooth submanifold. If the normal bundle ν_X is positive, then $M - X$ is k-convex.*

The main property that we will use, is the following generalization of Hartog's extension Theorem:

Corollary 1.3. *Let M be a compact, complex manifold and let $X \subset M$ be a compact, k-codimensional smooth submanifold, where $k \leq n-1$ and such that $M - X$ is k-convex. Let $\mathcal{E}$ be a locally free analytic sheaf. Then the restriction map*

$$\Gamma(M; \mathcal{E}) \to \Gamma(U; \mathcal{E})$$

is surjective for any open set $U \supset X$ such that $M - U$ is connected.

The main idea in the proof of theorem (3.4) and (3.5), consists in the following: First, we will prove the positivity of the normal bundle of a Kupka component, this implies, by theorem (1.2), that $M - K$ is 2-convex, and then, we will use the corollary (1.3), in order to find an integrating factor as in [**C–L**].

2. Codimension one Foliations.

As usual, by a codimension one holomorphic foliation with singularities in a complex manifold M, we mean a family of holomorphic 1-forms ω_α defined on an open cover $\{U_\alpha\}$ of M, satisfying the *integrability condition* $\omega_\alpha \wedge d\omega_\alpha = 0$, and the *cocycle condition* $\omega_\alpha = \lambda_{\alpha\beta}\omega_\beta$ when $U_\alpha \cap U_\beta \neq \varnothing$, where $\lambda_{\alpha\beta}$ are never vanishing holomorphic functions defined on $U_\alpha \cap U_\beta$. Let L be the holomorphic line bundle on M obtained with the cocycles $\{\lambda_{\alpha\beta}\}$, then the 1-forms $\{\omega_\alpha\}$ glue to a holomorphic section of the vector bundle $T^*M \otimes L$.

2.1. Definition. *A codimension one holomorphic foliation $\mathcal{F}$ with singularities in the connected complex manifold M is an equivalence class of sections $\omega \in \Gamma(M, T^*M \otimes L)$ where L is a holomorphic line bundle such that ω does not vanishes identically on M and satisfies the integrability condition $\omega \wedge d\omega = 0$.*

The *singular set* of the foliation represented by the section ω, is the set of points $S_\omega = \{p \in M \,|\, \omega(p) = 0\}$. The *leaves of the foliation with singularities*, are the leaves of the non–singular foliation defined on $M - S_\omega$. When a leaf $\mathcal{L}$ of the foliation $\mathcal{F}$ is such that its closure $\overline{\mathcal{L}}$ is a closed analytic subspace of M of codimension 1, we will also call $\overline{\mathcal{L}}$ a leaf of the foliation, and by abuse of language, we will say that $\overline{\mathcal{L}}$ is a *compact leaf* of the foliation.

We will say that a codimension one holomorphic foliation $\mathcal{F}$ on M is *positive* (resp. *very ample*), if it is represented by an integrable section $\omega \in \Gamma(T^*M \otimes L)$, and the holomorphic line bundle L, is *positive* (resp. *very ample*).

Recall that an integrating factor of a foliation represented by a section $\omega \in \Gamma(M; T^*M \otimes L)$, is a holomorphic section $\varphi \in H^0(M; \mathcal{O}(L))$ such that the meromorphic 1–form $\Omega = \omega/\varphi$ is closed.

We will use the following result, whose proof may be found in [**C**]. Let M be a compact complex manifold with $H^1(M;\mathbb{C}) = 0$. If the holomorphic section $\varphi = \varphi_1^{n_1} \cdots \varphi_k^{n_k}$ of the line bundle L is an integrating factor of $\omega \in H^0(M; \Omega^1(L))$, then

$$\frac{\omega}{\varphi} = \Omega = \sum_{i=1}^{k} \lambda_i \frac{d\varphi_i}{\varphi_i} + d\left(\frac{\psi}{\varphi_1^{n_1-1} \cdots \varphi_k^{n_k-1}}\right) \tag{1}$$

where ψ is a holomorphic section of the line bundle associated to the divisor

$$\sum_{i=1}^{k} (n_i - 1) \cdot [\varphi_i = 0].$$

If η_i denotes the fundamental class of the hypersurface $\{\varphi_i = 0\}$, by the residue theorem, we have the following relation:

$$\sum_{i=1}^{k} \lambda_i \cdot \eta_i = 0 \in H^2(M;\mathbb{C}). \tag{2}$$

We will use formula (1) in two situations: When $\varphi = \varphi_1 \cdot \varphi_2$, where φ_i are irreducible sections of positive line bundles. In this case; the equation (1) has the form

$$\frac{\omega}{\varphi} = \Omega = \lambda \frac{d\varphi_1}{\varphi_1} - \mu \frac{d\varphi_2}{\varphi_2},$$

in this case, by (2), we have $\mu\eta_1 = \lambda\eta_2 \in H^2(M;\mathbb{C})$, and then, $\mu/\lambda = p/q \in \mathbb{Q}$, where p, q are natural numbers relatively prime, and the foliation has the meromorphic first integral

$$\phi = \frac{\varphi_1^q}{\varphi_2^p} : M - \{\varphi_1 = 0\} \cap \{\varphi_2 = 0\} \to \mathbb{P}^1.$$

The second case that we will consider, is when $\varphi = \varphi_1^{q+1}$, and φ_1 is an irreducible section of a positive line bundle.

Now, formula (1) has the form

$$\frac{\omega}{\varphi} = \Omega = d\left(\frac{\psi}{\varphi_1^q}\right),$$

and the foliation has the meromorphic first integral

$$\phi = \frac{\psi}{\varphi_1^q} : M - \{\{\psi = 0\} \cap \{\varphi_1 = 0\} \to \mathbb{P}^1.$$

3. Kupka Type Singularities.

In this section, we are going to study the properties of the binding.

3.1. Definition. *Let $\omega \in \Gamma(M, T^*M \otimes L)$ be a codimension one holomorphic foliation. The* Kupka singular set, *$K_\omega \subset S_\omega$ is defined by:*

$$K_\omega = \{p \in S_\omega \mid \quad d\omega(p) \neq 0\}.$$

We say that $\omega \in H^0(M, \Omega^1(L))$ has a Kupka component *if the Kupka set has a compact connected component.*

It is known, (see [**Me**] for details), that the Kupka set is well defined and it is independent of the choice of the section ω which defines the foliation. The main property of the Kupka set, is the *local product property*, namely, for every connected component $K \subset K_\omega$ there exist a holomorphic 1-form η, called the *transversal type* at the component K, defined in a neighborhood V of $0 \in \mathbb{C}^2$ and vanishing only at 0, an open covering $\{U_\alpha\}$ of a neighborhood of K in M, and a family of submersions $\varphi_\alpha : U_\alpha \to \mathbb{C}^2$ such that $\varphi_\alpha^{-1}(0) = K \cap U_\alpha$, In particular, the Kupka set, is a codimension 2 locally closed, smooth submanifold of

M, furthermore, the family of 1–forms $\omega_\alpha = \varphi_\alpha^* \eta$ defines the foliation in the open set U_α.

The transversal type η, is well defined up to biholomorphism and multiplication by non–vanishing holomorphic function, as well its dual vector field X, since $d\omega \neq 0$ we have that $\mathrm{Div}X(0) \neq 0$, thus the linear part $\mathbf{L} = DX(0)$, which is well defined up to linear conjugation and multiplication by scalars, has at least one non–zero eigenvalue. We can normalize, by multiplying by a non–zero complex number, in order to have eigenvalues $1, \lambda$. We will say that $\mathbf{L}$ is the *linear type* at the component K.

Let K be a connected component of the Kupka set of a foliation $\mathcal{F}$, and let $\mathbf{L}$ be the linear type of K, with eigenvalues $1, \lambda$, we say that the linear transversal type is *Poincaré*, if $\lambda \notin (-\infty, 0)$, *Siegel*, if $\lambda \in (-\infty, 0)$ and degenerate if $\lambda = 0$. When $\lambda = 1$, and the canonical Jordan form of $\mathbf{L}$ is non semisimple, we say that the linear type is non–semisimple.

The local behavior of the leaves near the Kupka set, which depends on the transversal type of the component, and the non–triviality of the embedding of $K \subset M$, which can be measured with the Chern classes of the normal bundle, are strongly related, namely, [**GM–L**] have shown that if the first Chern class of the normal bundle $\nu_K(M)$ of a compact connected component K of the Kupka set is non–zero, then the linear transversal type must be semisimple, non degenerate, and with eigenvalues $(1, \lambda)$ where $\lambda \in \mathbb{Q}$.

As a consequence of the local product structure, a Kupka component of a foliation represented by a section $\omega \in \Gamma(M; T^*M \otimes L)$, is *subcanonically embedded* [**C–S**], that is, its canonical bundle, denoted by Ω_K^{n-2}, and the second exterior power of the normal bundle, $\wedge^2 \nu_K$; may be extended to a global holomorphic line bundle on M. In fact, we have the following two equivalent formulas:

$$\Omega_K^{n-2} = \Omega_M^n \otimes L|_K \quad \text{and} \quad \wedge^2 \nu_K = L|_K,$$

where Ω_M^n and Ω_K^{n-2} denote the canonical bundle of M and K respectively.

Now, let $\mathcal{F}$ be a foliation represented by a section $\omega \in \Gamma(M; T^*M \otimes$

L), with a compact connected component K of the Kupka set. When the line bundle L is very ample, the above formulae implies that the first Chern class of the normal bundle of $K \subset M$, does not vanishes, because it is given by the formulae

$$c_1(\nu_K) = i^* c_1(L) \in H^2(K; \mathbb{C}),$$

and then, the linear transversal must be diagonalizable with eigenvalues $1, \lambda \in \mathbb{Q}$, furthermore, if the linear transversal type belongs to the Poincaré domain, then the transversal type is linearizable and semisimple, [**GM–L**]. This remark gives the following result.

Proposition 3.2. *Let $\omega \in H^0(M; \Omega^1(L))$ be a very ample foliation with a Kupka component K. If the transversal type is not the radial vector field, then the normal bundle of K is positive.*

Proof. The above remark implies that the linear transversal type of the Kupka component K_ω is given by the 1–form

$$\eta = px\, dy - qy\, dx \quad \text{with} \quad p, q \in \mathbb{Z} \quad (p, q) \neq 1.$$

We are going to prove, that the linear transversal type belongs to the Poincaré domain:

In [**C–S**] it is shown that there exists a pair (E, σ), where $E \longrightarrow M$ is rank two holomorphic vector bundle over M, and σ is a holomorphic section of the bundle E. The zero locus of this section, is the Kupka set. Furthermore, the total Chern class of the vector bundle E is given by

$$\begin{aligned} c(E) &= \left(1 + \frac{p \cdot c_1(L)}{p+q}\right) \cdot \left(1 + \frac{q \cdot c_1(L)}{p+q}\right) \\ &= 1 + c_1(L) + [K] \in H^*(M; \mathbb{C}), \end{aligned}$$

whenever the linear transversal type is $px\, dy - qy\, dx$.

Now, consider the embedding induced by the line bundle L, the degree of the Kupka set under this embedding, which is a positive integer, is given by the formulae

$$\begin{aligned} \left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_M c_2(E) \wedge c_1(L)^{n-2} &= \frac{p \cdot q}{(p+q)^2} \left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_M c_1(L)^n \\ &= \frac{p \cdot q}{(p+q)^2} \cdot \mathbf{vol}(M) > 0. \end{aligned}$$

This implies that $p \cdot q > 0$, hence, the linear part belongs to the Poincaré domain.

Now, in **[GM–L]** **[C–L]**, it is shown, that the normal bundle ν_K splits in a direct sum of holomorphic line bundles, L_p and L_q, corresponding to the two eigendirections, that is: $\nu_K = L_p \oplus L_q$, moreover, its Chern classes satisfy the following relation:

$$p \cdot c(L_q) - q \cdot c(L_p) = \mathbf{0} \in H^2(K; \mathbb{C}).$$

Since $\wedge^2 \nu_K = L_p \otimes L_q = L|_K$, we have that both holomorphic line bundles are positive, and then, the normal bundle of the Kupka component $K \subset M$ splits in a direct sum of positive line bundles, and then, it is a positive vector bundle. □

Remark. If the transversal type is given by the form $\eta = x\,dy - y\,dx$, and the normal bundle admits the exact sequence of vector bundles

$$0 \to L_1 \to \nu_K \to L_1 \to 0,$$

then the normal bundle of K is positive.

In fact, since $L_1 \otimes L_1 = L_K$, which is positive, implies that L_1 is positive. This condition is true, if for example, there exists a smooth separatrix of the Kupka set.

Now, the main consequence of the proposition (3.2) is the following result:

Theorem 3.3. *Let $\mathcal{F}$ be a complex open book foliation of M, represented by a section $\omega \in H^0(M; \Omega^1(L))$. Let K be the compact Kupka component of $\mathcal{F}$. Then, if the transversal type of K is not the radial vector field, then $M - K$ is 2–convex.*

Proof. By definition of an complex open book foliation, the holomorphic line bundle L is very ample, and by the Theorem (3.2), the normal bundle $\nu_K(M)$ is positive, now, because the codimension of K in M is two, the Proposition (1.2) implies that $M - K$ is 2–convex. □

A codimension one foliation without singularities $\mathcal{F}$ on a manifold M has an *affine transversal structure*, if there exits an open cover $\{U_\alpha\}$ of

M, and a family of submersions $f_\alpha : U_\alpha \to \mathbb{C}$ which define the foliation on U_α, such that whenever $U_\alpha \cap U_\beta \neq \varnothing$ we have

$$f_\beta = a_{\alpha\beta} \cdot f_\beta + b_{\alpha\beta} \quad a_{\alpha\beta} \in \mathbb{C}^*, \quad b_{\alpha\beta} \in \mathbb{C}.$$

In our case, we consider codimension one holomorphic foliations such that, outside of an algebraic set, the foliation admits a transversal affine structure. In this case, it is shown in [**Sc**] and [**Sc1**], that a codimension one holomorphic foliation with singularities on a complex manifold M has a transversal affine structure, if and only if, the following holds:

Let Ω be a meromorphic 1–form which defines the foliation out of its poles, then there exists a closed meromorphic 1–form θ, such that:

$$d\Omega = \theta \wedge \Omega, \quad \text{and} \quad \theta_\infty = \Omega_\infty,$$

where Ω_∞ and θ_∞ denotes the divisor of poles of the 1-forms Ω and θ respectively.

Now, we are in a position to prove our main result. To do this, we are going to follow the exposition of [**Ce–L**].

Theorem 3.4. *Let $\mathcal{F}$ be a complex open book foliation on M, with Kupka component K, and represented by a section $\omega \in \Gamma(M; T^*M \otimes L)$, where L is a very ample line bundle. If the transversal type of K is not the radial vector field, then the foliation has at least one compact leaf, moreover, in the complement of this leaf, the foliation admits an affine transversal structure.*

Proof. We are going to find a transversal affine structure on a neighborhood of the Kupka set. Now, by proposition (3.3), we know that $M - K$ is two convex, and then, we can extend this structure to M.

To find the transversal structure, we know that the transversal type of the foliation is linearizable. This implies that there exits a covering of K by open sets $\{U_\alpha\}$ and coordinate systems $((x_\alpha, y_\alpha), \mathbf{z}_\alpha) : U_\alpha \to \mathbb{C}^2 \times \mathbb{C}^{n-2}$ such that:

(1) $K \cap U_\alpha = \{x_\alpha = y_\alpha = 0\}$.

(2) The foliation on U_α is defined by the form $\omega_\alpha = p x_\alpha \, dy_\alpha - q y_\alpha \, dx_\alpha$, where $1 \leq p < q$ are natural numbers.

(3) If $U_\alpha \cap U_\beta \neq \varnothing$, then it is connected and there exists a 1–cocycle $\lambda_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$

We are going to consider two cases:

(1) If the transversal type is $\eta = px\,dy - qy\,dx$ where $1 < p < q$ are relatively prime integers, then the meromorphic 1–form

$$\Omega_\alpha = \frac{\omega_\alpha}{x_\alpha y_\alpha},$$

defines the foliation on U_α out of its poles, moreover, as was pointed in [**C–L**], those meromorphic forms satisfy the equality $\Omega_\alpha = \Omega_\beta$ when $U_\alpha \cap U_\beta \neq \varnothing$.

This shows that $\{x_\alpha y_\alpha\}$ defines a holomorphic section of the line bundle L on a neighborhood of K, and then, it defines an integrating factor for the foliation ω, on this neighborhood, since $M - K$ is 2–convex, this section can be extended to M. The compact leaves are extensions of the divisors $x_\alpha y_\alpha = 0$.

(2) In the second case, i. e., if the transversal type is of the form $\eta = x\,dy - qy\,dx$, we can applied the same argument as before to show that $\{x_\alpha^{q+1} = 0\}$ defines a divisor invariant by the foliation, which can be extended to M, because $M - K$ is two convex; but in this case, we have the following equation:

$$\frac{\omega_\alpha}{x_\alpha^{q+1}} = c_{\alpha\beta}\frac{\omega_\beta}{x_\beta^{q+1}}, \quad c_{\alpha\beta} \in \mathbb{C}^*.$$

A direct calculation [**C–L**], shows that

$$\Omega_\alpha = \frac{\omega_\alpha}{x_\alpha^{q+1}} = d\left(\frac{y_\alpha}{x_\alpha^q}\right) \quad \Rightarrow \quad \left(\frac{y_\alpha}{x_\alpha^p}\right) = c_{\alpha\beta} \cdot \left(\frac{y_\beta}{x_\beta^p}\right) + b_{\alpha\beta},$$

for some $c_{\alpha\beta} \in \mathbb{C}^*$ and $b_{\alpha\beta} \in \mathbb{C}$.

Now, let Ω be a meromorphic 1–form which defines the foliation out of its polar divisor, then

$$\Omega|_{U_\alpha} = h_\alpha \cdot d\left(\frac{y_\alpha}{x_\alpha^q}\right),$$

for some meromorphic function $h_\alpha : U_\alpha \to \mathbb{C}$.

When $U_\alpha \cap U_\beta \neq \varnothing$, it is showed in [**C–L**] that $h_\alpha = c_{\alpha\beta} \cdot h_\beta$, and then, the meromorphic 1–form θ locally defined in the open set U_α by

$$\theta_\alpha = \frac{dh_\alpha}{h_\alpha},$$

satisfies that

$$\theta_\alpha = \theta_\beta \quad \text{and} \quad d\Omega = \theta \wedge \Omega.$$

Then, it defines the affine transversal structure on the neighborhood $U = \bigcup U_\alpha$ of the Kupka set, which can be extended to M because $M - K$ is two convex. □

With an additional topological condition on M, we can prove our last result.

Theorem 3.5. *Let $\mathcal{F}$ be a complex open book foliation of M, with Kupka component K, and represented by a section $\omega \in \Gamma(M; T^*M \otimes L)$, where L is a very ample line bundle. If the transversal type of K is not the radial vector field, and $H^1(M;\mathbb{C}) = 0$, then the foliation has a meromorphic first integral.*

Proof. We are going to use the same notation as above.

As in [**Ce–L p. 128**] we are going to find an integrating factor, and then, by using the normal form of foliations with an integrating factor, we are going to find the meromorphic first integral.

Now, if the transversal type is $\eta = px\,dy - qy\,dx$ with $1 < p < q$, we have seen that $x_\alpha y_\alpha$ defines an integrating factor of the foliation.

If the transversal type is $\eta = x\,dy - qy\,dx$, let Ω be a meromorphic 1–form which defines the foliation out of its pole, and let θ the closed 1–form which defines the transversal structure,

$$d\Omega = \theta \wedge \Omega, \qquad \theta_\infty = \Omega_\infty.$$

Since $H^1(M,\mathbb{C})$ vanishes, we can show as in [**C–L**], that $\theta = \frac{df}{f}$, for some meromorphic function, this implies that f is a meromorphic integrating factor of Ω, and this implies that ω has an integrating factor.

Now, let φ be the integrating factor of the foliation represented by ω, in the case that the transversal type is $\eta = px\,dy - qy\,dx, \quad 1 < p < q$, the closed meromorphic 1-form

$$\Omega = \frac{\omega}{\varphi},$$

on the open set U_α, where the foliation is defined by the 1–form $\omega_\alpha = p x_\alpha\, dy_\alpha - q y_\alpha\, dx_\alpha$ the 1-form Ω has the expression

$$\Omega_\alpha = \frac{\omega_\alpha}{\varphi_\alpha} = p\frac{dx_\alpha}{x_\alpha} - q\frac{dy_\alpha}{y_\alpha},$$

by comparing the residues, we have that the integrating factor decomposes as $\varphi = \varphi_1 \cdot \varphi_2$ and $\varphi_1|_{U_\alpha} = x_\alpha$ and $\varphi_2|_{U_\alpha} = y_\alpha$, by the normal form of foliations with an integrating factor, we see that the foliation has a meromorphic first integral.

If the transversal type is $\eta = x\,dy - qy\,dx$, we have seen that $x_\alpha^{q+1} = 0$ defines an invariant divisor which can be extended to M, moreover, the fundamental class of this divisor coincides with the Chern class of the line bundle L and hence, because $H^1(M, \mathbb{C}) = 0$, and M is Kälher (M is algebraic), the Chern class determines the line bundle, and then, $\{x_\alpha^{q+1} = 0\}$ defines a holomorphic section of the line bundle L.

Now, let φ the integrating factor of ω, then on the open set U_α we have that $\varphi_\alpha = a_\alpha \cdot x_\alpha^{q+1}$ for some never vanishing holomorphic function a_α.

Now

$$\Omega_\alpha = \frac{\omega_\alpha}{\varphi_\alpha} = a_\alpha^{-1} \cdot \frac{\omega_\alpha}{x_\alpha^{q+1}} = a_\alpha^{-1} \cdot \widetilde{\Omega_\alpha}, \quad \widetilde{\Omega_\alpha} = d\left(\frac{y_\alpha}{x_\alpha^q}\right),$$

since both forms Ω_α and $\widetilde{\Omega_\alpha}$ are closed, we have that

$$0 = \frac{da_\alpha}{a_\alpha} \wedge \widetilde{\Omega},$$

and then, a_α is a holomorphic first integral of the foliation ω and hence it is a constant, then the zero locus of the integrating factor $\{\varphi = 0\} = \{x_\alpha^{q+1} = 0\}$.

Again, from the normal form of foliations with integrating factor, we obtain a meromorphic first integral. □

Let $\mathcal{K}(M, L, \eta)$ be the set of codimension one holomorphic foliations, having a compact connected component of the Kupka set with transversal type η, if L is very ample, the transversal type is $\eta_{p,q} = px\,dy - qy\,dx$ for some positive integers $1 \leq p < q$, or $\eta = x\,dy - y\,dx$, then we have the following corollary:

Corollary 3.6. *Let M be a compact complex manifold of complex dimension ≥ 3 and with $H^1(M, \mathbb{C}) = 0$. If L is a very ample line bundle, the set $\mathcal{K}(M, L, \eta_{p,q})$, where $1 \leq p < q$, is an irreducible component of $Fol(M, L)$, and its generic element is structurally stable.*

Proof. It follows from Theorem 3.5, that the foliation has a meromorphic first integral, and the generic element in this case, is a Branched Lefschetz Pencil [**C**]. □

Corollary 3.7. *Let M be a compact complex manifold of complex dimension ≥ 3 and with $H^1(M, \mathbb{C}) = 0$. Let $\mathcal{F} \in \mathcal{K}(M, L, \eta_{p,q})$, $1 \leq p < q$ be a foliation with Kupka component K. Then the foliation is locally structural stable near its Kupka component K.*

Proof. It follows from the existence of the meromorphic first integral of any foliation $\mathcal{F} \in \mathcal{K}(M, L, \eta_{p,q})$. □

Remark. When the transversal type is the radial vector field, and the Kupka set has a smooth separatrix, then the foliation admits an affine transversal structure, this is given as follows: set $x_\alpha = 0$ be the equation of the separatrix on the open set U_α, as in the proof of the theorem 2.5, we can show that

$$\Omega_\alpha = \frac{\omega_\alpha}{x_\alpha^2},$$

defines a meromorphic section of $T^*M \otimes L_0$, where $L_0 \in Pic_0(M)$, and having a order two pole along a divisor D which is invariant by the foliation, and which is a separatrix of the Kupka set, moreover, this section is locally represented by closed forms.

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