On A Class of Quaternion Functions*

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Abstract

A special class of functions of one quaternion variable is studied. These functions have properties similar to the properties of analytic functions of one complex variable and contain these functions in particular. The applications of these functions to the analysis of Space-Time structures which are described in terms of quaternions, is suggested.

I – Quaternions

The quaternion algebra Q can be described as the set of elements of the form $X = \sum X_\alpha e^\alpha$, $\alpha = 0, 1, 2, 3$, where the e^α's are the elements of the basis of the algebra and satisfy the multiplication table

$$\begin{aligned} e^i e^j &= -\delta^{ij} + \sum \varepsilon^{ijk} e^k \\ e^i e^0 &= e^0 e^i = e^i . \end{aligned} \tag{1-1}$$

δ^{ij} is the Kronecker delta and ε^{ijk} is the Levi-Civita symbol.

In general Greek indices will run from 0 to 3 while small Latin indices will run from 1 to 3. Unless otherwise indicated the simbol $\sum$ indicates a summation over the repeated indices. The X_u's are the components of the quaternion X in the basis e^α and are assumed to be real. These numbers can be thought of as coordinates of a point in some 4-dimensional manifold. The module of X is defined by $|X|^2 = X\bar{X} = \sum X_\alpha X_\alpha$, where $\bar{X}$ denotes the quaternion conjugate of $X : \bar{X} = X_0 e^0 - \sum X_i e^i$. Thus, $\overline{e^0} = e^0$ and $\overline{e^i} = -e^i$. In particular we have $|e^i|^2 = e^i \overline{e^i} = e^0$. Assuming that $|X| \neq 0$, there is an inverse X^{-1} such that $XX^{-1} = X^{-1}X = e^0$. Therefore,

$$X^{-1} = \bar{X}/|X|^2 . \tag{1-2}$$

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In virtue of the non commutative character of the algebra we have that $XY^{-1} \neq Y^{-1}X$. Thus,

$$e^i(e^j)^{-1} = -e^i e^j = \delta^{ij} - \sum \varepsilon^{ijk} e^k$$

$$(e^j)^{-1} e^i = -e^j e^i = \delta^{ij} + \sum \varepsilon^{ijk} e^k. \tag{1-3}$$

We can write a general quaternion as $X = X_0 e^0 + \xi$, where $\xi = \sum X_i e^i$, corresponds to a vector in a three-dimensional Euclidean-space, with norm $|\xi|^2 = (\xi, \xi) = X_1^2 + X_2^2 + X_3^2$. It follows from (1-1) that $\xi\xi = \xi^2 = -(\xi, \xi)$. If we put

$$\xi = |\xi| \lambda = (\sqrt{\sum X_i^2})\, \lambda,$$

then $\lambda^2 = -(\lambda, \lambda) = -1$. We can also write a quaternion in polar form. Define $\gamma = \cos^{-1}(X_0/|X|)$. From $X\bar{X} = |X|^2 = X_0^2 + \sum X_i^2$, we get $\sum X_i^2 = |X|^2 \operatorname{Sen}^2 \gamma$. Thus, if $X = X_0 + \xi = X_0 + (\sum X_i^2)^{1/2} \lambda$, it follows that $X = |X| (\operatorname{Cos}\gamma + \lambda \operatorname{Sen}\gamma)$. Defining $e^{\lambda\gamma} = \operatorname{Cos}\gamma + \lambda \operatorname{Sen}\gamma$ we get

$$X = |X| e^{\lambda\gamma}. \tag{1-4}$$

II – Quaternion Functions

Now consider a function $f: Q \to Q$ such that for $X \in Q$,

$$f(X) = \sum U_\alpha(X) e^\alpha \in Q$$

where $U_\alpha(X) = U_\alpha(X_0, X_1, X_2, X_3)$ are the components of $f(X)$. The norm of the quaternions define a topology in Q which is the same as that of $\mathbf{R}^4$. Thus we can speak of limits and continuity as in real analysis. We shall assume that the $U's$ are real and of class C^1. As the quaternion algebra is non commutative we can define two distinct kinds of derivatives: the right derivative

$$f'(X) = \lim_{\Delta X \to 0} [f(X + \Delta X) - f(X)](\Delta X)^{-1} \tag{2-1}$$

and the left derivative

$$'f(X) = \lim_{\Delta X \to 0} (\Delta X)^{-1} [f(X + \Delta X) - f(X)]. \tag{2-2}$$

Using polar representation of quaternions we can write $\Delta X = |\Delta X| e^{\lambda\gamma}$. Thus if $|\Delta X| = 0$ we have $\Delta X = 0$ without imposing any condition on γ,

which remains arbitrary. This means that we can have an indefinite value for the two derivates above at each point X. The function $f(X)$ is said to be holomorphic at right and left if the respective derivates (2-1), (2-2) are well defined at each point.

Taking in particular $\Delta X = \Delta X_\beta e^\beta$ we get

$$f'(X)_{(\beta)} = \lim_{(\Delta X)_\beta \to 0} \sum_\alpha [U_\alpha(X + \Delta X) - U_\alpha(X)]\, e^\alpha (\Delta X_\beta e^\beta)^{-1}$$

$$= \frac{\partial U_0}{\partial X_\beta} e^0 (e^\beta)^{-1} + \sum_i \frac{\partial U_i}{\partial X_\beta} e^i (e^\beta)^{-1}. \tag{2-3}$$

Similarly

$$'f(X)_{(\beta)} = \frac{\partial U_0}{\partial X_\beta} (e^\beta)^{-1} e^0 + \sum_i \frac{\partial U_i}{\partial X_\beta} (e^\beta)^{-1} e^i. \tag{2-4}$$

Therefore

$$f'(X)_{(0)} = \frac{\partial U_0}{\partial X_0} + \sum_i \frac{\partial U_i}{\partial X_j} e^i = {}'f(X)_{(0)},$$

$$f'(X)_{(j)} = -\frac{\partial U_0}{\partial X_j} e^j + \sum_i \frac{\partial U_i}{\partial X_j} (\delta^{ij} - \sum_k \varepsilon^{ijk} e^k), \tag{2-5}$$

$$'f(X)_{(j)} = -\frac{\partial U_0}{\partial X_j} e^j + \sum_i \frac{\partial U_i}{\partial X_j} (\delta^{ij} + \sum_k \varepsilon^{ijk} e^k),$$

where the indice (α) in the derivative indicates the particular direction on which the differentiation is being carried out. On comparing these derivatives we can select vatious classes of quaternion functions:

(A) – The class defined by the conditions:

$$f'(X)_{(0)} = f'(X)_{(i)} \quad \text{and} \quad f'(X)_{(i)} = f'(X)_{(j)},$$

which leads to

$$\partial_0 U_0 = \partial_i U_i, \quad \partial_0 U_i = -\partial_i U,$$

$$\partial_j U_i = -\partial_i U_j, \quad \partial_0 U_i = \partial_k U_j, \quad i, j, k \quad \text{cyclic} \tag{2-6}$$

where $\partial_\alpha = \partial / \partial X_\alpha$.

(B) – The class defined by

$$'f(X)_{(0)} = 'f(X)_{(i)} \quad \text{and} \quad 'f(X)_{(i)} = 'f(X)_{(j)}$$

i.e.

$$\partial_0 U_0 = \partial_i U_i, \qquad \partial_0 U_i = -\partial_i U_0,$$

$$\partial_j U_i = -\partial_i U_j, \qquad \partial_0 U_i = -\partial_k U_j, \qquad i, j, k \quad \text{cyclic} \tag{2-7}$$

(C) – The class of functions for which

$$f'(X)_{(\alpha)} = 'f(X)_{(\alpha)},$$

or

$$\partial_i U_j = \partial_j U_i. \tag{2-8}$$

Introducing the quaternionic differential operator

$$\partial = \sum e^\alpha \partial_a = \sum e^\alpha \frac{\partial}{\partial X_\alpha}, \tag{2-9}$$

we have

$$\partial f(X) = \sum e^\alpha \partial_\alpha f(X) \tag{2-10}$$

$$= (\partial_0 U_0 + \sum U_i e^i) - \left(- \sum \partial_i U_0 e^i + \sum_{i,j} \partial_j U_i (\delta^{ij} - \sum \varepsilon^{ijk} e^k) \right),$$

or

$$\partial f(X) = f'(X)_{(0)} - \sum_j f'(X)_{(j)}$$

On the other hand ∂ can act on $f(X)$ at right:

$$f(X)\partial = \sum \partial_\alpha f(X) e^\alpha \tag{2-11}$$

$$= (\partial U_0 + \sum \partial_0 U_j e^j) - \left(- \sum \partial_i U_0 e^i + \sum_{i,j} \partial_i U_j (\delta^{ij} + \sum \varepsilon^{ijk} e^k) \right).$$

That is,

$$f(X)\partial = 'f(X)_{(0)} - \sum_j 'f(X)_{(j)}.$$

With this we can define three other classes of functions:

(D) – The class of functions for which $\partial f(X) = 0$. That is, from (2-10),

$$\partial_0 U_0 - \sum \delta^{ij} \partial_j U_i = 0$$

$$\partial_i U_0 + \partial_0 U_i + \sum \varepsilon^{ijk} \partial_j U_k = 0. \tag{2-12}$$

(E) – The class defined by $f(X)\partial = 0$. We get from (2-11) the equations

$$\partial_0 U_0 - \sum \delta^{ij} \partial_j U_i = 0$$

$$\partial_i U_0 + \partial_0 U_i - \sum \varepsilon^{ijk} \partial_j U_k = 0. \tag{2-13}$$

(F) – The class defined by $\partial f(X) = f(X)\partial$. That is, from (2-10) and (2-11)

$$\partial_i U_j = \partial_j U_i. \tag{2-14}$$

We notice that the classes (A) and (B) differ only in the last equations of (2-6) and (2-7). In both cases the first three equations are generalizations of the Cauchy-Riemann equations. Thus we may consider the class (A) as the class of right analytic functions and the class (B) as the class of left analytic functions. If we unify these two classes to obtain analytic functions at left and right simultaneously we get the conditions $\partial U_i / \partial X_0 = = \partial U_j / \partial X_i = 0$. Hence, introducing second order derivatives we get also $\partial^2 U_i / \partial X_i \partial X_j = 0$ which gives $U_i = aX_i + b$ (a, b constant quaternions). Therefore the only possible solutions are the linear quaternion functions. A reference to this theorem is found in Ketchum (1928). The functions of classes (A), (B) and ($A \cap B$) were extensively studied by Fueter (1928-1937).

We notice that the conditions of definition of the classes (D) and (E) are equivalent to

$$f'(X)_{(0)} = \sum_i f'(X)_{(i)}, \tag{2-15}$$

$$'f(X)_{(0)} = \sum_i 'f(X)_{(i)}, \tag{2-16}$$

respectively. Furthermore the class (D) differs from the class (A) by the substituition of the condition $f'(X)_{(0)} = f'(X)_{(j)}$, by (2-15). An analogous difference exists between the classes (E) and (B). Observe that (2-15) and (2-16) contain in particular case the ordinary Cauchy-Riemann conditions. Combining (2-12), (2-14) we get another class of functions, denoted by H, which satisfy the conditions:

$$\partial_0 U_0 = \sum \partial_i U_i, \quad \partial_i U_0 = -\partial_0 U_i, \quad \partial_i U_j = \partial_j U_i. \tag{2-17}$$

As we can see these equations also contain the Cauchy-Riemann conditions in particular. However, from (2-10) and (2-11) we see that the functions of class H are not analytic in the same sense of Cauchy-Riemann. The direction X_0 is singled out so that the derivative of a H-function along X_0 is the same as the derivative of the function along a vector of the coordinate space spanned by X_1, X_2, X_3, in the neighborhood of a point. The derivative $f'(X)$ or $'f(X)$ of a H-function can be taken to be $'f(X)_{(0)}$ or $\sum f'(X)_{(i)}$ or $'f(X)_{(0)}$ or $\sum 'f(X)_{(i)}$. We shall see that the functions of class H can be developed in power series so that in a certain sense they can be labelled as analytic. One example of H-function is

$$f(X) = (1-X)^{-1} = \frac{\overline{1-X}}{|1-X|^2} = U_0 e_0 + U_i e^i, \qquad X \neq 1, \tag{2-18}$$

where

$$U_0 = (1-X_0)/(1+2X_0+|X|^2), \qquad U_i = X_i/(1+2X_0+|X|^2).$$

It is an easy matter to see that these components satisfy (2-17). If a function is of class H it is also harmonic. In fact the first equation in (2-17) gives

$$-\sum \delta^{ij} \frac{\partial^2 U}{\partial X_i \partial X_j} = \frac{\partial^2 U_0}{\partial X_0^2};$$

defining $\square^2 = \partial\bar{\partial}$, we get $\square^2 U_0 = 0$. On the other hand we have also from (2-17)

$$\frac{\partial^2 U_k}{\partial X^2} + \sum \delta^{ij} \frac{\partial^2 U_k}{\partial X_i \partial X_j} = 0$$

or

$$\square^2 U_k = 0.$$

Therefore if $f(X)$ is a H-function,

$$\square^2 f(X) = 0. \tag{2-19}$$

III – Diferential and Integral Relations

In analogy with the complex functions we can define the general form of the differential of $f(X) = \sum U_\alpha e^\alpha$,

$$df(X) = \sum \frac{\partial U_\alpha}{\partial X_\beta} dX_\beta e^\alpha$$

$$= \left(\frac{\partial U_0}{\partial X_0} dX_0 + \sum \frac{\partial U_0}{\partial X_i} dX_i\right) e^0 + \sum \left(\frac{\partial U_i}{\partial X_0} dX_0 + \frac{\partial U_i}{\partial X_j} dX_j\right) e^i.$$

From (2-4) we have for a fixed β

$$f'(X)_{(\beta)} e^\beta = \sum \frac{\partial U_\alpha}{\partial X_\beta} e^\alpha,$$

so that

$$df(X) = \sum \frac{\partial U_\alpha}{\partial X_\beta} e^\alpha dX_\beta = \sum f'(X)_{(\beta)} e^\beta dX_\beta \tag{3-1}$$

which is the quaternion formula equivalent to the complex differential $df(Z) = f'(Z)dZ$. By commuting e^β with $f'(X)_{(\beta)}$ we get another differential expression

$$\delta f(X) = \sum e^\beta f'(X)_{(\beta)} dX_\beta = \sum \frac{\partial X_\alpha}{\partial X_\beta} e^\beta e^\alpha (e^\beta)^{-1}. \tag{3-2}$$

On the other hand replacing the right derivative by the left one we get two other differentials:

$$f(X)d = \sum 'f(X)_{(\beta)} e^\beta dX_\beta = \sum \frac{\partial U_\alpha}{\partial X_\beta} (e^\beta)^{-1} e^\alpha e^\beta dX_\beta, \tag{3-3}$$

$$f(X)\delta = \sum e^\beta\, 'f(X)_{(\beta)} dX_\beta = \sum \frac{\partial U_\alpha}{\partial X_\beta} e^\alpha dX_\beta. \tag{3-4}$$

Comparing (3-1) and (3-4) we see that $df(X) = f(X)\delta$, for any $f(X)$. Furthermore from (3-2) and (3-3) we see that $\delta f(X) = f(X)d$ for any $f(X)$, since $e^\beta e^\alpha (e^\beta)^{-1} = (e^\beta)^{-1} e^\alpha e^\beta$, whichever be α, β. If $f(X)$ is of class H we also have $df(X) = \delta f(X)$. In fact from (3-1) and (3-2) we have

$$\Delta = df(X) - \delta f(X) = \sum \frac{\partial U_\alpha}{\partial X_\beta} (e^\alpha - e^\beta e^\beta (e^\beta)^{-1})\, dX_\beta$$

$$= \sum \frac{\partial U_j}{\partial X_i} (e^j - e^i e^j (e^i)^{-1}\, dX_i\,.$$

Using (1-6) we get

$$\Delta = \sum \frac{\partial U_j}{\partial X_i} \varepsilon^{jik} e^i e^k dX_i\,.$$

If $f(X)$ is of class H this vanishes in view of the last equation (2-17). Therefore if the function $f(X)$ is of class H we have only one differential form which we denote by $df(X)$ and it is given by (3-1). From now on we shall consider that the functions $f(X)$ are of class H (we shall refer to them as H-functions).

In direct analogy with the theory of complex functions we can define line, surface and volume integrals. The integral of $f(X)$ along a curve γ is simply

$$\int_\gamma f(X)\, dX = \sum e^\alpha e^\beta \int_\gamma U_\alpha\, dX_\beta\,.$$

But we could also have

$$\int_\gamma dX\, f(X) = \sum e^\beta e^\alpha \int_\gamma U_\alpha\, dX_\beta\,.$$

The difference between these two integrals is

$$\Delta I = \sum (e^i e^j - e^j e^i) \int_\gamma U_i\, dX_j = - \sum \int_\gamma U_i\, dX_i\,.$$

Using Stokes theorem and denoting by dS_{ij} the components of the oriented surface element limited by γ, one gets.

$$\Delta I = \int_S \left(\frac{\partial U_i}{\partial X_j} - \frac{\partial U_j}{\partial X_i} \right) dS_{ij} = 0,$$

when $f(X)$ is a H-function.

Note that in the four dimensional space the 2-surface element correspond to the quaternion

$$dS_n = \sum dX_{\alpha\beta} e^\alpha e^\beta = 2 \sum dS_{oi} e^i + \sum \varepsilon^{ijk} dS_{ij} e^k.$$

The 3-surface element is a quaternion dv_n with components

$$dv_0 = dX_1 dX_2 dX_3, \quad dv_1 = dX_0 dX_2 dX_3,$$
$$dv_2 = dX_0 dX_1 dX_3, \quad dv_3 = dX_0 dX_1 dX_2,$$

and the 3-surface integral of an arbitrary quaternion function is

$$\int_v f(X)\, dv_n = \int_v \sum U_\alpha e^\alpha dv_\beta e^\beta$$

$$= \int_v \{ (U_0 dv_0 - \sum \delta^{ij} U_i dv_j) + \sum (U_i dv_0 + U_0 dv_i) + \sum \varepsilon^{ijk} U_i dv_j e^k \}.$$

On the other hand we have the 4-volume element $d\tau = dX_0 dX_1 dX_2 dX_3$ and the integral extended over the volume τ bounded by a 3-surface v:

$$\int \partial f(X) d\tau = \int \sum \partial_\alpha U_\beta e^\alpha e^\beta\, d\tau$$

$$= \int_v \{ (U_0 dv_0 - \sum \delta^{ij} U_i dv_j) + \sum (U_i dv_0 + U_0 dv_i) e^i + \sum \varepsilon^{ijk} U_i dv_j e^k \}.$$

Therefore we get the Boundary Theorem

$$\int_v f(X)\, dv_n = \int_\tau \partial f(X)\, d\tau \qquad (3\text{-}5)$$

and similarly

$$\int_v dv_n\, f(X) = \int_\tau f(X)\partial d\tau. \qquad (3\text{-}6)$$

Using the theorem of tue rotational, the difference between these two integrals is

$$\sum \varepsilon^{ijk} e^k \int_v (U_i\, dv_j - U_j\, dv_i) = \sum \varepsilon^{ijk} e^k \int_\tau \left(\frac{\partial U_i}{\partial X_j} - \frac{\partial U_j}{\partial X_i}\right) d\tau,$$

so that if $f(X)$ is of class H this difference vanishes.

In this case it follows from the right hand sides of (3-5) and (3-6), that

$$\int_v f(X)\, dv_n = \int_v dv_n f(X) = 0. \qquad (3\text{-}7)$$

This result can be regarded as a generalisation of Cauchy's Theorem. The second Cauchy's Theorem follows immediately. Let $f(X)$ be a H-function in a region τ bounded by a simple, closed 3-surface v. Let P be an arbitrary point inside v. Then the second Cauchy's theorem would be generalized by the expression

$$f(P) = -\frac{1}{\pi^2}\int_v f(X)(X-P)^{-3}\, dv_n.$$

To see that this is true we notice firstly that the function $f(X)(X-P)^{-3}$ fails to be of class H at P, so that the integral does not necessarily vanish. However, constructing a small sphere C of radius ε with center at P and connecting its surface with v by means of a "cut" such that along which the integrals vanish by symmetry we get as in complex analysis:

$$\int_v f(X)(X-P)^{-3}\, dv_n = \int_C f(X)(X-P)^{-3}\, dv_n. \qquad (3\text{-}8)$$

Now we can use the polar form of quaternions to write $n = e^{\lambda\gamma}$ and $X - P = \varepsilon e^{\lambda\gamma} = \varepsilon n$ for X on the sphere C. Considering that $f(X)$ has the form $\sum U_\alpha(X_u) e^\alpha$ and that it is of class H, we can expand each component in Taylor series around the point P:

$$f(X) = f(P) + \varepsilon \sum \eta_\alpha\, \alpha_\alpha f(X)\Big|_P + \frac{\varepsilon^2}{2!}\sum_{\alpha,\beta} \eta_\alpha \eta_\beta\, \partial_\alpha \partial_\beta f(X)\Big|_P + \ldots, \qquad (3\text{-}9)$$

where η_α are the components of the unit quaternion n. With this expansion the integral over C becomes

$$\int_C f(X)(X-P)^{-3}\, dv_n =$$

$$f(P)\int_C (X-P)^{-3}\, dv_n + \varepsilon \sum \int_C \eta_\alpha \partial_\alpha f(X)\Big|_P (X-P)^{-3}\, dv_n + \ldots$$

In the limit $\varepsilon \to 0$ we obtain

$$f(P) = \int_C f(X)(X-P)^{-3}\, dv_n \left[\int_C (X-P)^{-3}\, dv_n\right]^{-1} + 0(\varepsilon). \qquad (3\text{-}10)$$

We can calculate $\int_C (X-P)^{-3}\, dv_n$ using the spherical coordinates in $\mathbf{R}^4$:

$$X_0 = \rho \operatorname{Cos}\gamma, \qquad X_1 = \rho \operatorname{Sin}\gamma \operatorname{Sin}\theta \operatorname{Cos}\phi$$

$$X_2 = \rho \operatorname{Sin}\gamma \operatorname{Sin}\theta \operatorname{Sin}\phi, \qquad X_3 = \gamma \operatorname{Sin}\gamma \operatorname{Cos}\theta.$$

The volume element is $d\tau = dX_0\, dX_1\, dX_2\, dX_3 = J d\rho\, d\theta\, d\phi\, d\gamma$, where J is the Jacobian determinant of the coordinate transformation: $J = \rho^3 \operatorname{Sin}^2\gamma \operatorname{Sin}\theta$. On the other hand the 3-surfacc element can be written as $dv_n = \eta dv$, $dv = |dv_n|$, $\eta = e^{\lambda\gamma}$. In particular the 3-surface element for a 3-sphere of radius R is $dv_n = \eta dv = e^{\lambda\gamma}\dfrac{d\tau}{dR} = e^{\lambda\gamma} d\theta\, d\phi\, d\gamma$. Therefore we have for the 3-sphere of radius ε

$$\int (X-P)^{-3}\, dv_n = \int_C e^{-2\lambda\gamma} \operatorname{Sin}^2\gamma \operatorname{Sin}\theta\, d\theta\, d\phi\, d\gamma = -\pi^2,$$

Where we put $e^{-2\lambda\gamma} = \operatorname{Cos} 2\gamma - \operatorname{Sin} 2\gamma$ and $0 \le \theta \le 2\pi, 0 \le \gamma \le \pi, 0 \le \phi \le \pi$. Hence (3-10) becomes for an arbitrary point Y

$$f(Y) = -\frac{1}{\pi^2} \int f(X)(X-Y)^{-3}\, dv_n. \qquad (3\text{-}11)$$

From this it can be easily seen that the n^{th} derivate at right of $f(Y)$ is

$$f^{(n)}(Y) = -\frac{1}{\pi^2} \int f(X)[(X-Y)^{-3}]^{(n)}\, dv_n. \qquad (3\text{-}12)$$

IV – Power Series and Singularities

Consider initially the H-function $(1-X)^{-3}$

The series

$$S(X) = \sum_{n=0}^{\infty} \frac{(2+n)!}{2n!} X^n = 1 + 3X + 6X^2 + \ldots + \frac{(n+1)!}{2(n-1)!} X^{n-1} + \ldots \qquad (4\text{-}1)$$

converges to $(1-X)^{-3}$. In fact, we have

$$(1-X)^{-3} = \frac{\overline{(1-X)^3}}{(|1-X|^2)^3} = \overline{(1-X)^3}\, \frac{1}{(1-r)^3},$$

where we have put $|1-X|^2 = 1 - 2X_0 + |X|^2 = 1 - r$.

Assuming that $|X|^2 < 1$, we have also $r < 1$. Thus we can develop the real series

$$\frac{1}{(1-r)^3} = 1 + 3r + 6r^2 + 10r^3 + \ldots + \frac{n(n+1)}{2} r^{n-1} + \ldots,$$

so that

$$(1-X)^{-3} = \overline{(1-X)^3} \sum_{n=1}^{\infty} \frac{n(n+1)}{2} r^{n-1}.$$

Taking the module of the difference between the n^{th} term of the two series and considering a fundamental property of quaternions which is

$$|XX'| = |X|\,|X'|, \qquad (4\text{-}2)$$

together with the triangular inequality, one gets that this difference goes to zero as $n \to \infty$, so that we can write

$$(1-X)^3 = \sum_{n=1}^{\infty} \frac{n(n+1)}{2} X^{n-1}. \qquad (4\text{-}3)$$

Starting with this simple example we can construct more general power series for a H-function.

Consider that we have a quaternion $f(X)$ which is of class H in a certain region of the four dimensional manifold. Let Q be a point inside this region and let R be the radius of the largest sphere with center at Q and which is inside the region considered. We wish to show that there is a power series of the form $\sum a_n (X-Q)^n$ which converges to $f(X)$ for all X inside the sphere. In fact, consider a point P inside the sphere. Then $|P-Q| = R_1 < R$ and by (3-11)

$$f(P) = -\frac{1}{\pi^2} \int_{C} f(X)\,(X-P)^{-3}\, dv_n,$$

where C_1 is the sphere with radius R_1 and center at Q. We can also write

$$f(P) = -\frac{1}{\pi^2} \int_{C_1} f(X)(X-Q)^{-3}[1-(X-Q)^{-1}(P-Q)]^{-3}\, dv_n.$$

As we have seen,

$$[1-(X-Q)^{-1}(P-Q)]^{-3} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} (X-Q)^{-n}(P-Q)^n, \qquad (4\text{-}4)$$

so that

$$f(P) = -\frac{1}{\pi^2} \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} \int_{C_1} f(X)\,(X-Q)^{-3-n}(P-Q)^n\, dv_n.$$

$(P-Q)^n$ is a constant quaternion and can be written as $(P-Q)^n = \varepsilon^n e^{n\lambda\gamma}$. Since $dv_n = e^{\lambda\gamma}\, dv$, we have

$$(P-Q)^n\, dv_n - dv_n (P-Q)^n = \varepsilon^n\, dv\,(e^{n\lambda\gamma} e^{\lambda\gamma} - e^{\lambda\gamma} e^{n\lambda\gamma}) = 0 \qquad (4\text{-}5)$$

so that

$$f(P) = -\frac{1}{\pi^2}\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}\left[\int_{C_1} f(X)\,(X-Q)^{-3-n}\, dv_n\right](P-Q)^n.$$

Defining

$$a_t = \frac{1}{\pi^2}\frac{(t+1)(t+2)}{2}\int_{C_1} f(X)(X-Q)^{-3-t}\, dv_n, \qquad (4\text{-}6)$$

we get

$$f(P) = \sum_0^{\infty} a_t (A-A)^t. \qquad (4\text{-}7)$$

Therefore we have a power expansion for a general H-function $f(X)$, provided the integral (4-6) is finite. Conversely, every power series of the form (4-7) represents a H-function defined in a certain domain. We can also have power expansions involving negative powers. Consider that $f(X)$ is of class H inside a region limited by two concentric spheres c and C of radius r and R respectively and center at Q. Thus $f(X)$ is H for $r \le |X - X_0| \le R$ and by (3-11) (Cauchy's formula) we have

$$f(P) = -\frac{1}{\pi^2}\int_C f(X)(X-P)^{-3}\, dv_n + \frac{1}{\pi^2}\int_c f(X)(X-P)^{-3}\, dv_n$$

$$= -\frac{1}{\pi^2}\int_C f(X)[(X-Q)-(P-Q)]^{-3} dv_n - \frac{1}{\pi^2}\int_c f(X)[(P-Q)-(X-Q)]^{-3} dv_n$$

$$= \frac{1}{\pi^2}\int_C f(X)(X-Q)^{-3}[1-(X-Q)^{-1}(P-Q)]^{-3}\, dv_n -$$

$$-\frac{1}{\pi^2}\int_c f(X)(P-Q)^{-3}[1-(P-Q)^{-1}(X-Q)]^{-3}\, dv_n.$$

As in (4-4) we replace the brackets by the corresponding expansions to get

$$f(P) = \frac{1}{\pi^2}\int_C f(X)\,(X-Q)^{-3}\left\{\sum_0^{\infty}\frac{(t+1)(t+2)}{2}(X-Q)^{-t}(P-Q)^t\right\} dv_n -$$

$$-\frac{1}{\pi^2}\int_c f(X)\,(P-Q)^{-3}\left\{\sum_0^{\infty}\frac{(s+1)(s+2)}{2}(P-Q)^{-s}(X-Q)^s\right\} dv_n. \qquad (4\text{-}8)$$

In virtue of (4-5) we may commute $P-Q$ with dv_n. Using the definition (4-6) the first integral can be written as

$$\sum_0^{\infty} a_t\,(P-Q)^t.$$

On the other hand, putting $f(X) = U_\alpha e^\alpha$ and commuting with $(P-Q)^{-3-s}$ we get

$$f(X)(P-Q)^{-3-s} = (P-Q)^{-3-s} f(X) + \varepsilon^{ijk}\, U_i((P-Q)^{-3-s})_j e^k.$$

so that the second integral (4-8) becomes

$$\frac{1}{\pi^2}\sum\frac{(s+1)(s+2)}{2}(P-Q)^{-3-s}\int_c f(X)\,(X-Q)^s\, dv_n +$$

$$\frac{1}{\pi^2}\sum\frac{(s+1)(s+2)}{2}\int_c \varepsilon^{ijk}\, U_i((P-Q)^{-3-s})_j\, e^k (X-Q)^s\, dv_n.$$

Defining

$$a_s = -\frac{1}{\pi^2}\frac{(s+1)(s+2)}{2}\int_c f(X)(X-Q)^s\, dv_n \qquad (4\text{-}9)$$

$$a_{s(i)} = -\frac{1}{\pi^2}\frac{(s+1)(s+2)}{2}\int_c U_i(X-Q)^s\, dv_n \qquad (4\text{-}10)$$

the expression (4-8) can be written as

$$f(P) = \sum_0^\infty a_t(P-Q)^t + \sum_0^\infty (P-Q)^{-3-s}\, a_s + \sum_0^\infty \varepsilon^{ijk}\, e^k\, ((P-Q)^{-3-s})_j\, a_{s(i)} \tag{4-11}$$

But since $(X-Q)^s$ is of class H in the interior of c (that is, outside the spherical ring considered) it follows from (3-5) that (4-10) vanishes. Thus, putting $-3-s=-r$, $s=0,1,2,\dots$ $r=3,4,5,\dots$ and calling $a_s = a_{-3+r} = b_{-r}$, the expression (4-11) becomes for $P=X$

$$f(X) = \sum_{t=0}^\infty a_t(X-Q)^t + \sum_{r=3}^\infty (X-Q)^{-r}\, b_{-r}, \tag{4-12}$$

with

$$b_{-r} = -\frac{1}{\pi^2}\,\frac{(r-1)(r-2)}{2}\int_c f(X)(X-Q)^{r-3}\, dv_n. \tag{4-13}$$

The expression (4-12) is the equivalent to the Laurent expansion for complex functions. We can say that the two spheres C, c define the convergence "ring" for $f(X)$ around Q. In analogy with the complex case we call Q the 3-pole and the first negative coefficient in (4-12):

$$b_{-3} = -\frac{1}{\pi^2}\int_C f(X)\, dv_n, \tag{4-14}$$

is called the 3-residue of $f(X)$ at Q.

Now we can prove the "Residue Theorem" for H-functions. If $f(X)$ is of class H in a given region except in a finite number of 3-poles $Q^{(j)}$ which do not belong to the boundary V of the region, then

$$\int_V f(X)\, dv_n = -\pi^2\left(\sum_j b_{-3}{}^{(j)} + \sum_j C_{-3}{}^{(j)}\right), \tag{4-15}$$

where $b_{-3}{}^{(j)}$ are 3-residues given by (4-14) for each pole j given by the quaternion $Q^{(j)}$:

$$b_{-3}{}^{(j)} = -\frac{1}{\pi^2}\int_{c_j} f(X)\, dv_n{}^{(j)}, \tag{4-16}$$

where c_j is a small sphere constructed with center at $Q^{(j)}$ and

$$C_{-3}{}^{(j)} = \int_{c_j} (X-Q^{(j)})^{-3}\, [b_{-3}{}^{(j)}\, dv_n{}^{(j)}]_{(-)}, \tag{4-17}$$

where [ ] denotes the commutatior. To prove (4-15) we construct the spheres c_j around each 3-pole $Q^{(j)}$ and use (3-7):

$$\int_V f(X)\, dv_n = \sum_j \int_{c_j} f(X)\, dv_n{}^{(j)}.$$

Expanding $f(X)$ in power series around each pole we have

$$f(X)^{(j)} = \sum_m a_m{}^{(j)}\, (X-Q^{(j)})^m + \sum_l (X-Q^{(j)})^{-l}\, b_{-l}{}^{(j)},$$

so that

$$\begin{aligned} f(X)\, dv_n &= \sum_j \int_{c_j} f(X)^{(j)}\, dv_n{}^{(j)} \\ &= \sum_j\sum_m a_m{}^{(j)} \int_{c_j} (X-Q^{(j)})^m\, dv_n{}^{(j)} + \sum_j\sum_l \int (X-Q^{(j)})^{-l}\, b_{-l}{}^{(j)}\, dv_n{}^{(j)} \\ &= \sum_j\sum_l a_m{}^{(j)} \int (X-Q^{(j)})^m\, dv_n{}^{(j)} + \sum_j\sum_l \int (X-Q^{(j)})^{-1}\, dv_n{}^{(j)}\, b_{-l}{}^{(j)} \\ &\quad + \sum_j\sum_l \int_{c_j} (X-Q^{(j)})^{-l}\, [b_{-1}{}^{(j)}\, dv_n{}^{(j)}]_{(-)}. \end{aligned}$$

Using the polar form we can put $(X-Q^{(j)})^m = \varepsilon_{(j)}^m\, e^{m\lambda_j\gamma_j}$, where $\varepsilon_{(j)}$ is the radius of the j^{th} sphere and

$$dv_n{}^{(j)} = e^{\lambda_j\gamma_j}\, dv_n{}^{(j)},$$

so that

$$\int_{c_j} (X-Q^{(j)})^m\, dv_n{}^{(j)} = \varepsilon_{(j)}^{m+3} \int_{c_j} e^{(m+1)\lambda_j\gamma_j}\, \mathrm{Sin}^2\gamma\, \mathrm{Sin}\,\theta\, d\gamma\, d\theta\, d\phi,$$

$$\int_{c_j} (X-Q^{(j)})^{-l}\, dv_n{}^{(j)} = \varepsilon_{(j)}^{-l+3} \int_{c_j} e^{(-l+1)\lambda_j\gamma_i}\, \mathrm{Sin}^2\gamma\, \mathrm{Sin}\,\theta\, d\gamma\, d\theta\, d\phi$$

and

$$\int_{c_j} (X-Q^{(j)})^{-l}[b_{-l}^{(j)} \underset{(-)}{dv_n^{(j)}}]$$

$$= \varepsilon^{-l+3} \int_{c_j} e^{-l\lambda_i\gamma_j}[b_{-l}^{(j)} e^{\lambda_j\gamma_j}] \mathrm{Sin}^2\gamma \, \mathrm{Sin}\theta \, d\gamma \, d\theta \, d\phi.$$

When $\varepsilon \to 0$ the first integral vanishes for any m while the last two integrals are different from zero when $l = 3$. Thus

$$\int f(X)\, dv_n = -\pi^2 \left(\sum_j b_{-3}^{(j)} + \sum_j \int_{c_j} (X - Q^{(j)})^{-3} [b_{-3}^{(j)} \underset{(-)}{dv_n^{(j)}}] \right).$$

In general b_{-3} is a quaternion given by (4-14). In particular it may have only the e^0 component and in this case it commutes with dv_n so that $C_{-3}^{(j)}$ vanishes and (4-15) reduces to the form

$$f(X)\, dv_n = -\pi^2 \sum_j b_{-3}^{(j)}. \tag{4-18}$$

The maximum negative power in (4-12) is called the order of the 3-pole Q. When this maximum power is -3 the 3-pole is said to be a zero order pole (note that there is no negative power less than 3). In the case of a zero order pole we have from (4-12)

$$b_{-3} = \lim_{X \to Q} \{(X-Q)^3 f(X)\}. \tag{4-19}$$

On the other hand if the pole is of finite order larger than zero (i.e. if the largest negative power is $-s-3$ for some finite s) then, again from (4-12),

$$(X-Q)^{s+3} f(X) = \sum_t (X-Q)^{t+s+3} a_t + (X-Q)^s b_{-3} + (X-Q)^{s-1} b_{-4} + \ldots + b_{-s-3},$$

so that

$$b_{-3} = \lim_{X \to Q} \frac{d^s}{dX^s} \{(X-Q)^{s+3} f(X)\}. \tag{4-20}$$

We may use the same terminology of the complex theory. For example if there is no negative powers in (4-12) the power series coincide with the Taylor's series for quaternion functions and in this case Q is called a removable singularity. On the ohter hand for the case when we have an infinite number of negative powers, Q is called an essential singularity.

V – The Applications

We shall consider here the uses of the theory of H-functions in Special and General Relativity theory. Quaternion formalism has been the object of renewed interest in these branches of theoretical Physics (see for instance Rastall (1964)) and the theory of H-functions can be useful in treating some problems. Let us consider initially the case of Special Relativity. In this case the manifold to be considered is the Minkowski space-time for which the metric tensor is $\eta^{ij} = \mathrm{diag}(-1, 1, 1, 1)$. A vector belonging to this space-time has norm

$$(X, X) = -X_0^2 + X_1^2 + X_2^2 + X_3^2 = \sum \eta^{ij} X_i X_j. \tag{5-1}$$

We can represente such vector by a quaternion $X = X_\mu \sigma^\mu$ where σ^μ is a new basis defined by

$$\sigma^i = ie^i \qquad \sigma^0 = e^0. \tag{5-2}$$

With this, $X\bar{X} = |X|^2 = (X, X)$. The multiplication table for the new basis is

$$\begin{aligned} \sigma^i \sigma^j &= \delta^{ij} + i\varepsilon^{ijk}\sigma^k \\ \sigma^i \sigma^0 &= \sigma^0 \sigma^i = \sigma^i \end{aligned} \tag{5-3}$$

This table is satisfied by the set of Pauli matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5-4}$$

In Minkowski Space-time a vector transforms according to the Lorentz group. As it is known, the group of automorphisms of the quaternion algebra is isomorphic to SO(3) only (see eg. Boerner (1963)). Therefore the quaternion algebra is not invariant under the Lorentz group.

We can represent a Lorentz transformation in terms of quaternions by

$$X' = qXq^+, \tag{5-5}$$

where $X = X_\mu \sigma^\mu$ and q^+ means the hermitian conjugate of q. In the special case where q is unitary we get one automorphism but this occur only for the subgroup SO(3) of the Lorentz group. We can modify the quaternion algebra in such a way that the automorphisms of the resulting algebra represent the Lorentz group. This can be obtained by asking for a set of 4 algebraic elements S^μ so that $Q\sigma^\mu Q^+ = QS^\mu Q^{-1}$. That is, $S^\mu = \sigma^\mu Q^+ Q$ where Q is a specific Lorentz transformation (Lord (1966)). These elements form the basis of the modified quaternion algebra. To relate to our original algebra we set $\sigma^\mu = (e^0 \delta_0^\mu + i \sum e^i \delta_i^\mu)$ and

$$S^\mu = (e^0 \delta_0^\mu + i \sum e^i \delta_i^\mu) Q^+ Q. \tag{5-6}$$

Conversely $\sigma^\mu = S^\mu (Q^+ Q)^{-1}$ and

$$e^\mu = (S^0 \delta_0^\mu - i \sum S^i \delta_i^\mu)(Q^+ Q)^{-1}. \tag{5-7}$$

Theorefore to use the theory of H-functions in Minkowski Space-Time we just plug in the above expression for the $e^{\mu'}s$. It is important to notice that if Q is unitary we obtain $S^i = ie^i$, $S^0 = e^0$, and we say that we have a special frame for the modified algebra. The unitary transformations map these special frames into each other. The theory of H-functions developed in the previous sections can be thought of as being done in one of these special frames.

Conformal motions are also considered in Special Relativity. Consider the quaternion formulation of a conformal transformation (Gürsey (1956)) $dX' = \pm \Gamma^+(X) dX\, \Gamma(X)$, where $\Gamma(X) = \sqrt{\beta}(X-A)^{-1} q$ and where β is a constant, q is a Lorentz transformation and A is a quaternion which gives a translation. In analogy with the complex conformal transformation we should expect that $\Gamma(X)$ is a H-function. The components of such function are

$$U_0(X) = \sqrt{\beta}\, \frac{X_0 - A_0}{|X-A|^2}\, q, \quad U_i(X) = -\sqrt{\beta}\, \frac{X_i - A_i}{|X-A|^2}\, q.$$

By direct calculation we see that (2-17) are satisfied.

In the case of General Relativity we have to introduce further modifications on the original quaternion algebra. We may consider the tangent bundle structure where each fibre is a Minkowski space where we can apply (5-6, 7). In order to obtain the algebra in the Riemannian manifold we use a tetrad system $h_\mu^\alpha(x)$ so that the modified algebra is translated to $\zeta^\alpha = \sum h_\mu^\alpha(x) S^\mu$ and, using (5-6)

$$\zeta^\alpha = \sum h_\mu^\alpha(x)(e^0 \delta_0^\mu + i \sum e^i \delta_i^\mu) Q^+ Q, \tag{5-8}$$

or conversely $S = \Delta h_\alpha^\mu \zeta^\alpha$, so that, from (5-7)

$$e^\mu = \sum (h_\alpha^0 \zeta^\alpha \delta_0^\mu - i \sum h^i \zeta^\alpha \delta_i^\mu)(Q^+ Q)^{-1}. \tag{5-9}$$

With this expression we may translate the theory of H-functions to Riemanian Manifolds. The tetrad system can be conveniently choosen. We suggest the one used by Newmann & Penrose (1962) which is closely related to the quaternion formalism. The algebraic forms which can be associated to a tensor are obtained by contraction with the base elements ζ^α. Thus, for example the Ricci tensor corresponds to five algebraic functions

$$R_\beta = R_{\alpha\beta} \zeta^\alpha = R_{\alpha\beta} h_\mu^\alpha (e^0 \delta_0^\mu + i \sum e^i \delta_i^\mu) Q^+ Q,$$

$$R = R_{\alpha\beta} \zeta^\alpha \zeta^\beta = R_{\alpha\beta} h_\mu^\alpha h_\nu^\beta (e^0 \delta_0^\mu + i \sum e^i \delta_i^\mu)(e^0 \delta_0^\nu + i \sum e^i \delta_j^\nu) Q^+ Q.$$

These function can be written is the form $U_\alpha(x) e^\alpha$ and be studied from the point of view of the theory of H-functions. One possible application is the study of the properties of singularities in General Relativity.

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