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Topological lower bounds on the distance between area preserving diffeomorphisms

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Abstract. Area preserving diffeomorphisms of the 2-disk which are Identity near the boundary form a group which can be equipped, using the L^2-norm on its Lie algebra, with a right invariant metric. In this paper we give a lower bound on the distance between diffeomorphisms which is invariant under area preserving changes of coordinates and which improves the lower bound induced by the Calabi invariant. In the case of renormalizable and infinitely renormalizable maps, our estimate can be improved and computed.

Keywords: Area preserving maps, group of diffeomorphisms, invariant metric, topological invariant.

1 Introduction

The group of volume preserving diffeomorphisms on a compact oriented Riemannian manifold carries a structure of metric space. The (right invariant) metric can be defined, using the L^2-norm on the Lie algebra, as the minimal length of an arc connecting two diffeomorphisms (see section 2 for a complete definition). In [12], A. Shnirelman proved that the diameter of the group of volume preserving diffeomorphisms of the unit cube in $\mathbb{R}^n$ is finite when $n \geq 3$. The situation turns to be drastically different in the 2-dimensional case: the diameter of the group of area preserving diffeomorphisms of the square is infinite. Actually, Y. Eliashberg and T. Ratiu [7] proved that, on any compact symplectic manifold, the diameter of the connected component of Identity of the group of symplectomorphisms is infinite. The main steps in their proof are:

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- one can reduce the problem to the case of symplectomorphisms of the unit $2n$-ball $\mathbf{B}_{2n}$ which are Identity near the boundary $\partial\mathbf{B}_{2n}$;
- in this case, the distance of a symplectomorphism to Identity is bounded from below by the modulus of its Calabi invariant up to multiplication by a constant[1](see section 2 for a complete definition);
- it is easy to find symplectomorphisms with arbitrarily large values of their Calabi invariant.

In this paper we shall concentrate on the case of the 2-disk. In this case, the Calabi invariant can be seen as the averaged linking number of pairs of orbits (see [8] and [10]), results which are summarized in section 2.

Our aim is to introduce another quantity which gives a better lower bound of the distance to Identity. We call this number *the asymptotic crossing number* of the diffeomorphism. It is related to the crossing number of pairs of orbits. In section 3, we make precise the definition of this asymptotic crossing number and discuss its main properties. In particular, we show that this number is a differentiable invariant, i.e. invariant under conjugacy by an area preserving diffeomorphism.

In section 4, we consider the case when the diffeomorphism is renormalizable, that is to say when it permutes a collection of disks. This new invariant allows us to bound from below the distance to Identity normalized by the area of each disk by a constant which depends only on the braid type of this collection. When the diffeomorphism is infinitely renormalizable, this last estimate shows restrictions on the geometry of the invariant Cantor set imposed by the braid type of the infinite system of permuted disks.

Our work has been inspired by results due successively to V. Arnold [1] and M. Freedmann and Z.-X. He [9] on lower bounds for the energy of a divergence-free vector field in a domain of $\mathbb{R}^3$. In Section 5, we show how our results lead to better lower bounds of the energy when the domain is the solid torus and the divergence-free vector field a suspension of an area preserving diffeomorphism of the 2-disk.

2 Area preserving diffeomorphisms of the 2-disk

Let us call $\mathcal{D}_2$ the group of smooth area preserving diffeomorphisms of the 2-disk which are Identity near the boundary of the disk . A tangent vector to $\mathcal{D}_2$ at

[1] This implies that the Calabi invariant is a continuous function of the symplectomorphism.

a point ϕ is a divergence free vector field X_ϕ (see [6]) whose L^2-norm is defined by:

$$\|X_\phi\|_2 = (\int_{\mathbf{D}^2} \|X_\phi(x)\|^2 dx)^{1/2},$$

where $\|.\|$ stands for the standard Euclidean norm. More generally in the sequel, $\| - \|_p$ will stand for the L^p-norm.

Consider a path $\{\phi_t\}_{t\in[0,1]}$ in $\mathcal{D}_2$ connecting two maps ϕ_0 and ϕ_1. The length of the path $\{\phi_t\}$ is given by the formula:

$$l_2(\{\phi_t\}) = \int_0^1 \|\frac{d\phi_t}{dt}\|_2 dt.$$

Any two maps in $\mathcal{D}_2$ being connected by a path with finite length, this length function induces a distance function on $\mathcal{D}_2$ defined by $d_2(\phi_0, \phi_1) = \inf l_2(\{\phi_t\})$, where the infimum is taken over all paths joining ϕ_0 to ϕ_1. Notice that this distance is right invariant: $d_2(\phi_0 \circ \phi, \phi_1 \circ \phi) = d_2(\phi_0, \phi_1)$.

The set $\mathcal{D}_2$ being now a metric space, it makes sense to speak about the diameter of a subset S:

$$diam_2(S) = \sup_{\phi_0,\phi_1\in S} d_2(\phi_0, \phi_1).$$

Let us recall now the definition of the *Calabi invariant* (see [5]) $C : \mathcal{D}_2 \to \mathbb{R}$.

Consider a map ϕ in $\mathcal{D}_2$ and a 1-form α on $\mathbf{D}^2$ which is a primitive of the area 2-form. Since ϕ is area preserving, the form $\phi^\star\alpha - \alpha$ is closed and vanishes near the boundary of $\mathbf{D}^2$. Thus, there exists a unique function $H(\phi, \alpha) : \mathbf{D}^2 \to \mathbb{R}$, which vanishes near the boundary of $\mathbf{D}^2$ and such that $dH(\phi,\alpha) = \phi^\star\alpha - \alpha$. The Calabi invariant of ϕ is defined by:

$$C(\phi) = \int_{\mathbf{D}^2} H(\phi, \alpha).$$

The reader can check out (see for instance [2] or [10]) the fact that this integral does not depend on the choice of the primitive α and that the Calabi invariant is actually a morphism from $\mathcal{D}_2$ to $\mathbb{R}$: $C(\phi_0 \circ \phi_1) = C(\phi_0) + C(\phi_1)$.

It is known that every element in $\mathcal{D}_2$ is the the "time 1" map of a Hamiltonian isotopy $\{\phi_t\}_{t\in[0,1]}$, i.e. there exists a smooth Hamiltonian $\mathcal{H}_t : \mathbf{D}^2 \to \mathbb{R}$, which vanishes near the boundary of the 2-disk, depends on time and satisfies for all t in $[0, 1]$:

$$d\mathcal{H}_t(-) = area(-, \frac{\partial\phi_t}{\partial t})$$

This Hamiltonian isotopy yields a second definition of the Calabi invariant:

$$C(\phi) = 2\int_0^1\int_{\mathbf{D}^2} \mathcal{H}_t(x)dxdt.$$

For the equivalence of these two definitions, we refer for example to [2] where explicit calculations are done. Notice that on one hand, the Calabi invariant does not depend on the choice of the Hamiltonian isotopy and that on the other hand any isotopy in $\mathcal{D}_2$ is Hamiltonian (see [7]). This second definition of the Calabi invariant allows us to bound from below the distance of any diffeomorphism in $\mathcal{D}_2$ to Identity:

$$l_2(\{\phi_t\}) = \int_0^1 \|\frac{d\phi_t}{dt}\|_2 dt = \int_0^1 \|\nabla\mathcal{H}_t(x)\|_2 dt.$$

Thanks to Poincaré inequality:

$$l_2(\{\phi_t\}) \geq C\int_0^1 \|\mathcal{H}_t(x)\|_2 dt$$

for a constant C which depends only on the domain of integration (here the unit 2-disk).

From Schwarz inequality (normalizing the area of the unit 2-disk to 1):

$$\int_0^1 \|\mathcal{H}_t(x)\|_2 dt \geq \int_0^1 \|\mathcal{H}_t(x)\|_1 dt \geq |\int_0^1\int_{\mathbf{D}^2} \mathcal{H}_t(x)dt| = \frac{1}{2}|C(\phi)|.$$

Using the right invariance of the path length and the additivity of the Calabi invariant, we get the Lipschitz estimate:

$$|C(\phi_0) - C(\phi_1)| \leq \frac{2}{C}d_2(\phi_0, \phi_1).$$

There is a third definition of the Calabi invariant which is due to A. Fathi [8] (see also [10]) and which can be seen as an estimate in terms of braiding of pairs of orbits. Consider an isotopy $\{\phi_t\}_{t\in[0,1]}$ in $\mathcal{D}_2$ connecting Identity to ϕ. The map:

$$Ang_\phi : \mathbf{D}^2 \times \mathbf{D}^2 \setminus \Delta \to \mathbb{R},$$

(where Δ stands for the diagonal) which associates to any pair of points $x \neq y$ in $\mathbf{D}^2$ the angular variation of the vector $\overrightarrow{\phi_t(x)\phi_t(y)}$ when t goes from 0 to 1, does not depend on the choice of the isotopy and is bounded where it is defined. The Calabi invariant is the integral of this function, i.e.:

$$C(\phi) = \int_{\mathbf{D}^2\times\mathbf{D}^2} Ang_\phi(x, y)dxdy.$$

This third definition allows us to prove (see [10]) that the Calabi invariant is a topological invariant, that is to say that if h is a homeomorphism of the 2-disk which is Identity near the boundary, preserves the area 2-form and conjugates two maps ϕ and ψ in $\mathcal{D}_2$, then: $C(\psi) = C(\phi)$. Notice that when ϕ is a homeomorphism of $\mathbf{D}^2$ which is Identity near the boundary of $\mathbf{D}^2$, the map Ang_ϕ remains defined and continuous on $\mathbf{D}^2 \times \mathbf{D}^2 \setminus \Delta$. However in this last case it is not necessarily integrable.

3 Asymptotic crossing number

Let us introduce now the new quantity we wish to study. Fix again an element ϕ in $\mathcal{D}_2$ and a (Hamiltonian) isotopy $\{\phi_t\}_{t\in[0,1]}$ in $\mathcal{D}_2$ connecting Identity to ϕ. To any pair of distinct points $x,\ y$ in $\mathbf{D}^2$ and to every $t \in [0, 1]$, we associate the unit vector:

$$u(t, x, y) = \frac{\phi_t(y) - \phi_t(x)}{\|\phi_t(y) - \phi_t(x)\|}.$$

Lemma 1. *The integral*

$$\mathcal{G}(\{\phi_t\}) = \frac{1}{2\pi}\int_0^1\int_{\mathbf{D}^2\times\mathbf{D}^2} \|\frac{du}{dt}(t, x, y)\|dxdydt$$

is well defined.

Proof. An easy calculation yields:

$$\|\frac{du}{dt}(t, x, y)\| = \frac{\|(\dot\phi_t(y) - \dot\phi_t(x)) \wedge (\phi_t(y) - \phi_t(x))\|}{\|\phi_t(y) - \phi_t(x)\|^2},$$

where $\wedge$ is the wedge product and $\dot\phi_t = \frac{\partial\phi_t}{\partial t}$. It follows that

$$\|\frac{du}{dt}(t, x, y)\| \leq \frac{2}{\|\phi_t(y) - \phi_t(x)\|} \sup_{(t,x)\in[0,1]\times\mathbf{D}^2} \|\dot\phi_t(x)\|.$$

The quantity $\sup_{(t,x)\in[0,1]\times\mathbf{D}^2} \|\dot\phi_t(x)\|$ is bounded. Hence it is enough to remark that since ϕ_t is area preserving for every t in $[0, 1]$:

$$\int_{\mathbf{D}^2\times\mathbf{D}^2} \frac{dxdy}{\|\phi_t(y) - \phi_t(x)\|} = \int_{\mathbf{D}^2\times\mathbf{D}^2} \frac{dxdy}{\|y - x\|}$$

and that this last integral converges. □

We now introduce the quantity $\mathcal{G}(\phi) = \inf \mathcal{G}(\{\phi_t\})$, where the infimum is taken over all isotopies in $\mathcal{D}_2$ joining Identity to ϕ.

Lemma 2. *The map $\mathcal{G} : \mathcal{D}_2 \to \mathbb{R}^+$ is subadditive, i.e. for all ϕ and ψ in $\mathcal{D}_2$:*

$$\mathcal{G}(\phi \circ \psi) \leq \mathcal{G}(\phi) + \mathcal{G}(\psi).$$

Proof. The proof is straightforward. Let us choose two isotopies $\{\phi_t\}_{t\in[0,1]}$ and $\{\psi_t\}_{t\in[0,1]}$ in $\mathcal{D}_2$ joining respectively Identity to ϕ and Identity to ψ and consider the isotopy $\{\rho_t\}_{t\in[0,1]}$ joining Identity to $\phi \circ \psi$ obtained by concatenating $\{\phi_t \circ \psi\}$ after $\{\psi_t\}$ and time rescaling. The quantity $\mathcal{G}(\{\rho_t\})$ is equal to:

$$\mathcal{G}(\{\rho_t\}) = \mathcal{G}(\{\psi_t\}) +$$
$$+\frac{1}{2\pi}\int_0^1\int_{\mathbf{D}^2\times\mathbf{D}^2} \frac{\|(\dot{\phi}_t(\psi(y)) - \dot{\phi}_t(\psi(x))) \wedge (\phi_t(\psi(y)) - \phi_t(\psi(x)))\|}{\|\phi_t(\psi(y)) - \phi_t(\psi(x))\|^2} dxdydt.$$

Since ψ is area preserving, we get:

$$\mathcal{G}(\{\rho_t\}) = \mathcal{G}(\{\psi_t\}) + \mathcal{G}(\{\phi_t\}).$$

Thus $\mathcal{G}(\phi \circ \psi) \leq \mathcal{G}(\phi) + \mathcal{G}(\psi)$. □

This subadditivity property implies that the sequence $(\frac{1}{n}\mathcal{G}(\phi^n))_{n>0}$ converges when n goes to $+\infty$. We denote by $\mathcal{L}(\phi)$ this limit and call it *the asymptotic crossing number* of ϕ. It satisfies:

$$\mathcal{L}(\phi) = \lim_{n\to+\infty} \frac{1}{n}\mathcal{G}(\phi^n) = \inf_{n>0} \frac{1}{n}\mathcal{G}(\phi^n) \leq \mathcal{G}(\phi).$$

Theorem 1. *The asymptotic crossing number satisfies the following three properties:*

i) it is a differentiable invariant, i.e. for any maps ϕ and ψ in $\mathcal{D}_2$:

$$\mathcal{L}(\psi \circ \phi \circ \psi^{-1}) = \mathcal{L}(\phi);$$

ii) there exists a constant $K > 0$ such that for any map ϕ in $\mathcal{D}_2$:

$$\mathcal{L}(\phi) \leq K d_2(Id, \phi);$$

iii) for any map ϕ in $\mathcal{D}_2$: $|C(\phi)| \leq \mathcal{L}(\phi)$.

Proof. **i)** The conjugacy equality reads: $\mathcal{G}((\psi\circ\phi\circ\psi^{-1})^n) = \mathcal{G}(\psi\circ\phi^n\circ\psi^{-1})$. It follows that:

$$|\mathcal{G}((\psi\circ\phi\circ\psi^{-1})^n) - \mathcal{G}(\phi^n)| \le \mathcal{G}(\psi) + \mathcal{G}(\psi^{-1}).$$

Dividing by n and letting n go to $+\infty$ we get $\mathcal{L}(\psi\circ\phi\circ\psi^{-1}) = \mathcal{L}(\phi)$.

ii) The quantity $\mathcal{G}(\{\phi_t\})$ satisfies:

$$\mathcal{G}(\{\phi_t\}) \le \frac{1}{\pi}\int_0^1\int_{\mathbf{D}^2\times\mathbf{D}^2} \frac{\|\dot{\phi}_t(x)\|}{\|\phi_t(y) - \phi_t(x)\|}dxdydt.$$

The Cauchy-Schwarz inequality gives:

$$G(\{\phi_t\}) \le \frac{1}{\pi}\int_0^1 \|\dot{\phi}_t\|_2\|I_t\|_2 dt,$$

where:

$$I_t(x) = \int_{\mathbf{D}^2} \frac{dy}{\|\phi_t(y) - \phi_t(x)\|}.$$

For every t in $[0, 1]$ the map ϕ_t is area preserving; consequently the L_2-norm of I_t satisfies:

$$\|I_t\|_2 = (\int_{\mathbf{D}^2}(\int_{\mathbf{D}^2} \frac{dy}{\|x - y\|})^2 dx)^{1/2} < +\infty.$$

Thus, for $K = \frac{1}{\pi}\|I_t\|_2$ we get $\mathcal{G}(\{\phi_t\}) \le Kl_2(\{\phi_t\})$. It follows that $\mathcal{L}(\phi) \le Kd_2(Id, \phi)$.

iii) We choose an integer n, a map ϕ in $\mathcal{D}_2$ and an isotopy $\{\phi_t^{(n)}\}_{t\in[0,1]}$ connecting Identity to ϕ^n. For every pair of distinct points x and y in $\mathbf{D}^2$, the unit vector:

$$u^{(n)}(t, x, y) = \frac{\phi_t^{(n)}(y) - \phi_t^{(n)}(x)}{\|\phi_t^{(n)}(y) - \phi_t^{(n)}(x)\|}$$

can be written as $u^{(n)}(t, x, y) = \exp(2i\pi\theta(t, x, y))$. This yields:

$$\|\frac{du^{(n)}}{dt}(t, x, y)\| = 2\pi|\frac{d\theta}{dt}(t, x, y)|,$$

which gives by integration:

$$|Ang_{\phi^n}(x, y)| \le \frac{1}{2\pi}\int_0^1 \|\frac{du^{(n)}}{dt}(t, x, y)\|dt.$$

By integrating over $\mathbf{D}^2\times\mathbf{D}^2$ we get:

$$|C(\phi^n)| = n|C(\phi)| \le \mathcal{G}(\{\phi_t^{(n)}\}).$$

Since this is true for all isotopies joining Identity to ϕ^n, we have $n|C(\phi)| \le \mathcal{G}(\phi^n)$. Dividing by n and letting n go to $+\infty$ we get: $|C(\phi)| \le \mathcal{L}(\phi)$. □

The asymptotic crossing number of a map ϕ in $\mathcal{D}_2$ can be interpreted as follows. For an isotopy $\{\phi_t\}_{t\in[0,1]}$ connecting Identity to ϕ and a pair of distinct points $x,\ y$ in $\mathbf{D}^2$, we consider the map:

$$\begin{aligned} u_{x,y} : [0,1] &\rightarrow \mathbf{S}^1 \\ t &\mapsto u(t,x,y). \end{aligned}$$

The change of variables induced by the map $u_{x,y}$ leads to the equality:

$$\int_0^1 \|\frac{du_{x,y}}{dt}(t)\|dt = \int_{\mathbf{S}^1} \sharp\{u_{x,y}{}^{-1}(\omega)\}d\omega,$$

where $\sharp$ stands for cardinality. By integrating we get:

$$\mathcal{G}(\{\phi_t\}) = \frac{1}{2\pi}\int_{\mathbf{D}^2\times\mathbf{D}^2}\int_{\mathbf{S}^1} \sharp\{u_{x,y}{}^{-1}(\omega)\}d\omega dx dy.$$

For any ω in $\mathbf{S}^1$, when the arcs $\gamma_y : t \mapsto \phi_t(y)$ and $\gamma_x : t \mapsto \phi_t(x)$ are projected in the direction of ω onto a plane orthogonal to ω, γ_y overcrosses $\sharp\{u_{x,y}{}^{-1}(\omega)\}$ times γ_x. The integral:

$$Cr_{\{\phi_t\}}(x,y) = \frac{1}{2\pi}\int_{\mathbf{S}^1} \sharp\{u_{x,y}{}^{-1}(\omega)\}d\omega$$

is the averaged number of times the arc γ_y overcrosses γ_x over all directions ω. $\mathcal{G}(\{\phi_t\})$ is then the spatial average $\int_{\mathbf{D}^2\times\mathbf{D}^2} Cr_{\{\phi_t\}}(x,y)dxdy$.

Changing the isotopy $\{\phi_t\}$ may reduce the averaged quantity of overcrossings $\mathcal{G}(\{\phi_t\})$; the infimum is given by $\mathcal{G}(\phi)$.

Remark. The group $\mathcal{D}_2$ has infinite diameter. This is also the case for the kernel of the Calabi invariant (see [7]), even if in this last case a direct estimate of the distance to Identity using the Calabi invariant is useless. The estimate of the distance to Identity using the asymptotic crossing number turns to give a simple proof of this last result and is a direct consequence of the following lemma:

Lemma 3. *There exist on $Ker(C)$ maps with arbitrary large asymptotic crossing number.*

Proof. Consider on $\mathbf{D}^2$ a Hamiltonian H which vanishes near the boundary of the 2-disk and satisfies $H(-x_1,x_2) = -H(x_1,x_2)$, $H(x_1,x_2) \geq 0$ when $x_1 \geq 0$ and $H(x_1,x_2) = 0$ when x_1 is close to 0. The "time 1" map ϕ of the corresponding Hamiltonian flow is in $Ker(C)$ (indeed $C(\phi) = 2\int_{\mathbf{D}^2} H(x)dx = 0$). Let $\mathbf{D}^2_0$ and $\mathbf{D}^2_1$ be the two 1/2-disks of $\mathbf{D}^2$ corresponding respectively to $x_1 \leq 0$ and $x_1 \geq 0$.

The Hamiltonian H can be written as the sum of two Hamiltonians H_0 and H_1 with supports respectively in $\mathbf{D}_0^2$ and $\mathbf{D}_1^2$. We denote respectively by $\phi_{(0),t}$ and $\phi_{(1),t}$ the "time t" maps of H_0 and H_1. Let $\{\phi_t\}_{t\in[0,1]}$ be an isotopy connecting Identity to ϕ. We have:

$$\begin{aligned}
\mathcal{G}(\{\phi_t\}) &= \frac{1}{2\pi}\int_0^1\int_{\mathbf{D}^2\times\mathbf{D}^2}\|\frac{du}{dt}(t,x,y)\|dxdydt \\
&\geq \frac{1}{2\pi}\int_0^1\int_{\mathbf{D}_0^2\times\mathbf{D}_0^2}\|\frac{du}{dt}(t,x,y)\|dxdydt \; + \\
&\qquad + \frac{1}{2\pi}\int_0^1\int_{\mathbf{D}_1^2\times\mathbf{D}_1^2}\|\frac{du}{dt}(t,x,y)\|dxdydt \\
&\geq |\int_{\mathbf{D}_0^2\times\mathbf{D}_0^2} Ang_\phi(x,y)dxdy| + |\int_{\mathbf{D}_1^2\times\mathbf{D}_1^2} Ang_\phi(x,y)dxdy|
\end{aligned}$$

Let us recall that $Ang_\phi(x,y)$ stands for the angular variation of the vector $\overrightarrow{\eta_t(x)\eta_t(y)}$ where $\{\eta_t\}_{t\in[0,1]}$ is an isotopy connecting Indentity to ϕ and that it does not depend on the choice of the isotopy. If x and y are in $\mathbf{D}_i^2$ then $Ang_\phi(x,y)$ can be computed using only $\{\phi_{(i),t}\}_{t\in[0,1]}$.

Thus:

$$|\int_{\mathbf{D}_i^2\times\mathbf{D}_i^2} Ang_\phi(x,y)dxdy| = |\int_{\mathbf{D}_i^2\times\mathbf{D}_i^2} Ang_{\phi_{(i),1}}(x,y)dxdy|.$$

Hence:

$$|\int_{\mathbf{D}_i^2\times\mathbf{D}_i^2} Ang_\phi(x,y)dxdy| \geq |\int_{\mathbf{D}^2\times\mathbf{D}^2} Ang_{\phi_{(i),1}}(x,y)dxdy| - 2I,$$

where

$$I = |\int_{\mathbf{D_0}^2\times\mathbf{D_1}^2} Ang_{\phi_{(i),1}}(x,y)dxdy| \leq \pi.$$

Then, we get:

$$\begin{aligned}
\mathcal{G}(\{\phi_t\}) &\geq |C(\phi_{(0),1})| + |C(\phi_{(1),1})| - 4\pi \\
&= 2(\int_{\mathbf{D}^2}|H_0(x)|dx + \int_{\mathbf{D}^2}|H_1(x)|dx) - 4\pi.
\end{aligned}$$

Consequently:

$$\mathcal{G}(\phi) \geq 2\int_{\mathbf{D}^2}|H(x)|dx - 4\pi.$$

Using the linearity of the Calabi invariant, the same calculation for the "time n" map ϕ^n of the Hamiltonian H leads to:

$$\mathcal{G}(\phi^n) \geq 2n \int_{\mathbf{D}^2} |H(x)| dx \; - \; 4\pi. \qquad \square$$

4 Renormalizable maps

A map ϕ in $\mathcal{D}_2$ is *renormalizable* if there exist a disk Δ in $\mathbf{D}^2$ and an integer $n > 0$ such that:

- for $0 < i < n$, $\phi^i(\Delta) \cap \Delta = \emptyset$;
- $\phi^n(\Delta) = \Delta$.

In this section we focus on this particular type of maps which play a central role in Dynamics. We derive, in this case, an estimate from below of the asymptotic crossing number which depends only on the area of the disk Δ and on a topological invariant associated to the way the n first images of the disk Δ are permuted. In order to give a more precise statement, let us recall some notations, definitions and results concerning braids.

The *Artin Braid group* B_n is a group given by the set of generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and the relations:

$$\sigma_i \sigma_j \; = \; \sigma_j \sigma_i,$$

$$\sigma_i \sigma_{i+1} \sigma_i \; = \; \sigma_{i+1} \sigma_i \sigma_{i+1};$$

for all i, j in $\{1, \ldots, n\}$ with $|i - j| \geq 2$. An element of this group is called *a braid.*

A geometrical way to represent braids (see for instance [11]) consists in fixing the $2n$ points $P_i = (i/(n-1) - 1/2, 0, 1)$ and $Q_i = (i/(n-1) - 1/2, 0, 0)$ for $i = 0, \ldots, n-1$ in the solid cylinder $\mathbf{D}^2 \times [0, 1]$. A braid β can be seen as the isotopy class of a system of n non-intersecting arcs, joining each point P_i to a point $Q_{\tau_\beta(i)}$, where τ_β is a permutation on $\{0, \ldots, n-1\}$, and such that the intersection of any of these arcs with any disk $\mathbf{D}^2 \times \{t\}$, t in $[0, 1]$, consists in a unique point. With this representation, it turns easy to understand what the group law, the generators and the relations mean (see for instance [11]).

The *crossing number* of a braid β in B_n is the minimal number $cr(\beta)$ of generators σ_i and their inverse that can be used to write β. Two braids β and β' in B_n are *conjugated* if there exists a third braid γ in B_n such that $\beta' = \gamma\beta\gamma^{-1}$. The conjugacy class $\hat{\beta}$ of a braid β is called a *closed braid.* The *crossing number* $cr(\hat{\beta})$ of a closed braid is the minimal crossing number among all braids in the conjugacy class $\hat{\beta}$. Closed braids (that are the conjugacy classes in B_n) possess also a geometrical interpretation. Consider the standard revolution solitorus $\mathbf{D}^2 \times \mathbf{S}^1$ embedded in $\mathbb{R}^3$; a closed braid can be seen as the isotopy class of a collection of curves in the solitorus which intersect the disk $\mathbf{D}^2 \times \{\theta\}$ transversally in exactly n points for all θ in $\mathbf{S}^1$. It is plain that any orthogonal projection of a collection of curves representing a closed braid $\hat{\beta}$ possesses at least $cr(\hat{\beta})$ double points and that there exists a representative and an orthogonal projection with exactly $cr(\hat{\beta})$ double points. This last point will be used in the proof of Lemma 4.

Finally, the *asymptotic crossing number* $ac(\hat{\beta})$ of a closed braid $\hat{\beta}$ is the limit:

$$ac(\hat{\beta}) = \lim_{n \to +\infty} \frac{1}{n} cr(\widehat{\beta^n}).$$

It is easy to check that this last definition does not depend on the element β of the conjugacy class $\hat{\beta}$ and that, for any integer n in $\mathbf{Z}$, we have the inequality:

$$cr(\widehat{\beta^n}) \leq |n| cr(\hat{\beta}),$$

which insures that our limit exists and satisfies: $ac(\hat{\beta}) \leq cr(\hat{\beta})$. It does not seem to be known whether this inequality can be strict. In consequence, it is interesting to find a lower bound for the asymptotic crossing number $ac(\hat{\beta})$ which can be computed directly from β without looking at the whole conjugacy class and at the asymptotic behavior.

This can be done as follows. There is a canonical morphism from the Artin braid group B_n onto the symmetric group Σ_n which maps any braid β on the permutation τ_β. The kernel of this morphism is called the *pure braid group* and denoted by P_n. For any pair of distinct integers i, j in $\{0, \ldots, n-1\}$ we consider the forgetful map $f_{i,j} : P_n \to P_2$ which associates to any pure braid β in P_n the pure braid obtained by keeping only the strands i and j of a system of arcs representing β. The braid group B_2 is the free group generated with a single element σ. For any braid β in B_2, $cr(\beta)$ is the absolute value of the integer n in $\mathbf{Z}$ such that $\beta = \sigma^n$. Notice that $cr(\beta)$ does not depend on the choice of the generator σ. Then, to any pure braid β in P_n we can associate the quantity:

$$cl(\hat{\beta}) = \sum_{i \neq j} cr(f_{i,j}(\beta)),$$

which does not depend on the element chosen in the conjugacy class of β and satisfies $cl(\widehat{\beta^m}) = |m| cl(\hat{\beta})$ for any integer m in $\mathbf{Z}$. For a braid β in B_n, we get:

$$ac(\hat{\beta}) \geq \frac{1}{p(\hat{\beta})} cl(\widehat{\beta^{p(\hat{\beta})}}),$$

where $p(\hat{\beta})$ stands for the smallest positive integer p such that $\hat{\beta}^p$ is a pure braid class.

There is a third way to interpret the Artin braid group which is related to Dynamics. Consider the n-punctured disk $\mathbf{D}_n$ construct by taking away from the 2-disk $\mathbf{D}^2$ n non intersecting disks B_i centered around the points with coordinates $(i/(n-1)-1/2, 0)$ for $i = 0, \ldots, n-1$). J. Birman [4] proved that the Artin Braid group is isomorphic to the subgroup of automorphisms of the fundamental group $\pi_1(\mathbf{D}_n)$ deriving from homeomorphisms of $\mathbf{D}_{n+1}$ which are Identity near the boundary of the 2-disk.

Consider now a diffeomorphism ϕ in $\mathcal{D}_2$ which is renormalizable and let Δ be a disk in $\mathbf{D}^2$ and $n > 0$ an integer such that:

- for $0 < i < n$, $\phi^i(\Delta) \cap \Delta = \emptyset$;
- $\phi^n(\Delta) = \Delta$.

Let h be a homeomorphism of $\mathbf{D}^2$ which is Identity near the boundary of $\mathbf{D}^2$ and such that, for each i in $\{0, \ldots, n\}$, $h(\phi^i(\Delta)) = B_i$. The map $\psi = h \circ \phi \circ h^{-1}$ leaves the n disks B_i globally invariant. Thus, thanks to J. Birman's isomorphism, we can associate to ψ a braid $\gamma(\phi, h)$. Another choice of homeomorphism h' leads to a braid $\gamma(\phi, h')$ which is conjugated to $\gamma(\phi, h)$. It follows that we have described a way to associate to any renormalizable diffeomorphism ϕ, a closed braid $\hat{\gamma}(\phi)$. Notice also that the closed braid $\hat{\gamma}(\phi)$ depends only on the conjugacy class of ϕ in $\mathcal{D}_2$.

Theorem 2. $\mathcal{L}(\phi) \geq ac(\hat{\gamma}(\phi))(area(\Delta))^2$.

Proof. Consider an isotopy $\{\phi_t\}_{t\in[0,1]}$ in $\mathcal{D}_2$ connecting Identity to ϕ. The integral $\mathcal{G}(\{\phi_t\})$ reads (see Section 3):

$$\mathcal{G}(\{\phi_t\}) = \int_{\mathbf{D}^2\times\mathbf{D}^2} Cr_{\{\phi_t\}}(x, y)dxdy,$$

and consequently:

$$\begin{aligned}\mathcal{G}(\{\phi_t\}) &\geq \sum_{i\neq j} \int_{\phi^i(\Delta)\times\phi^j(\Delta)} Cr_{\{\phi_t\}}(x, y)dxdy \\ &= \int_{\Delta\times\Delta} (\sum_{i\neq j} Cr_{\{\phi_t\}}(\phi^i(x), \phi^j(y)))dxdy.\end{aligned}$$

For $(x, y, \omega) \in \Delta \times \Delta \times \mathbf{S}^1$, let $\mathcal{U}_\phi(x, y, \omega)$ be the minimum over all isotopies in $\mathcal{D}_2$ connecting Identity to ϕ of the quantity:

$$\sum_{i \neq j} \sharp\{u^{-1}_{\phi^i(x),\phi^j(y)}(\omega)\},$$

and consider:

$$\mathcal{T}_\phi = \min_{(x,y)\in\Delta\times\Delta} \min_{\omega\in\mathbf{S}^1} \mathcal{U}_\phi(x, y, \omega).$$

It is clear that: $\mathcal{G}(\phi) \geq \mathcal{T}_\phi(area(\Delta))^2$.

Lemma 4. *There exists a constant C which depends only on the way the n disks* $\Delta, \phi(\Delta) \ldots, \phi^{n-1}(\Delta)$ *are embedded in* $\mathbf{D}^2$ *such that:*

$$\mathcal{T}_\phi \geq cr(\hat{\gamma}(\phi)) - C$$

Proof of Lemma 4. Let $x_0, \ldots, x_{n-1}$ be n points respectively in $\Delta, \ldots, \phi^{n-1}(\Delta)$ and $\{\phi_t\}_{t\in[0,1]}$ be an isotopy in $\mathcal{D}_2$ connecting Identity to ϕ. In the cylinder $\mathbf{D}^2 \times [0, 1]$, we consider the system of arcs:

$$S = \{(\phi_t(x_i), t) \mid t \in [0, 1],\ i \in \{0, \ldots, n-1\}\}.$$

We perform now a surgery on this system of arcs which consists in adding to each arc $\{(\phi_t(x_i), t) \mid t \in [0, 1], \}$ an arc in $\phi^{i+1 mod n}(\Delta)$ connecting $(\phi(x_i), 1)$ to $(x_{i+1 mod\, n}, 1)$. This yields a system of n arcs S' in $\mathbf{D}^2 \times [0, 1]$ connecting the point $(x_i, 0)$ to the point $(x_{i+1}, 1)$ for $i = 0, \ldots, n-1$ and the point $(x_n, 0)$ to the point $(x_0, 1)$. Identifying the top and the bottom of the cylinder, transforms the system S' in a simple closed curve in the solid torus whose isotopy class corresponds the closed braid $\hat{\gamma}(\phi)$. It follows that any generic orthogonal projection of S' parallel to a direction $\omega \in \mathbf{S}^1$ yields a system of arcs which intersect at least in $cr(\hat{\gamma}(\phi))$ points. The number of extra crossing points forced by the surgery is bounded by a constant C which depends only on the way the n disks $\Delta, \phi(\Delta) \ldots, \phi^{n-1}(\Delta)$ are embedded in $\mathbf{D}^2$, and consequently the number of crossing points of the projection of S is at least $cr(\hat{\gamma}(\phi)) - C$. This yields the inequality:

$$\sum_{i \neq j} \sharp\{u^{-1}_{\phi^i(x),\phi^j(y)}(\omega)\} \geq cr(\hat{\gamma}(\phi)) - C$$

and achieves the proof of lemma 4.

Since any iterate ϕ^q (with q and n coprime) possesses the same set of invariant disks as ϕ, we obtain by replacing in Lemma 4 ϕ by ϕ^q:

$$\mathcal{G}(\phi^q) \geq (cr(\hat{\gamma^q}(\phi)) - C)(area(\Delta))^2.$$

Dividing by q and letting q go to $+\infty$ yields the theorem. □

We say that a map ϕ in $\mathcal{D}_2$ is *infinitely renormalizable* if there exist a sequence of nested disks $\mathbf{D}^2 \supset \Delta_1 \supset \Delta_2 \supset \Delta_3 \supset \cdots \supset \Delta_m \ldots$ and a sequence of integers $(a_n)_{n>0}$ with $a_n > 1$ such that:

- for $0 < i < q_m$, $\phi^i(\Delta_m) \cap \Delta_m = \emptyset$;
- $\phi^{q_m}(\Delta_m) = \Delta_m$,

where $q_1 = a_1$ and $q_m = a_m q_{m-1}$ for $m > 1$.

Area preserving infinitely renormalizable maps appear naturally in Dynamics, for instance when perturbing the "time t" map of a Hamiltonian flow.

Let $(\hat{\gamma}_m(\phi))_{m>0}$ be the sequence of closed braids associated to a infinitely renormalizable map by using J. Birman's identification. Theorem 2 gives:

$$Kd_2(Id, \phi) \geq \mathcal{L}(\phi) \geq cr(\hat{\gamma}_m(\phi))(area(\Delta_m))^2.$$

It follows that for an infinitely renormalizable map in $\mathcal{D}_2$ which is at bounded distance from Identity, there is a relation between the way the area of the nested disks $(\Delta_m)_{m>0}$ goes to zero and the complexity of the sequence of closed braids $(\hat{\gamma}_m)_{m>0}$.

5 Suspension of maps of the 2-disk

Let ϕ be a map in $\mathcal{D}_2$ and $\{\phi_t\}$ an isotopy from Identity to ϕ and consider the solid torus $\mathbb{T} = \mathbf{D}^2 \times \mathbb{R}/\mathbb{Z}$. The volume form $area \wedge dt$ on $\mathbf{D}^2 \times \mathbb{R}$ induces a volume form vol on the solid torus. The vector field in $\mathbf{D}^2 \times \mathbb{R}$ with coordinates $(\frac{\partial \phi_t}{\partial t}(x), 1)$ over any point (x, t) yields a vector field $X_{\{\phi_t\}}$ in $\mathbb{T}$ whose flow preserves the volume form vol: $i_{X_\phi} vol = 0$. It is tangent to the boundary $\partial\mathbb{T} \approx \mathbf{S}^1 \times \mathbf{S}^1$ which is foliated by periodic orbits with length 1. Consider now the embedding i mapping $\mathbb{T}$ onto $\mathcal{T}$, the standard solitorus of revolution in $\mathbb{R}^3$:

$$\begin{aligned} i : \mathbf{D}^2 \times \mathbb{R}/\mathbb{Z} &\to \mathbb{R}^3 \\ (r \exp 2i\pi\theta, t) &\mapsto ((R + r\cos 2\pi\theta)\cos 2\pi t, (R + r\cos 2\pi\theta)\sin 2\pi t, r \sin 2\pi\theta), \end{aligned}$$

where the constant R is chosen so that the pullback by i of the standard volume form in $\mathbb{R}^3$ coincides with the 3-form vol. The above construction allows us to associate to any map ϕ in $\mathcal{D}_2$ a divergence-free vector field $X(\{\phi_t\})$ in the solitorus $\mathcal{T}$ tangent to the boundary $\partial\mathcal{T}$.

In [1] V. Arnold studies the more general case of divergence-free vector fields in a domain $\mathcal{D}$ of $\mathbb{R}^3$ with smooth boundary $\partial\mathcal{D}$ which are tangent to this boundary. He finds a lower bound for the energy of such vector fields which can be interpreted in terms of linking of the orbits. More precisely, let X be a divergence-free vector field in $\mathcal{D}$ tangent to the boundary and ϕ_t the flow associated to this vector field. To any point x in $\mathcal{D}$ and any time T we consider the closed curve $k(x, T)$ constituted of the arc of orbit joining x to $\phi_T(x)$ followed by the arc of geodesic joining $\phi_T(x)$ to x. V. Arnold proved the following:

- for almost every pair of points (x, y) and almost every time T_1 and T_2, the two closed curves $k(x, T_1)$ and $k(y, T_2)$ are knots and we denote by $l(x, y, T_1, T_2)$ their linking number;
- for almost every pair of points (x, y) the limit when T_1 and T_2 go to $+\infty$ of the quantity $\frac{1}{T_1 T_2} l(x, y, T_1, T_2)$ exits and is denoted by $l_X(x, y)$;
- the map $(x, y) \mapsto l_X(x, y)$ is integrable and the integral

$$\mathcal{A}_X = \int_{\mathcal{D}\times\mathcal{D}} l_X(x, y) dx dy$$

  is called the *Asymptotic Hopf invariant* of the vector field;
- the energy of a vector field is bounded from below by its asymptotic Hopf invariant, i.e. there exists a positive constant C' which depends only on the geometry of the domain $\mathcal{D}$ such that for any divergence-free vector field X in $\mathcal{D}$ tangent to the boundary:

$$\|X\|_2^2 \geq C'|\mathcal{A}_X|; \qquad (*)$$

- the asymptotic Hopf invariant is invariant under volume preserving conjugacy, i.e. for any volume preserving diffeomorphism ψ on $\mathcal{D}$ and for any divergence-free vector field X in $\mathcal{D}$ tangent to the boundary:

$$\mathcal{A}_{\psi_\star X} = \mathcal{A}_X,$$

  where $\psi_\star X(x) = d\psi(X(\psi^{-1}(x))$ for any x in $\mathcal{D}$.

This last point is particularly important. Indeed, the energy of the vector field is not invariant under volume preserving change of variables and thus V. Arnold's result gives a lower bound of the energy among all volume preserving conjugacy of the vector field.

In the particular case of a divergence-free vector field $X(\{\phi_t\})$ in the solitorus $\mathcal{T}$ tangent to the boundary $\partial\mathcal{T}$ induced by a map ϕ in $\mathcal{D}_2$ and an isotopy $\{\phi_t\}$ joining Identity to ϕ a simple calculation gives:

$$\|X(\{\phi_t\})\|_2^2 = E_0 + \int_0^1 \|\frac{d\phi_t}{dt}\|_2^2 dt \geq E_0 + (l_2(\{\phi_t\}))^2 \geq E_0 + (d_2(Id, \phi))^2,$$

where E_0 is the energy of the vector field induced by the Identity in $\mathcal{D}_2$ and the constant isotopy from Identity to Identity. On the other side, it is shown in [10] that the asymptotic Hopf invariant $\mathcal{A}_{X(\{\phi_t\})}$ does not depend on the isotopy $\{\phi_t\}$ but only on the map ϕ and that we actually have: $\mathcal{A}_{X(\{\phi_t\})} = C(\phi)$. It follows that the estimate $(\star)$ can be improved in our particular case in:

$$\|X(\{\phi_t\})\|_2^2 \geq E_0 + (\frac{2}{C}C(\phi))^2, \qquad (**)$$

a lower bound that grows at least quadratically in term of asymptotic Hopf invariant.

In [9] M. Freedmann and Z.-X. He improved the Arnold's result by considering the crossing number of two piecewise differentiable closed oriented curves $k_1 : \mathbf{S}^1 \to \mathbb{R}^3$ and $k_2 : \mathbf{S}^1 \to \mathbb{R}^3$ parameterized by an angular value in $[0, 1]$. This crossing number is the quantity:

$$cr(k_1, k_2) = \int_0^1 \int_0^1 \frac{|(dk_1(t)/dt, dk_2(s)/ds, k_1(t) - k_2(s))|}{\|k_1(t) - k_2(s)\|^3} dtds.$$

It differs from the linking number of the two curves k_1 and k_2 by the absolute value in the numerator, and unlike the linking number it is not a topological invariant. However it can be interpreted geometrically as follows:

- choose a direction u in $\mathbb{R}^3$, i.e. a point on the 2-sphere $\mathbf{S}^2$, and project k_1 and k_2 following that direction onto a plane orthogonal to u;
- count the number $cr(k_1, k_2, u)$ of times the projection of k_1 overcrosses the projection of k_2;
- the crossing number is then the average over all direction u of the overcrossing number $cr(k_1, k_2, u)$.

M. Freedmann and Z.-X. He proved the following:

- for almost every pair of points (x, y) the limit when T_1 and T_2 go to $+\infty$ of the quantity $\frac{1}{T_1 T_2} cr(x, y, T_1, T_2)$ exits and is denoted by $cr_X(x, y)$;
- the map $(x, y) \mapsto cr_X(x, y)$ is integrable; the integral

  $$\mathcal{K}_X = \int_{\mathcal{D}\times\mathcal{D}} cr_X(x, y) dxdy$$

  is called the *asymptotic crossing number* of the vector field;

- for any divergence-free vector field tangent in $\mathcal{D}$ tangent to the boundary, we have $\mathcal{K}_X \geq |\mathcal{A}_X|$;

- the energy of a vector field is bounded from below by its asymptotic crossing number, i.e. there exists a positive constant C' which depends only on the geometry of the domain $\mathcal{D}$ such that for any divergence-free vector field X in $\mathcal{D}$ tangent to the boundary:

$$\|X\|_2^2 \geq C'\mathcal{K}_X; \qquad (*')$$

- the asymptotic crossing number is invariant under volume preserving conjugacy.

The advantage of the asymptotic crossing number on the asymptotic Hopf invariant is that the quantity $cr_X(x, y)$ is positive when defined, consequently the asymptotic crossing number of the vector field is always greater than or equal to the asymptotic crossing number computed on any subset of $\mathcal{D}$ invariant under the action of the flow.

In the particular case of a divergence-free vector field $X(\{\phi_t\})$ in the solitorus $\mathcal{T}$ tangent to the boundary $\partial\mathcal{T}$ induced by a map ϕ in $\mathcal{D}_2$ and an isotopy $\{\phi_t\}$ joining Identity to ϕ, the estimate $(*')$ can be improved in:

$$\|X(\{\phi_t\})\|_2^2 \geq E_0 + (\frac{2}{C}\mathcal{L}(\phi))^2, \qquad (*'*')$$

a lower bound that grows at least quadratically in term of asymptotic crossing number of the map ϕ. However, unlike the equality $\mathcal{A}_{X(\{\phi_t\})} = C(\phi)$, there is apriori no way to compare the two numbers $\mathcal{K}_{X(\{\phi_t\})}$ and $\mathcal{L}(\phi)$. The reason is that the quantity $Cr_{\{\phi_t\}}(x, y)$ estimated in section 3 is an average of overcrossings over all directions in $\mathbf{S}^1$ and the quantity $Cr_{X(\{\phi_t\})}(x, y)$ is an average of overcrossings over all directions in $\mathbf{S}^2$.

At this point it is worth noticing that the fact that the energy of a divergence-free vector field grows at least like the square of some average of the number of crossing points of projection of orbits has been already pointed out by M. A. Berger [3] who cleverly derived such estimates in the special case of a magnetic field in a cylinder.

Finally M. Freedmann and Z.-X. He also give a minimization of the asymptotic crossing number of a divergence-free vector field which possesses a knotted invariant tube. For this purpose, they introduce a knot invariant called the *asymptotic crossing number of the knot* which is smaller or equal to the minimal crossing number of the knot and conjecturally equal. In the particular case of a divergence-free vector field $X(\{\phi_t\})$ in the solitorus which is the suspension of a renormalizable diffeomorphism of the 2-disk, our Theorem 2 gives a similar estimate. However, in our particular case, the estimate we get is better in the following sense. We introduced a quantity defined for closed braids: the asymptotic crossing number of a closed braid, which is smaller than or equal to the minimal crossing number of the closed braid and conjecturally equal. The minimal crossing number of a closed braid can be of course strictly bigger than the minimal crossing number of the corresponding knot; it is also easy to show examples where the asymptotic crossing number of a closed braid is strictly bigger that the asymptotic crossing number of the corresponding knot.

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