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Centralizers of finite Blaschke products

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Abstract. In this article we consider the set of finite Blaschke products F of degree $n > 1$. We give sufficient conditions for F to commute only with their own powers among all rational functions of the Riemann sphere, in terms of the derivate around the fixed points.

Keywords: Blaschke products, centralizers, rational functions.

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1. Introduction

A finite Blaschke product F of degree n is a rational function of the Riemann sphere $\bar{\mathbb{C}}$ defined by the equation

$$F(z) = a_0 \prod_{i=1}^{n} \frac{z - \bar{a}_i}{1 - a_i z}, \quad n > 1, \quad |a_0| = 1, \quad |a_i| < 1, \quad for \quad i > 0.$$

Notice that F/S^1 is a n-to-1 endomorphism on the circle S^1, which is determined, up to a rotation, by the zeros $\bar{a}_1, ..., \bar{a}_n$. Moreover if a continuos surjective endomorphism of S^1 has a holomorphic extension in the open unit disk, then it can be realized as the restriction on S^1 of a finite Blaschke product.

For a finite Blaschke product F of degree $n > 1$, its centralizer $Z(F)$ is defined as the set of rational functions of $\bar{\mathbb{C}}$ that commute with F. We say that F has trivial centralizer if $Z(F)$ is reduced to the iterates $\{F^k, k \in \mathbb{N}\}$ of F.

The purpose of this paper is to investigate wheather the finite Blaschke products have trivial centralizers. The problem of finding conditions under which two given rational maps commute was investigated and resolved by Fatou, Julia, and

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Ritt ([Fa], [Ju], [Ri]). Here we give conditions for a finite Blaschke product to have trivial centralizer in terms of the derivates at the fixed points on S^1.

Theorem A. *Let F be a finite Blaschke product of degree $n > 3$ such that $F'(x) \neq F'(y)$ for all different fixed points x, y of F on S^1. Then the centralizer of F is trivial.*

Theorem A is false if degree $F = 2$ or 3. Consider

$$F_1(z) = z\left(\frac{z-a}{1-az}\right), \qquad F_2(z) = z^2\left(\frac{z-a}{1-az}\right), \quad a \in \mathbb{R}, 0 < |a| < 1$$

F_1 satisfies the hypothesis of Theorem A because 1 is the unique fixed point of F_1 on S^1. Also F_2 satisfies the hypothesis of Theorem A because 1 and -1 are the fixed points of F_2 on S^1 and

$$F_2'(1) = 2 + \frac{1+a}{1-a} \neq 2 + \frac{1-a}{1+a} = F_2'(-1)$$

It is easy to check that for $i = 1, 2$, the rational function defined by $G_i(z) = \frac{1}{F_i(z)}$ commutes with $F_i(z)$, however it is not a power of F_i.

For finite Blaschke products of degrees 2 and 3, we have the following results:

Theorem B. *Let $F = a_0\prod_{i=1}^{3}\frac{z-\bar{a}_i}{1-a_i z}$ be a finite Blaschke product of degree 3 such that $a_0 \neq 1$, and $F'(x) \neq F'(y)$, for all different fixed points x, y of F on S^1. Then $Z(F)$ is trivial.*

Theorem C. *Let F be a finite Blaschke product of degree 2 satisfying the following conditions:*

(i) *$a_0 \neq 1, -1$;*

(ii) *$F'(x) \neq F'(y)$ for all different fixed points x, y of F on S^1;*

(iii) *There exists a fixed point x of F on S^1 such that $F'(x) \neq 2$.*

Then $Z(F)$ is trivial.

The rational functions F_1 and F_2 considered above show that we cannot take $a_0 = 1$ in Theorems B and C. Theorem C is false without hypothesis (iii). Consider $F(z) = -iz^2$, which satisfies hypothesis (i) and (ii) of Theorem C, commutes with $G(z) = z^5$, however G is not a power of F.

Theorems A, B, and C generalize a previous work of the author of this paper [Ar]. In that paper we consider the set of finite Blaschke products F for which the restriction to S^1 are expanding and we give conditions for F to commute only with their own powers among all finite Blaschke products.

In the proof of the Theorems we use some results about the dynamic of finite Blaschke products F which makes possible to reduce the proof of the Theorems to the proof of similar results for the restriction of F to its Julia set.

2. Preliminary results

Recall that an endomorphism $f : S^1 \to S^1$ of the circle S^1 is an immersion if $f'(x) \neq 0$ for all $x \in S^1$. It follows from basic dynamical properties of immersions that given an immersion f of S^1 of degree $n \neq 1$, there exists a continuos map $h : S^1 \to S^1$ of degree 1 such that

$$h \circ f = f_n \circ h,$$

where $f_n : S^1 \to S^1$ is defined by $f_n(z) = z^n$.

Moreover h is monotone, i.e. for every $z \in S^1$, $h^{-1}(\{z\})$ is either a unique point or an interval $[a, b]$ with $a \neq b$. In the last case (a, b) is called a plateau of f. Denote $J(f)$ the complement of the union of the plateaux of f. Using the map h and basic properties of $J(f)$ (see [Ma]) it is easy to check the following propertie

Lemma 2.1. *Let $f : S^1 \to S^1$ be an immersion of degree $n \neq 1$, and let z_0 be a fixed point of f. Then there exists an unique monotone continuous map $h : S^1 \to S^1$ of degree 1 such that $h(z_0) = 1$ and $h \circ f = f_n \circ h$.*

As usual we say that an endomorphism $f : S^1 \to S^1$ is expanding if there exist $k > 0$ and $\lambda > 1$ such that

$$|(f^n)'(x)| > k\lambda^n,$$

for all $x \in S^1, n \in \mathbb{N}$.

The following lemmas relates a finite Blaschke product and its restriction to S^1. The first one is proved in $[Mar]$.

Lemma 2.2. *A finite Blaschke product F of degree $n > 1$ determines an expanding endomorphism of S^1 iff F has an unique fixed point in the open unit disc $\Delta = \{|z| < 1\}$.*

Lemma 2.3. *A finite Blaschke product F of degree $n > 1$ determines an orientation-preserving immersion on S^1.*

Proof. Let F be a finite Blaschke product defined by

$$F(z) = a_0 \prod_{i=1}^{n} \frac{z - \bar{a}_i}{1 - a_i z}, \quad n > 1, \quad |a_0| = 1, \quad |a_i| < 1 \quad \text{for} \quad i > 0,$$

Let $f : \mathbb{R} \to \mathbb{R}$ be the lift of the restriction of F on S^1. From the relation

$$F(e^{i\theta}) = e^{if(\theta)}$$

we deduce that

$$DF(e^{i\theta})(sin\theta, cos\theta) = f'(\theta)(sinf(\theta), cosf(\theta)),$$

where $DF(e^{i\theta})$ denotes the Jacobian matrix of F at $(sin\theta, cos\theta)$. Then by the Cauchy - Riemann equations we have that

$$|f'(\theta)| = |F'(e^{i\theta})|,$$

for all $\theta \in \mathbb{R}$. This and the fact that for $z \in S^1$,

$$\frac{zF'(z)}{F(z)} = \sum_{i=1}^{n} \frac{1 - |a_i|^2}{|z - a_i|^2}$$

imply that

$$|f'(\theta)| = \sum_{i=1}^{n} \frac{1 - |a_i|^2}{|e^{i\theta} - a_i|^2} \neq 0,$$

so F/S^1 is an immersion. Moreover, since F has a fixed point on S^1, we have that if $F(e^{i\theta}) = e^{i\theta}$ for some $\theta \in \mathbb{R}$, then

$$F'(e^{i\theta}) = \sum_{i=1}^{n} \frac{1 - |a_i|^2}{|e^{i\theta} - a_i|^2},$$

so $F'(e^{i\theta})$ is a real number and by the Cauchy - Riemann equations we deduce that

$$f'(\theta).(sin\theta, cos\theta) = DF(e^{i\theta})(sin\theta, cos\theta) = \sum_{i=1}^{n} \frac{1 - |a_i|^2}{|e^{i\theta} - a_i|^2}(sin\theta, cos\theta),$$

thus $f'(\theta) > 0$. Since f is strictly monotone, we conclude that f is an orientation-preserving immersion of S^1, and Lemma 2.3 is proved.

Recall that the Fatou set of a rational map R is defined to be the set of points $z \in \bar{\mathbb{C}}$ such that $\{R^n\}$ is a normal family in some neighborhood of z. The Julia set $\mathcal{J}$ of R is the complement of the Fatou set. It follows from basic complex dynamic that the Julia set of a finite Blaschke product of degree $n > 1$ is either the unit circle S^1 or a Cantor set on S^1 (see [CG], pg 58).

3. Proof of the Theorems

Let

$$F(z) = a_0 \prod_{i=1}^{n} \frac{z - \bar{a}_i}{1 - a_i z}$$

be a finite Blaschke product of degree $n > 1$. The following results are key steps in the proof of the Theorems.

Lemma 3.1. *The rational function $\frac{1}{F(z)}$ commutes with F iff $a_0 = 1$ or -1, and for each $i = 1, ..., n$; a_i and $\bar{a}_i$ are zeros of F of the same order.*

Proof. Let $G(z) = \frac{1}{F(z)}$. Then

$$G(F(z)) = \frac{1}{a_0} \prod_{i=1}^{n} \frac{1 - a_i F(z)}{F(z) - \bar{a}_i}, \qquad F(G(z)) = a_0 \prod_{i=1}^{n} \frac{1 - \bar{a}_i F(z)}{F(z) - a_i}.$$

First suppose that G commutes with F. For each $j = 1, ...n$, we choose z_j such that $F(z_j) = a_j$. Then

$$\frac{1}{a_0} \prod_{i=1}^{n} \frac{1 - a_i a_j}{a_j - \bar{a}_i} = a_0 \prod_{i=1}^{n} \frac{1 - \bar{a}_i a_j}{a_j - a_i} = \infty.$$

This implies that $a_j = \bar{a}_i$ for some $i = 1, ...n$, so a_j is a zero of $F(z)$. Moreover

$$\frac{1 - a_j F(z)}{F(z) - \bar{a}_j} \quad \text{and} \quad \frac{1 - \bar{a}_j F(z)}{F(z) - \bar{a}_i}$$

are common factors of $G \circ F(z)$ and $F \circ G(z)$. It follows that a_j and $\bar{a}_j$ are zeros of $F(z)$ of the same order, and $\frac{1}{a_0} = a_0$, so $a_0 = 1$ or -1.

For the converse, we have that the hypothesis imply that

$$\frac{1 - a_i F(z)}{F(z) - \bar{a}_i} \quad \text{and} \quad \frac{1 - \bar{a}_i F(z)}{F(z) - a_i}$$

are common factors of $F \circ G(z)$ and $G \circ F(z)$ with the same multiplicity. This and the fact that $a_0 = 1$ or -1 imply that $F \circ G(z) = G \circ F(z)$ and Lemma 3.1 is proved.

Corollary 3.2. *Let $n =$ degree F.*

(a) *If $n = 3$ and F satisfies the hypothesis of Theorem B, then $\frac{1}{F(z)} \notin Z(F)$.*

(b) *If $n = 2$ and F satisfies the hypothesis of Theorem C, then $\frac{1}{F(z)} \notin Z(F)$.*

Proof. (a) Suppose by contradiction that $\frac{1}{F(z)}$ commutes with $F(z)$. Then by Lemma 3.1,

$$F(z) = -\frac{z-\bar{a}_1}{1-a_1 z}\frac{z-a_1}{1-\bar{a}_1 z}\frac{z-a_3}{1-a_3 z},$$

where $a_3 \in \mathbb{R}$. It follows that $F(1) = 1$ and $F(-1) = 1$. Since $n = 3$, F has at least two fixed points on S^1, so there exists a fixed pont $z_0 \neq 1, -1$ of F on S^1. Then $\bar{z_0}$ is a fixed point of F because

$$F(\bar{z_0}) = F(\frac{1}{z_0}) = \frac{1}{F(z_o)} = \frac{1}{z_0} = \bar{z_0}.$$

This and the fact that for $z \in S^1$,

$$\frac{zF'(z)}{F(z)} = \sum\nolimits_{i=1}^{n} \frac{1-|a_i|^2}{|z-a_i|^2}$$

imply that

$$F'(z_0) = \sum\nolimits_{i=1}^{3} \frac{1-|a_i|^2}{|z_0-a_i|^2} = \sum\nolimits_{i=1}^{3} \frac{1-|\bar{a}_i|^2}{|\bar{z_0}-\bar{a}_i|^2} = F'(\bar{z_0}),$$

which contradicts the hypothesis.

(b) It is immediately from Lemma 3.1.

Lemma 3.3. *Suppose that $F'(x) \neq F'(y)$ for all different fixed points x, y of F on S^1. If G is a rational function of degree 1 that commutes with F, then G is the identity map of $\bar{\mathbb{C}}$.*

Proof. Since degree $G = 1$, G is a Mobius transformation. By Lemma 2.3, for all $k \in \mathbb{N}$, F^k determines a n^k-to-1 immersion on the S^1, so we deduce that F^2 has at least three different fixed points $x_1, x_2, x_3 \in S^1$. Since G commutes with F^2, $G(x_i)$ is a fixed point of F^2 for each $i = 1, 2, 3$. By the Denjoy-Wolff Theorem ([De], [W]), F^2 has at the most two different fixed points on $\bar{\mathbb{C}} - S^1$, so $G(x_i) \in S^1$ for some $i = 1, 2, 3$. This implies that for each $z \in S^1$ satisfying $F^2(z) = x_i$,

$$F^2(G(z)) = G(F^2(z)) = G(x_i),$$

Then $G(z) \in S^1$ and since x_i has at least three different F^2 - preimages on S^1, we deduce that $G(S^1) = S^1$. Now, let p be a fixed point of F on S^1. Then $G(p) \in S^1$ and it is a fixed point of F satisfying $F'(p) = F'(G(p))$. It follows from hypothesis that $G(p) = p$. Thus G has at least three different fixed points, so G is the identity map of $\bar{\mathbb{C}}$ and Lemma 3.3 is proved.

Lemma 3.4. *Suppose that F satisfies the following conditions:*

(i) $F'(x) \neq F'(y)$ *for all different fixed points x, y of F on S^1;*

(ii) *There exists a fixed point x of F on S^1 such that $F'(x) \neq n$.*

Let G be a rational map of degree $m > 1$ that commutes with F. Then $G(S^1) = S^1$ and $G^i = F^j$ for some integers $i, j \geq 1$.

Proof. Since F and G are permutable, they have a common Julia set $\mathcal{J}$ (see [Le]). By the Denjoy-Wolff Theorem, there exists a fixed point p, $|p| \leq 1$, such that

$$\lim_{n \to \infty} F^n(z) = p$$

for all z in the open unit disc Δ.

If $|p| < 1$, then by Lemma 2.2, F/S^1 is expanding. From this and the fact that $\mathcal{J} \subset S^1$, we conclude that $\mathcal{J} = S^1$, and S^1 separates the Fatou set of G in two components. Thus $G(S^1) = S^1$ and $G^2(\Delta) = \Delta$. This implies that G^2 is a finite Blaschke product (see [Ru], pg 310). This fact together with the hypothesis of Lemma 3.4 and the Theorem of [Ar], imply that G^2 is a power of F.

If $|p| = 1$, then F has no periodic point in Δ. This, the fact that a point $z \in \Delta$ is a fixed point of a finite Blaschke product iff $\frac{1}{\bar{z}}$ is also a fixed point; and the fact that for all $k \in \mathbb{N}$, F^k is a finite Blaschke product imply that F has no periodic point in $\bar{\mathbb{C}} - S^1$. It follows that the set $\bar{\mathbb{C}} - \bar{\Delta}$ is contained in a basin of attraction of a fixed point of F on S^1.Thus, for all $z \in \bar{\mathbb{C}} - S^1$, the F- orbit of z accumulates on S^1. From this and the fact that F has not critical points on S^1 we deduce that the forward orbits of the critical points of F are not finite. Then by [Le], $G^i = F^j$ for some integers $i, j \geq 1$. Moreover, we also deduce that S^1 is the unique closed curve of $\bar{\mathbb{C}}$ which is invariant by F because $\mathcal{J}$ is either S^1 or a totally disconnected subset of S^1. Since

$$F(G(S^1)) = G(F(S^1)) = G(S^1),$$

we conclude that $G(S^1) = S^1$ and Lemma 3.4 is established.

Let F be as in Lemma 3.4 and let G be a rational map of degree $m > 1$ that commutes with F. Let f and g denote the restriction of F and G to S^1 respectively. By Lemma 2.3, f is an immersion, so by Lemma 3.4, we deduce that g is also an immersion. The following result reduces the proof of the Theorems to the proof of a similar result for f.

Lemma 3.5. *Let F be as in Lemma 3.4 and let G be a rational map of degree $m > 1$ that commutes with F. If g preserves orientation, then G is a power of F.*

Proof. It is clear that the proof of Lemma 3.5 is reduced proving that $g/\mathcal{J}$ is a power of $f/\mathcal{J}$. Since g is an immersion on S^1 that commutes with f,

$$f'(g(z)) = f'(z)$$

for all fixed point z of f. Hence by hypothesis (i) of Lemma 3.4, g fixes the fixed points of f. From this we obtain that $m \geq n$ and f and g have at least a common fixed point z_0. By Lemma 2.3, f preserves orientation. Then by Lemma 2.1, there exist monotone continuous maps h_1, h_2 of S^1 satisfying

$$h_1(z_0) = h_2(z_0) = 1,$$

and

$$h_1 \circ f = f_n \circ h_1, \quad h_2 \circ g = f_m \circ h_2.$$

Since $f^j = g^i$, $n^j = m^i$. Then $f_n^j = f_m^i$. Hence $h_1 \circ g^i = f_m^i \circ h_1$, so, by Lemma 2.1, $h_1 = h_2$. Let $h = h_1 = h_2$

We claim that f_m fixes the fixed points of f_n. In fact, let z be a fixed point of f_n. Since h is monotone, there exists $x \in h^{-1}(z)$ such that $f(x) = x$. Then $g(x) = x$ and $h(x) = z$ is a fixed point of f_m, and the Claim is established. From the Claim and the fact that a point $z = e^{2\pi i t}$ is a fixed point of a map $f_d(z) = z^d$ iff $t = \frac{l}{d-1}$, $l = 0, 1, \ldots, d-2$, we obtain that $\frac{m-1}{n-1}$ is an integer. On the other hand, since $m^i = n^j$, we obtain that $m = ln^k$, for some $k \geq 1$, $1 \leq l < n$. Hence,

$$\frac{m-1}{n-1} = \frac{ln^s - 1}{n-1} = \frac{l(n^s-1)}{n-1} + \frac{l-1}{n-1},$$

so, $\frac{l-1}{n-1}$ is an integer. This implies that $l = 1$, because $l - 1 < n - 1$. Therefore $m = n^k$ and for all $z \in S^1$,

$$h(g(z)) = f_m(h(z)) = f_{n^k}(h(z)) = f_n^k(h(z)) = h(f^k(z)).$$

This and the fact that h is injective in a dense subset of $\mathcal{J}$ imply that

$$g(z) = f^k(z)$$

for all $z \in \mathcal{J}$, and Lemma 3.5 is proved.

Now we prove the Theorems. Let F be a finite Blaschke product of degree $n > 1$ satisfyng the hypothesis of one of Theorems A, B or C, and let G be a rational function of degree $m \geq 1$ which commutes with F. For $m = 1$, the Theorems follow from Lemma 3.3 because $G = F^0$. Thus we assume that $m > 1$. Notice that if $n \geq 3$, then F has at least two fixed points on S^1, so by hypothesis, $F'(z) \neq n$ for some fixed point z of F on S^1. Therefore, by Lemma 3.5, the proof of the Theorems is reduced proving that $g = G/S^1$ preserves orientation. This will be proved by contradiction.

First we assume that F is under hypothesis of Theorem A. If g is orientation-reversing, then by Lemma 2.1 and similar arguments as Lemma 3.5, we obtain a monotone continuous map h of S^1 such that

$$h \circ f = f_n \circ h \qquad h \circ g = f_{-m} \circ h.$$

Moreover, since g^2 preserves orientation, we deduce as in Lemma 3.5 that $G^2 = F^j$, for some $j \in \mathbb{N}$. Then $m^2 = n^j$, so $m = ln^k$, for some $k \geq 1$, $1 \leq l \leq n$, and l is a divisor of n. By similar arguments as Lemma 3.5, we deduce that $\frac{l+1}{n-1}$ is a natural number. Since $n \geq 4$, $l = n - 2$. Then for some $p \in \mathbb{N}$, $n = pl = p(n-2)$. It follows that

$$\frac{2}{n} = 1 - \frac{1}{p} \geq \frac{1}{2}.$$

This is false if $n > 4$, so Theorem A is hold for $n > 4$.
If $n = 4$, then $l = 2$, so $m^2 = l^2 n^{2k} = n^{2k+1}$. Thus

$$g^2 = f^{2k+1}$$

Let $\tilde{g}$ and $\tilde{f}$ be liftings of g and f respectively. Since g and f are commuting immersions, we have that $\tilde{g}$ and $\tilde{f}$ are commuting diffeomorphisms. Thus

$$\tilde{f} = (\tilde{g} \circ \tilde{f}^{-k})^2$$

Let $\tilde{u} = \tilde{g}\tilde{f}^{-k}$. Clearly $\tilde{u}$ is a lifting of an immersion $u : S^1 \to S^1$ of degree 2 with $f = u^2$. Then there exists a fixed point z_0 of f which is not a fixed point of u. Thus, z_0 and $u(z_0)$ are different fixed points of f and

$$f'(u(z_0)) = u'(z_0)u'(u(z_0)) = f'(z_0).$$

This is a contradiction with the hypothesis, so Theorem A is proved.

Now we assume that F satisfies the hypothesis of one of Theorems B or C. Since $n = 2$ or 3, we conclude as above that for some $k \in \mathbb{N}$, $m^2 = n^k$, and $g^2 = f^{2k}$. Then for the liftings $\tilde{g}$ and $\tilde{f}$ of g and f respectively, we have that

$$(\tilde{g} \circ \tilde{f}^{-k})^2(x) = x,$$

for all $x \in \mathbb{R}$. Then either

$$\tilde{g} \circ \tilde{f}^{-k}(x) = x, \qquad \forall x \in \mathbb{R},$$

or

$$\tilde{g} \circ \tilde{f}^{-k}(x) = -x, \qquad \forall x \in \mathbb{R}.$$

Since f preserves orientation, we conclude that $\tilde{g}(x) = \tilde{f}^k(-x)$, for all $x \in \mathbb{R}$. Then

$$\tilde{f}^k(\tilde{f}^k(x)) = \tilde{g}^2(x) = \tilde{f}^k(-\tilde{f}^k(-x)) \qquad \forall x \in \mathbb{R}$$

Since $\tilde{f}^k$ is a diffeomorphism, we conclude that

$$\tilde{f}^k(x) = -\tilde{f}^k(-x) \qquad \forall x \in \mathbb{R},$$

so,

$$\tilde{g}(x) = -\tilde{f}^k(x), \qquad \forall x \in \mathbb{R}.$$

This implies that $g(x) = \frac{1}{f^k(x)}$, so $G = \frac{1}{F^k}$. It follows that for $w = F^{k-1}(z)$,

$$F \circ \frac{1}{F}(w) = F \circ G(z) = G \circ F(z) = \frac{1}{F} \circ F(w),$$

a contradiction with Corollary 3.2. Thus Theorems B and C are proved.

Notice that in the proof of Theorems B and C, we proved that under hypothesis (i) and (ii) of Lemma 3.4, the centralizer of a finite Blaschke product F is contained in the set $\{F^k; k \in \mathbb{N}\} \cup \{\frac{1}{F^k}; k \in \mathbb{N}\}$. From this fact and Lemma 3.1, we obtain immediately the following result

Theorem D. *Let $F = a_0 \prod_{i=1}^n \frac{z-\bar{a}_i}{1-a_i z}$ be a finite Blaschke product of degree $n = 2$ or 3, satisfyng the following conditions:*

(i) *$F'(x) \neq F'(y)$ for all different fixed points x, y of F on S^1;*

(ii) *There exists a fixed point x of F on S^1 such that $F'(x) \neq n$.*

Then $Z(F) \subset \{F^k; k \in \mathbb{N}\} \cup \{\frac{1}{F^k}; k \in \mathbb{N}\}$. Moreover the equality holds iff $a_0 = 1$, or -1, and for each $i = 1, ..., n$; a_i and $\bar{a}_i$ are zeros of F of the same order.

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