

A real Riemann-Hurwitz theorem

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Abstract. We prove a real version of the Riemann-Hurwitz theorem and apply it to solve a problem of enumerative geometry in the real case: the number of plane projective curves tangent to a line and passing through the appropriate number of points.

Keywords: Ramification points, plane curves.

1. Introduction

Let $\mathbb{K} = \mathbb{C}$ or \mathbb{R} and consider the following enumerative problem: let n be a positive integer and set $N_n = \frac{n(n+3)}{2}$, which is the dimension of the projective space $\mathbb{P}_{n,\mathbb{K}}$ of curves of degree n in the projective plane $\mathbb{P}_{\mathbb{K}}^2$. Then, given k lines ℓ_1, \dots, ℓ_k and $N_n - k$ points P_{k+1}, \dots, P_{N_n} in general position in $\mathbb{P}_{\mathbb{K}}^2$, there is a finite number of curves of degree n which are tangent to ℓ_1, \dots, ℓ_k and pass through P_{k+1}, \dots, P_{N_n} .

If we are working over the complex numbers, then the number s_k of such curves does not depend on the choice of the generic ℓ 's and P 's; as we shall see in § 1, if k is small enough so that there is no curve of degree n with a double component through the P 's (i.e. $k < 2n - 1$), then it follows from the Riemann-Hurwitz formula that $s_k = (2(n - 1))^k$. This is well known and easy to prove; it should provide support for what follows.

The aim of this paper is to show that in the real case, for $k = 1$, for any r such that $0 \leq r \leq n - 1$ there exist configurations of real points P_2, \dots, P_{N_n} and a line ℓ such that there are $2(n - 1) - 2r$ real curves of degree n passing through P_2, \dots, P_{N_n} and tangent to ℓ at a real point. Our main ingredient is a real version of the Riemann-Hurwitz theorem. Such questions have raised interest in recent years (see for example [4]).

I wish to address my thanks to the referee for her or his valuable suggestions.

2. Reduction to the Riemann-Hurwitz theorem in the complex case

Let $1 \leq N \leq N_n$. For $P_1, \dots, P_N \in \mathbb{P}_{\mathbb{K}}^2$. Set

$$\mathcal{L}_{P_1, \dots, P_N} = \{[f] \in \mathbb{P}_{n, \mathbb{K}} \mid f(P_1) = \dots = f(P_N) = 0\}.$$

One expects that if P_1, \dots, P_N are in general position, then $\mathcal{L}_{P_1, \dots, P_N}$ has codimension N in $\mathbb{P}_{n, \mathbb{K}}$. Proposition 1 below establishes that it makes sense to speak of a “generic choice” of a line and points for our enumerative problem.

Proposition 1. *For all $2 \leq N \leq N_n$ the set*

$$\Omega = \left\{ (\ell, P_2, \dots, P_N) \in \check{\mathbb{P}}_{\mathbb{K}}^2 \times (\mathbb{P}_{\mathbb{K}}^2)^{N-1} \mid \forall P \in \ell \dim(\mathcal{L}_{P, P_2, \dots, P_N}) = N_n - N \right\}$$

is non-empty and Zariski-open in $\check{\mathbb{P}}_{\mathbb{K}}^2 \times (\mathbb{P}_{\mathbb{K}}^2)^{N-1}$.

Proof. Clearly, the set Ω is Zariski-open. In § 3, Proposition 5, we will show explicit points Q_2, \dots, Q_{N_n} and a line ℓ such that $\dim(\mathcal{L}_{Q, Q_2, \dots, Q_{N_n}}) = 0$, for all $Q \in \ell$. \square

So let $(\ell, P_2, \dots, P_{N_n}) \in \Omega$, and define

$$Z = \{(P, [f]) \in \mathbb{P}_{\mathbb{K}}^2 \times \mathbb{P}_{n, \mathbb{K}} \mid P \in \ell, f(P) = f(P_2) = \dots = f(P_{N_n}) = 0\}.$$

We have the two natural projections

$$\begin{array}{ccc} Z & \xrightarrow{\pi_1} & \ell \\ \pi_2 \downarrow & & \\ \mathcal{L}_{P_2, \dots, P_{N_n}} & & \end{array}$$

and π_1 is an isomorphism because $(\ell, P_2, \dots, P_{N_n}) \in \Omega$. Therefore, we can define

$$\phi: \ell \rightarrow \mathcal{L}_{P_2, \dots, P_{N_n}}, \quad \phi(P) = \pi_2(\pi_1^{-1}(P)) = \text{the unique curve of degree } n \text{ through } P, P_2, \dots, P_{N_n}.$$

Clearly, P is a ramification point of ϕ if and only if the curve $\phi(P)$ is tangent to ℓ at the point P . Assume now that $\mathbb{K} = \mathbb{C}$. According to the Riemann–Hurwitz theorem the number of ramification points of ϕ equals

$$\begin{aligned} \text{degree of } \phi \times \text{Euler characteristic of } \mathcal{L}_{P_2, \dots, P_{N_n}} - \text{Euler characteristic of } \ell \\ = 2n - 2 = 2(n - 1), \end{aligned}$$

and so this is also the number of curves of degree n through P_2, \dots, P_{N_n} and tangent to ℓ .

This also shows that, for $2 \leq N \leq N_n$, the subvariety of $\mathbb{P}_{\mathbb{K}}^2$,

$$\mathcal{L}_{\ell, P_2, \dots, P_N} = \{([f] \in \mathbb{P}_{\mathbb{K}}^2 \mid \text{the curve } f(x) = 0 \text{ is tangent to } \ell \\ \text{and } f(P_2) = \dots = f(P_N) = 0\},$$

has degree $2(n-1)$. Then one is tempted to argue that, given k lines ℓ_1, \dots, ℓ_k and $N_n - k$ points P_{k+1}, \dots, P_{N_n} in general position, the set

$$\mathcal{L}_{\ell_1, \dots, \ell_k, P_{k+1}, \dots, P_{N_n}} = \{([f] \in \mathbb{P}_{\mathbb{K}}^2 \mid f \text{ is tangent to } \ell_1, \dots, \ell_k \\ \text{and } f(P_{k+1}) = \dots = f(P_{N_n}) = 0\}$$

consists of $(2(n-1))^k$ points, since

$$\mathcal{L}_{\ell_1, \dots, \ell_k, P_{k+1}, \dots, P_{N_n}} = \bigcap_{i=1, \dots, k} \mathcal{L}_{\ell_i, P_{k+1}, \dots, P_{N_n}}. \quad (\#)$$

However, this argument works only for $k < 2n-1$, because otherwise there is a curve g of degree n through P_{k+1}, \dots, P_{N_n} having a double line as component, and so g is tangent to any line in the plane (see [2]). In fact, this shows that in (#) there might be a residual intersection to account for. For example, in the case of conics, there are 2 conics through 4 points and tangent to one line, there are $4 = 2^2$ conics through 3 points and tangent to 2 lines, but the number of conics through 2 points and tangent to 3 lines is not $2^3 = 8$: it is 4, the same as the number of conics through 3 points tangent to three lines, as one can see by resorting to the dual conics. In [1] the case of cubics and in [5] the case of quartics are considered, over the complex numbers.

It would be desirable to know in addition that for a generic choice of $\ell, P_2, \dots, P_{N_n}$, the curves through P_2, \dots, P_{N_n} tangent to ℓ are simply tangent to ℓ (no double tangents, no inflection points, nor worse), which amounts to say that the map ϕ has only ordinary ramification points, with distinct values. This can be done by studying more closely the family of maps $\phi_{\ell, P_2, \dots, P_{N_n}}$, where now $\ell, P_2, \dots, P_{N_n}$ are movable parameters. In § 3 we will satisfy this desire by exhibiting $\ell, P_2, \dots, P_{N_n}$ such that the corresponding ϕ has $2(n-1)$ distinct ramification points, which must therefore be ordinary ramification points, with distinct values.

3. A real version of the Riemann-Hurwitz theorem

Let

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - 1 = 0\}$$

denote the topological circle. We shall say that the (disjoint) subsets F_1, \dots, F_h of S^1 are non interlaced if there exist open, disjoint subintervals $I_i, i = 1, \dots, h$ of S^1 such that $F_i \subset I_i, i = 1, \dots, h$.

Proposition 2. *Let $f: S^1 \rightarrow S^1$ be a C^1 map whose fibers are all finite. If there are distinct $y_1, \dots, y_h \in S^1$ such that $f^{-1}(y_1), \dots, f^{-1}(y_h)$ are not interlaced, denoting by n_i the number of points in $f^{-1}(y_i)$, there are at least*

$$\sum_{i=1}^h n_i - h$$

points where the derivative of f vanishes. More precisely, if $f^{-1}(y_i) \subset I_i$, where the I_i 's are disjoint intervals, and

$$f^{-1}(y_i) = \{a_1^i, \dots, a_{n_i}^i\} \quad \text{with} \quad a_1^i < \dots < a_{n_i}^i$$

for some ordering of the interval I_i , then there is at least one ramification point between a_j^i and a_{j+1}^i , for $j = 1, \dots, n_i - 1$.

Proof. f maps I_i into $S^1 \setminus \{y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_h\}$, which is diffeomorphic to a subset of \mathbb{R} . Then it follows from Rolle's theorem that for all $j \in \{1, \dots, n_i - 1\}$ there exists $c, a_j^i < c < a_{j+1}^i$, with $f'(c) = 0$. \square

Corollary. *Let $\Theta = [F : G]: \mathbb{P}R^1 \rightarrow \mathbb{P}R^1$, where $F(x, t)$ and $G(x, t)$ are homogeneous polynomials of degree n without common non-trivial zeroes. Let $N = [1 : 0]$ and $S = [0 : 1]$, and assume that $\Theta^{-1}(N) = \{P_1, \dots, P_n\}$, $\Theta^{-1}(S) = \{Q_1, \dots, Q_n\}$ and that these 2 sets are not interlaced. Then Θ has exactly $2(n - 1)$ ramification points. More precisely, after perhaps renumbering the P_i 's and the Q_i 's, we can choose $Z \in \mathbb{P}R^1 \setminus \{P_1, \dots, P_n, Q_1, \dots, Q_n\}$ and an order of $\mathbb{P}R^1 \setminus Z$ such that $P_1 < \dots < P_n < Q_1 < \dots < Q_n$. Then for $i = 1, \dots, n - 1$ there exist $A_i, B_i, P_i < A_i < P_{i+1}, Q_i < B_i < Q_{i+1}$, such that $\Theta'(A_i) = \Theta'(B_i) = 0$.*

Proof. Proposition 2 ensures the existence of ramification points as stated. There cannot be more, for otherwise $\Theta' \equiv 0$ and Θ would be constant. \square

On the opposite end, we have:

Proposition 3. *Let $\Theta = [F : G]$, $F(x, t)$ and $G(x, t)$ homogeneous of degree n without non-trivial common zeroes, be such that $\Theta^{-1}(N) = \{P_1, \dots, P_n\}$, $\Theta^{-1}(S) = \{Q_1, \dots, Q_n\}$, and assume that each component of $\mathbb{P}^1 \setminus \{Q_1, \dots, Q_n\}$ contains exactly one of the P_i 's (and hence each component of $\mathbb{P}^1 \setminus \{P_1, \dots, P_n\}$ contains exactly one of the Q_i 's). Then Θ has no real ramification point.*

Proof. $\mathbb{P}^1 \setminus \{N, S\} = I \cup J$, where I and J are disjoint intervals. For $y \in I$ near N , $\Theta^{-1}(y)$ has still n distinct preimages, one in each component of $\mathbb{P}^1 \setminus \{Q_1, \dots, Q_n\}$; therefore, as we move $y \in I$ towards S , it cannot happen that 2 distinct preimages come together. If y_0 is the first critical value that we encounter, then, since Θ is proper, $\Theta^{-1}(y_0)$ also has one preimage in each of the n components of $\mathbb{P}^1 \setminus \{Q_1, \dots, Q_n\}$, and one of these preimages has multiplicity, a contradiction. \square

In order to produce a morphism $\Theta: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ with any number of ramification points of the form $2(n-1) - 2r$, $0 \leq r \leq n-1$, we want to use the old trick that consists in joining a morphism Θ_{\max} with $2(n-1)$ ramification points to a morphism Θ_{\min} with no ramification points, through a “generic” path in the space of morphisms. Such a generic path will encounter 2 kinds of catastrophes: F and G can acquire a non-trivial common zero, or Θ can have a ramification point of order 2. One hopes that when crossing a catastrophic situation, the number of ramification points changes by ± 2 , which ensures that all values $2(n-1) - 2r$, $0 \leq r \leq n-1$ are taken when moving from Θ_{\max} to Θ_{\min} . The details are provided by Proposition 4.

We may assume without loss of generality that $F(1, 0) \neq 0$, $G(1, 0) \neq 0$ and that $[1 : 0]$ is not a critical point of Θ . Then the morphism Θ can be written in affine coordinates as follows:

$$\theta(x) = \frac{f(x)}{g(x)}, \quad f(x) = x^n + a_1 x^{n-1} + \dots + a_n, \quad g(x) = x^n + b_1 x^{n-1} + \dots + b_n.$$

Let $\varphi(x, f, g) = f'(x) \cdot g(x) - f(x) \cdot g'(x)$, so that $\theta'(x) = \varphi(x)/g(x)^2$. Note that $f'(x) \cdot g(x)$ and $f(x) \cdot g'(x)$ have same leading coefficient, therefore $\varphi(x)$ has degree $2(n-1)$. The zeroes of φ are exactly the ramification points of θ when f and g have no common factor, and vanishes for $x = \alpha$ if f and g have $x - \alpha$ as a common factor.

We denote by A_n the space of polynomials of degree $\leq n$ in x with real coefficients and leading coefficient 1; thus $f, g \in A_n$.

Proposition 4. Assume $n \geq 2$. There exists an algebraic subset $Z \subset A_n \times A_n$ of codimension at least 2 such that if $(f, g) \notin Z$, then

$$\varphi(x, f, g) = \frac{\partial \varphi}{\partial x}(x, f, g) = 0 \quad \text{implies that} \quad \frac{\partial^2 \varphi}{\partial x^2}(x, f, g) \neq 0.$$

Proof. First let $n = 2$, $f = x^2 + ax + b$, $g = x^2 + cx + d$. Then

$$\begin{aligned} \varphi(x, f, g) &= x^2(a - c) + 2x(b - d) + bc - ad, \\ \frac{\partial \varphi}{\partial x}(x, f, g) &= 2x(a - c) + (b - d), \\ \frac{\partial^2 \varphi}{\partial x^2}(x, f, g) &= 2(a - c), \end{aligned}$$

so that $\frac{\partial \varphi}{\partial x}(x, f, g) = \frac{\partial^2 \varphi}{\partial x^2}(x, f, g) = 0$ implies that $f = g$. If, for $n \geq 2$, we take Z_1 to be the Zariski closure of the set

$$\{(f, g) \in A_n \times A_n \mid f \text{ and } g \text{ have at least 2 common roots}\},$$

then Z_1 has codimension at least 2 in $A_n \times A_n$, and so for $n = 2$ we can take $Z = Z_1$.

Let now $n \geq 3$ and consider

$$W = \left\{ (x, f, g) \in \mathbb{R} \times A_n \times A_n \mid \varphi(x, f, g) = \frac{\partial \varphi}{\partial x}(x, f, g) = \frac{\partial^2 \varphi}{\partial x^2}(x, f, g) = 0 \right\}.$$

If $(x, f, g) \in W$ and $g(x) \neq 0$, we take the derivatives of the 3 equations defining W in the direction of a vector of the form $(0, \bar{f}, 0) \in \mathbb{R} \times A_n \times A_n$ and obtain:

$$\text{a) } \varphi(x, f, g) = f'(x) \cdot g(x) - f(x) \cdot g'(x) = 0 \quad \longrightarrow \quad \bar{f}'(x) \cdot g(x) - \bar{f}(x) \cdot g'(x) = 0$$

$$\begin{aligned} \text{b) } \frac{\partial \varphi}{\partial x}(x, f, g) &= f''(x) \cdot g(x) - f(x) \cdot g''(x) = 0 \\ &\longrightarrow \bar{f}''(x) \cdot g(x) - \bar{f}(x) \cdot g''(x) = 0 \end{aligned}$$

c)

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial x^2}(x, f, g) &= f'''(x) \cdot g(x) + f''(x)g'(x) - f'(x) \cdot g''(x) - f(x) \cdot g'''(x) = 0 \\ &\longrightarrow \bar{f}'''(x) \cdot g(x) + \bar{f}''(x) \cdot g'(x) - \bar{f}'(x) \cdot g''(x) - \bar{f}(x) \cdot g'''(x) = 0 \end{aligned}$$

Since \overline{f} is of degree at least 3, we can find \overline{f} such that:

a) $\overline{f}(x) = 0, \overline{f}'(x) \neq 0$, which does not satisfy a) above

a-b) $\overline{f}(x) = \overline{f}'(x) = 0, \overline{f}''(x) \neq 0$, which satisfies a) but not b)

a+b-c) $\overline{f}(x) = \overline{f}'(x) = \overline{f}''(x) = 0, \overline{f}'''(x) \neq 0$, which satisfies a) and b) but not c).

It follows that the three equations defining W are of maximal rank at (x, f, g) . If $g(x) = 0$ and $f(x) \neq 0$, the above argument can be repeated with the roles of f and g exchanged. We set

$$W_1 = \{(x, f, g) \in W \mid f \text{ and } g \text{ have no real common root}\}$$

and let Z_2 be the Zariski closure of $p(W_1)$, with $p: \mathbb{R} \times A_n \times A_n \rightarrow A_n \times A_n$ the natural projection. Then, since W_1 has codimension 3 in $\mathbb{R} \times A_n \times A_n$, Z_2 has codimension at least 2 in $A_n \times A_n$.

Assume now that f and g have one real common root $\alpha: f(x) = (x - \alpha)f_1(x)$, $g(x) = (x - \alpha)g_1(x)$, but f_1 and g_1 without common root. Then

$$\begin{aligned} \varphi(x, f, g) &= (x - \alpha)f_1(x)(g_1(x) + (x - \alpha)g_1'(x)) \\ &\quad - (f_1(x) + (x - \alpha)f_1'(x))(x - \alpha)g_1(x) \\ &= (x - \alpha)^2\varphi(x, f_1, g_1). \end{aligned}$$

We know by Equation a) of the first part of this proof that $\frac{\partial \varphi}{\partial f_1}(x, f_1, g_1) \neq 0$ or $\frac{\partial \varphi}{\partial g_1}(x, f_1, g_1) \neq 0$. Consequently, the Zariski closure Z_3 of the set of pairs $(f, g) \in A_n \times A_n$ having a common real root α , for which $\varphi(\alpha, f_1, g_1) = 0$, has codimension at least 2 in $A_n \times A_n$. Since $\frac{\partial^2 \varphi}{\partial x^2}(\alpha, f, g) = 2\varphi(\alpha, f_1, g_1)$, the set $Z = Z_1 \cup Z_2 \cup Z_3$ will have the required properties. \square

Corollary 1. *There exists an algebraic subset $Z_0 \subset \mathbb{R}^n \times \mathbb{R}^n$ of codimension at least 2 such that for $((\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)) \in \mathbb{R}^n \times \mathbb{R}^n \setminus Z_0$, if we set*

$$f(x) = \prod_{i=1, \dots, n} (x - \alpha_i), \quad g(x) = \prod_{i=1, \dots, n} (x - \beta_i) \text{ and } \varphi(x, \alpha, \beta) = \varphi(x, f, g),$$

then

$$\varphi(x, \alpha, \beta) = \frac{\partial \varphi}{\partial x}(x, \alpha, \beta) = 0 \quad \text{implies} \quad \frac{\partial^2 \varphi}{\partial x^2}(x, \alpha, \beta) \neq 0.$$

Proof. The map $\mathcal{N}: \mathbb{R}^n \rightarrow A_n$, $\mathcal{N}(\alpha_1, \dots, \alpha_n) = \prod_{i=1, \dots, n} (x - \alpha_i)$ has finite fibers, and so $Z_0 = (\mathcal{N} \times \mathcal{N})^{-1}(Z)$ has codimension 2 and has the required properties. \square

Corollary 2. *For any r , $0 \leq r \leq n - 1$, there exist $2n$ distinct real numbers $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ such that the morphism*

$$\Theta(\alpha, \beta) = \left[\prod_{i=1, \dots, n} (x - \alpha_i t) : \prod_{i=1, \dots, n} (x - \beta_i t) \right] : \mathbb{P}R^1 \rightarrow \mathbb{P}R^1$$

has exactly $2(n - 1) - 2r$ ordinary real ramification points.

Proof. Take $\alpha_{\max} = \{-n, \dots, -1\}$ and $\beta_{\max} = \{1, \dots, n\}$, and take $\alpha_{\min} = \{2k - 1, k = 1, \dots, n\}$ and $\beta_{\min} = \{2k, k = 1, \dots, n\}$. Then $\theta(\alpha_{\max}, \beta_{\max})$ has $2(n - 1)$ real ramification points by Proposition 2, and $\theta(\alpha_{\min}, \beta_{\min})$ has no real ramification point according to Proposition 3. If we take a path that joins $(\alpha_{\max}, \beta_{\max})$ to $(\alpha_{\min}, \beta_{\min})$ and avoids the subset Z_0 of corollary 1, the corresponding $\varphi(x, \alpha, \beta)$ will have at worse a double root at $x \in \mathbb{R}$, so that the number of roots will change by ± 2 along this path. \square

4. Configurations with the maximal number of real solutions

Consider in $\mathbb{P}_{\mathbb{R}}^2$ two sets of n distinct lines in general position, $\{\ell_0^1, \dots, \ell_{n-1}^1\}$ and $\{\ell_0^2, \dots, \ell_{n-1}^2\}$. Set $A_{i,j} = \ell_i^1 \cap \ell_j^2$.

Proposition 5. *Let ℓ be a line not containing any of the $A_{i,j}$'s. Then the line ℓ and the points:*

$$A_{i,j} \text{ where } i + j \leq n \text{ and } 0 \leq i, j \leq n - 1,$$

are in general position in the sense that for all $P \in \ell$, there is exactly one curve of degree n through P and all the $A_{i,j}$.

Proof. Set $I = \{(i, j) \mid i + j \leq n, 0 \leq i, j \leq n - 1\}$. Then

$$I = \{(i, j) \mid i + j \leq n, 0 \leq i, j\} \setminus \{(n, 0), (0, n)\}.$$

Furthermore, the cardinality of I is, as expected,

$$\frac{(n+1)(n+2)}{2} - 2 = \frac{n(n+3)}{2} - 1 = N_n - 1.$$

Assume first that $P \notin \cup_{i=0}^{n-1} \ell_i^1$. Let $[\phi_{i,j}] \in \check{\mathbb{P}}_{\mathbb{R}}^2$, $(i, j) \in I$, be the equation of a line such that $\phi_{i,j}(A_{i,j}) = 0$ and $\phi_{i,j}(A_{i',j'}) \neq 0$ for $(i, j) \neq (i', j')$ and such that $\phi_{i,j}(P) \neq 0$. Let ψ_i be the equation of ℓ_i^1 .

The curves $\prod_{j=0}^h \phi_{0,j}$ for $0 \leq h \leq n-2$ go through $A_{0,0}, \dots, A_{0,h}$, but no other of the $A_{i,j}$'s, nor P , and ψ_0 goes through $A_{0,0}, \dots, A_{0,n-1}$, but no other of the $A_{i,j}$'s, nor P .

Next, for $0 \leq h \leq n-2$, $\psi_0 \cdot \prod_{j=0}^h \phi_{1,j}$ goes through $A_{0,0}, \dots, A_{0,n-1}$, $A_{1,0}, \dots, A_{1,h}$, but no other of the $A_{i,j}$'s, nor P , and $\psi_0 \cdot \psi_1$ goes through $A_{0,0}, \dots, A_{0,n-1}$, $A_{1,0}, \dots, A_{1,n-1}$, but no other of the $A_{i,j}$'s, nor P .

In general, for $0 \leq r \leq n-2$ and $0 \leq h \leq n-r-2$, $\psi_0 \cdots \psi_r \cdot \prod_{j=0}^h \phi_{r+1,j}$ will go through $A_{0,0}, \dots, A_{r,n-r}$, $A_{r+1,1}, \dots, A_{r+1,h}$, but no others, nor P , and $\psi_0 \cdots \psi_{r+1}$ through $A_{0,0}, \dots, A_{r,n-r}$, $A_{r+1,1}, \dots, A_{r+1,n-r-1}$, for $r+1 \leq n-1$, but no other of the $A_{i,j}$'s, nor P ; the total degree will always be at most $r+1 + (n-r-2) + 1 = n$.

This proves that

$$\begin{aligned} \mathbb{P}_{n,\mathbb{R}} &\supsetneq \mathcal{L}_{A_{0,0}} \supsetneq \mathcal{L}_{A_{0,0}, A_{0,1}} \supsetneq \cdots \mathcal{L}_{A_{0,0}, \dots, A_{0,n-1}} \\ &\supsetneq \cdots \mathcal{L}_{A_{0,0}, \dots, A_{n-1,0}} \supsetneq \mathcal{L}_{A_{0,0}, \dots, A_{n-1,1}} \supsetneq \mathcal{L}_{A_{0,0}, \dots, A_{n-1,1}, P} \end{aligned}$$

If $P \in \cup_{i=0}^{n-1} \ell_i^1$, then $P \notin \cup_{i=0}^{n-1} \ell_i^2$ and we can repeat the above construction with the roles of $\{\ell_i^1\}$ and $\{\ell_i^2\}$ exchanged. So, in any case,

$$\dim(\mathcal{L}_{A_{0,0}, \dots, A_{n-1,1}, P}) = \dim(\mathbb{P}_{n,\mathbb{R}}) - \#(I) - 1 = 0,$$

as required. \square

Note. A similar argument shows that the points,

$$\alpha = (\alpha_0, \dots, \alpha_k) \in \mathbb{N}^{k+1} \text{ for } |\alpha| = \alpha_0 + \cdots + \alpha_k = n,$$

impose linearly independent conditions on the vanishing of homogeneous polynomials of degree n in $k+1$ variables. This was conjectured in [3].

Theorem. Let ℓ , ℓ_0^1 , $\ell_0^2 \in \check{\mathbb{P}}_{\mathbb{R}}^2$ be lines in general position. Let $\ell_1^1, \dots, \ell_{n-1}^1$ be close enough to ℓ_0^1 and $\ell_1^2, \dots, \ell_{n-1}^2$ close enough to ℓ_0^2 , so that the two sets of points $\{B_h^1 = \ell \cap \ell_h^1, h = 0, \dots, n-1\}$ and $\{B_h^2 = \ell \cap \ell_h^2, h = 0, \dots, n-1\}$ are not interlaced in ℓ . Then, perhaps after moving the line ℓ slightly, there are $2(n-1)$ distinct non singular curves of degree n through $A_{i,j} = \ell_i^1 \cap \ell_j^2$, $i+j \leq n$, $0 \leq i, j \leq n-1$ and tangent to ℓ .

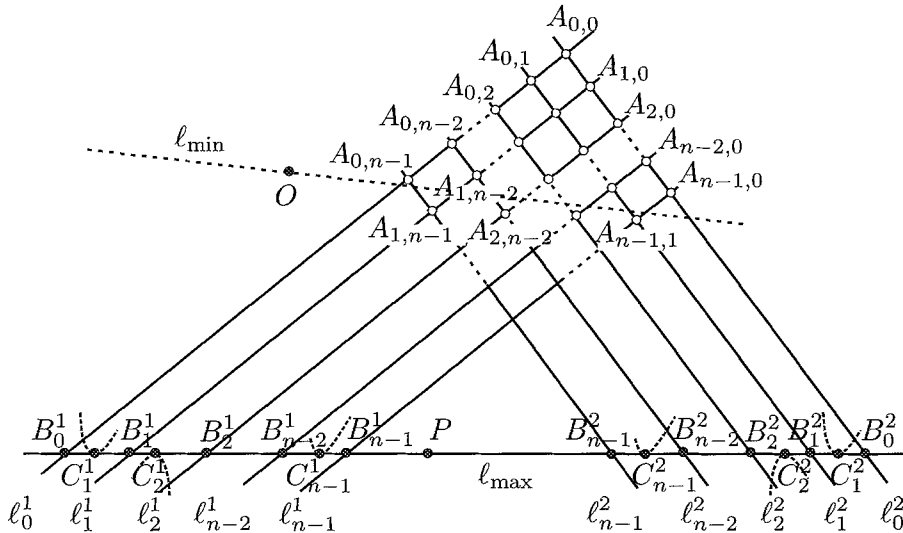


Figure 1

More precisely, let I_1, I_2 be intervals such that

$$B_0^1, \dots, B_{n-1}^1 \subset I_1, \text{ and } B_0^2, \dots, B_{n-1}^2 \subset I_2;$$

choose orders on I_1, I_2 , and assume $B_0^1 < \dots < B_{n-1}^1$ and $B_0^2 < \dots < B_{n-1}^2$. Then there exist points $C_h^1, C_h^2 \in \ell, h = 1, \dots, n-1$, with $B_{h-1}^1 < C_h^1 < B_h^1$ and $B_{h-1}^2 < C_h^2 < B_h^2$, such that for each B_h^1 or B_h^2 there exists a curve of degree n through the $A_{i,j}, 0 \leq i, j \leq n-1, i+j \leq n$, tangent to ℓ at B_h^1 or B_h^2 . Perhaps after moving the line ℓ slightly, these curves are non singular and distinct.

Proof. Set $I = \{(i, j) \mid i + j \leq n, 0 \leq i, j \leq n-1\}$. Consider the map,

$$\phi_\ell: \ell \rightarrow \mathcal{L}_{\{A_{i,j}\}_{(i,j) \in I}}, \quad \phi_\ell(P) = \text{the unique curve through } P \\ \text{and the } A_{i,j} \text{'s for } (i, j) \in I.$$

Denote by f_1 the curve constituted by the union of n lines, $\ell_0^1 \cup \dots \cup \ell_{n-1}^1$, and by f_2 the curve constituted by $\ell_0^2 \cup \dots \cup \ell_{n-1}^2$; then $f_1, f_2 \in \mathcal{L}_{\{A_{i,j}\}_{(i,j) \in I}}$ and

$$\phi_\ell^{-1}(f_1) = \{B_0^1, \dots, B_{n-1}^1\}, \quad \phi_\ell^{-1}(f_2) = \{B_0^2, \dots, B_{n-1}^2\}.$$

So we can apply Proposition 2 to infer that ϕ_ℓ has $2(n-1)$ critical points $C_1^1, \dots, C_{n-1}^1, C_1^2, \dots, C_{n-1}^2$, with $B_{h-1}^1 < C_h^1 < B_h^1$ and $B_{h-1}^2 < C_h^2 < B_h^2$ for $h = 1, \dots, n-1$ (see Figure 1).

Among the curves through the $A_{i,j}$'s, $(i, j) \in I$, there is only a finite number of singular ones. Also, the space of lines of $\mathbb{P}_{\mathbb{K}}^2$ that are tangent at 2 or more distinct points of one of these curves is of dimension 1. It follows that, perhaps after moving the line ℓ slightly, ϕ_ℓ has distinct critical values, corresponding to non singular curves. These are $2(n-1)$ distinct non singular curves through the $A_{i,j}$'s, $(i, j) \in I$, tangent to ℓ . \square

5. Configurations with an intermediate number of real solutions

Proposition 6. *There exist configurations of points P_2, \dots, P_{N_n} and a line ℓ such that there are exactly $2(n-1) - 2r$ curves of degree n through P_2, \dots, P_{N_n} , tangent to ℓ .*

Proof. The case $r = 0$ was done in the previous section. For $r = n-1$, we can take the line ℓ_{\min} shown in Figure 1; the points of intersection of ℓ_{\min} with the two sets of lines $\ell_0^1, \dots, \ell_{n-1}^1$ and $\ell_0^2, \dots, \ell_{n-1}^2$ are alternate, and it follows from Proposition 3 that the corresponding morphism $\phi_{\ell_{\min}}: \ell_{\min} \rightarrow \mathcal{L}_{\{A_{i,j}\}_{(i,j) \in I}}$ has no real ramification points.

There remain the cases $1 \leq r \leq n-2$; for these, we must invoke corollary 2 of Proposition 4. Take the affine line $\ell = \mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{R}$ and $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \ell$ such that the corresponding morphism,

$$\Theta(\alpha, \beta) = \left[\prod_{i=1, \dots, n} (x - \alpha_i t), \prod_{i=1, \dots, n} (x - \beta_i t) \right]: \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

has $2(n-1) - 2r$ real ramification points exactly. Choose lines in general position $\ell_0^1, \dots, \ell_{n-1}^1, \ell_0^2, \dots, \ell_{n-1}^2$, with $\ell_i^1 \cap \ell = \alpha_{i+1}$, $\ell_i^2 \cap \ell = \beta_{i+1}$, $i = 0, \dots, n-1$, and set $A_{i,j} = \ell_i^1 \cap \ell_j^2$, $i+j \leq n$, $0 \leq i, j \leq n-1$. Then the corresponding morphism $\phi_\ell: \ell \rightarrow \mathcal{L}_{\{A_{i,j}\}_{(i,j) \in I}}$ has exactly $2(n-1) - 2r$ real ramification points and we can argue as in the proof of the theorem of § 3 to conclude. \square

Remark. The mutual positions of the α_i 's and β_i 's determine a lower bound for the number of real ramification points. Namely, working in \mathbb{P}^1 rather than in \mathbb{R} , say

$$\mathbb{P}^1 \setminus \{\beta_1, \dots, \beta_n\} = I_1 \cup \dots \cup I_n, \text{ and } \mathbb{P}^1 \setminus \{\alpha_1, \dots, \alpha_n\} = J_1 \cup \dots \cup J_n$$

where the I_h 's are disjoint intervals, as well as the J_h 's. Then, if

$$m_h = \#(I_h \cap \{\alpha_1, \dots, \alpha_n\}), \quad n_h = \#(J_h \cap \{\beta_1, \dots, \beta_n\}) \text{ for } h = 1, \dots, n,$$

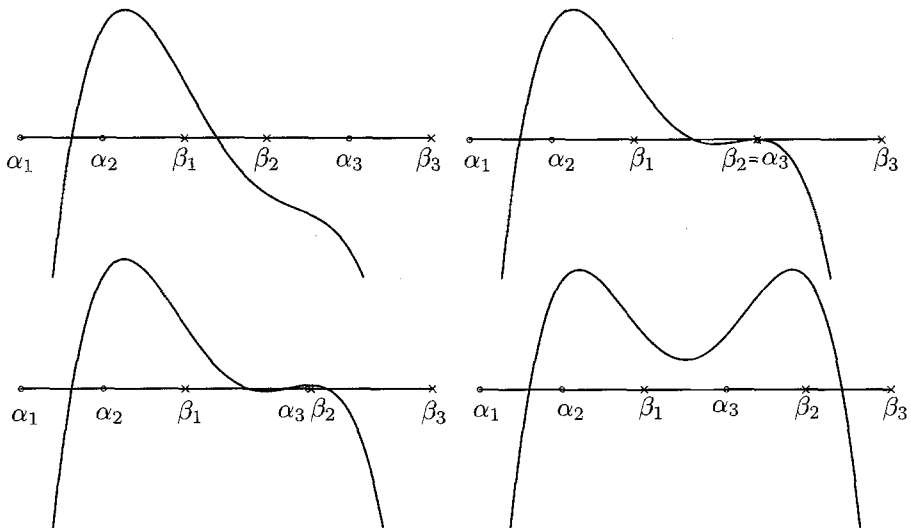


Figure 2

it follows from Proposition 2, or directly from Rolle's theorem, that the number of ramification points is at least,

$$\sigma_{\min} = \sum_{m_h \geq 1} (m_h - 1) + \sum_{n_h \geq 1} (n_h - 1). \quad (\heartsuit)$$

Only in the two extreme cases (the corollary of Propositions 2, and Proposition 3), can we deduce the exact number of real ramification points from the mutual positions of the α_i 's and β_i 's.

In fact, it is easy to produce explicitly all possible values for the Formula (\heartsuit) in the situation of Proposition 5. On Figure 1, we start with the line ℓ_{\min} and let it rotate with center at the point O . Then the line will cross successively the points $A_{n-1,1}, A_{n-2,2}, \dots, A_{1,n-1}$ and at each crossing the number σ_{\min} increases by 2. However, it is not at all clear how the number of real solutions to our enumerative problem will vary. Figure 2 shows an example with

$$\alpha_1 < \alpha_2 < \beta_1 < \beta_2 < \alpha_3 < \beta_3;$$

the α 's are represented by circles, the β 's by crosses and the graph represents $\varphi(x, \alpha, \beta)$. Then α_3 decreases, coincides with β_2 , then takes its place between β_1 and β_2 . When $\alpha_3 = \beta_2$, the graph is simply tangent to the x-axis, in accordance with Proposition 4. But when it decreases still a little bit, 2 additional ramification

points (or zeroes of φ) appear between β_1 and α_3 ; they disappear when the α 's and β 's are distributed more evenly.

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