Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol.31, No. 2, 205–224

Membranes of revolution under partially vanishing normal load

Jaime Arango and Hans Grabmüller

Abstract. This paper investigates the existence and uniqueness of solutions of some boundary value problems modeling the deformation of a membrane of revolution under a partially vanishing normal load. The frame of the membrane models we deal with is the Reissner theory of thin shells of revolution, in which strain–displacement relations are nonlinear, although it assumes a linear stress–strain relation.

Keywords: Nonlinear elasticity, finite rotation, nonlinear two point boundary value problems.

Mathematical subject classification: 73C05, 73K15, 73H99, 34B15.

1. Introduction

The goal of this paper is to study the solvability of a boundary value problem modeling the deformation of a membrane of revolution under the assumption of a fully nonlinear strain–displacement relation. The model we consider follows from E. Reissner's work on thin shells of revolution with negligible bending stiffness (see [5], [7]), assumes a linear strain–stress constitutive law, and consists of a coupled nonlinear system of three ODEs plus an edge condition. It has also been derived directly by Clark and Narayanaswamy [2] without reference to the shell theory. We study the case in which the external forces causing the deformation are rotationally symmetric, act normally on the surface of the membrane, and vanish on a neighborhood of the apex. This case applies to the inflation of a membrane of revolution against a rigid floor; on the contact surface the pressure of inflation cancels out the force exerted by the floor. By

Received 11 January 2000.

well-known mechanical reasons we look for solutions of the boundary value problems granting nonnegative radial stresses.

As in our previous papers on curved membranes of revolution under a partially vanishing vertical load [3], [4], we use the fact that an explicit analysis can be performed where the load vanishes; however we employ a completely different technique to treat the equations related to the loaded part of the membrane. Indeed, we apply a shooting argument to obtain existence and uniqueness results of the problem.

This paper is organized as follows. In Section 2 we outline the derivation of differential equations modeling the stress state of a membrane of revolution under a radially symmetric surface load, and specialize this setting to the case in which the load vanishes on a neighborhood of the apex. In Section 3 our equations are transformed, appropriated mathematical assumptions are stated, and the boundary value problems to be studied in this paper are posed. In Section 4 we develop the tools needed for the application of the shooting method to the boundary value problems we deal with. Though this is a somewhat technical section, we only use elementary results of the theory of ordinary differential equations. Section 5 states the main results of the article.

2. The model

In this section we introduce the equations and boundary conditions describing the deformation of a membrane of revolution subjected to an external, partially vanishing, normal load. The mechanical background of this is not novel; see for example [9] or [2] for a derivation.

Let the middle surface of the membrane in its undeformed state be generated by rotation of the meridian curve $r = r(s)$, $0 \leq s \leq s_1$, where $r(s)$ is the radial distance depending on the arc length s. We denote by $z(s)$ the vertical position and by $\phi(s)$ the angle of inclination (with respect to the radial direction) of the meridian curve. Since in our analysis we intend to consider deformations of dome-like shaped membranes, we assume in the following that $r(s)$ is positive and smooth enough on $(0, s_1]$, and $r(0) = 0$.

Suppose that under an external axisymmetric load with intensity $\widehat{q}(s)$ the membrane deforms again into a membrane of revolution (because of the symmetry of the loading). Thus, any meridian curve $r = r(s)$ of the original state is deformed, with no changes in the circumferential direction, into a meridian curve (which by rotation generates the middle surface of the deformed membrane) with radial distance, depending on the arc length S, given by $R = R(S)$, $0 \leq S \leq S_1$. For the description of the deformed membrane it is of interest the vertical position $Z(S)$ and the angle $\Phi(S)$ of inclination of the deformed meridian curve with respect to the radial direction. The vertical and radial components of the displacement are defined as

$$\widehat{u}(s) \equiv R(s) - r(s), \quad \widehat{w}(s) \equiv Z(s) - z(s),$$

respectively, and the meridional and the circumferential strains of the membrane as

$$\epsilon_\Phi \equiv -1 + \frac{dS}{ds},$$
$$\epsilon_\Theta \equiv -1 + \frac{dR}{dr} = \frac{\widehat{u}}{r},$$

respectively. Further, let S_Φ and S_Θ be the the meridional and radial stresses respectively, and suppose they are related to the already defined strains by *Hooke's law*:

$$\epsilon_\Phi = \frac{1}{E\,h}\left(S_\Phi - \nu\, S_\Theta\right),$$
$$\epsilon_\Theta = \frac{1}{E\,h}\left(S_\Theta - \nu\, S_\Phi\right),$$

where $E > 0$ denotes Young's modulus of elasticity, ν the Poisson's ratio, and $h > 0$ the thickness of the membrane.

According to R. A. Clark and O. S. Narayanaswamy [2] the state of the deformed membrane is described by

$$\frac{d}{ds}(r\, S_\Phi \cos\Phi) - S_\Theta + r\,h\,q_r = 0, \quad \frac{d}{ds}(r\, S_\Phi \sin\Phi) + r\,h\,q_z = 0, \tag{1}$$

and

$$\frac{d}{ds}\,(r\,S_\Theta) = S_\Phi\,\cos\Phi - \nu\, r\,h\,q_\Phi + E\,h\,\left(\cos\Phi - \cos\phi\right). \tag{2}$$

It is worth to note that the mathematical treatment of the model can be simplified by restricting the analysis to the shallow deformations in which $|\Phi|$ is very small and therefore the approximation $\sin\Phi = \Phi$ can be justified. Yet this simplification is not assumed in this paper; indeed, no assumption has been made regarding the strain-displacement relation. Thus, the model presented here applies to sufficient thin structures obeying a linear stress-strain relation and undergoing finite rotations.

On the other side, considerations of stability show that S_Φ and S_Θ have to be *tensile stresses*, i.e., $S_\Phi > 0$ and $S_\Theta > 0$, in order to prevent the membrane from wrinkling.

The radial and vertical displacements, from which the geometrical form of the deformed membrane can be obtained, are related to the stresses by

$$\widehat{u}(s) = \epsilon_\Theta r = r \frac{(S_\Theta - \nu S_\Phi)}{E h}, \tag{3}$$

$$\widehat{w}(s) = \widehat{w}(0) + \int_0^s - \sin\phi + \sin\Phi \left(1 + \frac{S_\Phi - \nu S_\Theta}{E h}\right) d\xi. \tag{4}$$

Equations (1) and (2) are quite general since they do not consider the nature of the external load. For the rest of this paper we carry out our analysis under the followings two assumptions on the external load

- it acts normally on the middle surface of the membrane,
- its intensity vanishes on a neighborhood of the apex the apex according to

$$\widehat{q}_n(s) = 0 \quad \text{for} \quad 0 < s < s_0.$$

The first assumption means that the vertical and radial components of load become related to the intensity of the normal load by

$$\widehat{q}_\Phi = 0, \quad \widehat{q}_z = -\widehat{q}_n \cos\Phi, \quad \widehat{q}_r = \widehat{q}_n \sin\Phi.$$

As a consequence of this, equations (1) and (2) turn out to be independent of the Poisson's ratio ν.

The second assumptions implies that the membranes flattens out where the external load vanishes. Indeed, from (1) it follows that $(r\, S_\Phi \sin\Phi)$ has to be constant on $0 \leq s \leq s_0$. Thus, $\Phi(s) = 0$ for $0 \leq s \leq s_0$ provided the stress S_Φ is finite. As a consequence of this, the vertical displacement is constant on $0 \leq s \leq s_0$. Because of (4) it readily follows that $\sin\phi(s) = 0$, on $0 < s < s_0$. Now, if we think of the external forces as the cause of the deformation, it is reasonable to assume the undeformed membrane, on which no external forces are acting, to be flat everywhere; that its to say

$$\sin\phi(s) = 0, \quad (\text{equivalently} \quad r(s) = s) \quad \text{for all} \quad s \in (0, s_1).$$

As an example, consider the inflation of a membrane of revolution against a rigid floor as indicated in figure 1. The membrane will touch the floor on

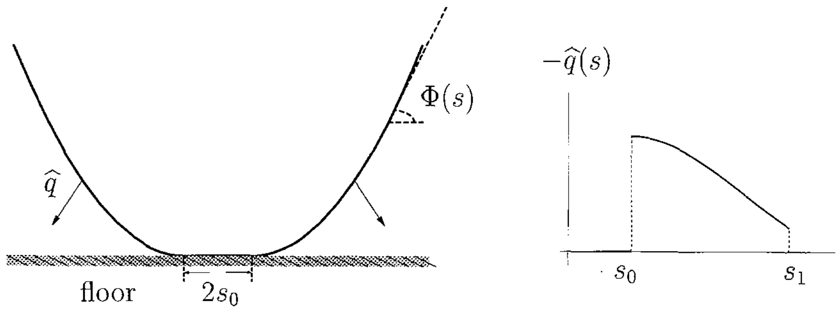

Figure 1: Inflation of a membrane against a rigid floor.

a circular section, say of radius s_0, on which the pressure exerted by the floor cancels out the pressure of inflation. On the rest of the inflated membrane, the resultant external force is exactly the pressure of inflation which acts normally to the surface of the membrane. In this example, the resultant external force is not continuously distributed, and despite of the physical evidence, it is not clear why the mathematical model predicts, as we shall show, a smooth deformation of the membrane.

Now, if we know both the depth l of the floor and the radius s_0 of the contact region, then the inflation problem is over posed since, apart of the necessary edge conditions (for example, fixed outer edge), we still have the additional (nontrivial) condition

$$l = \int_{s_0}^{s_1} \sin\Phi \left(1 + \frac{S_\Phi - \nu\, S_\Theta}{E\,h}\right) d\xi.$$

3. The mathematical problem

It is convenient to define the new variables:

$$x(t) = t\, S_\Phi(s)\, \cos\Phi(s), \qquad y(t) = t\, S_\Phi(s)\, \sin\Phi(s), \qquad z(t) = t\, S_\Theta(s),$$

with

$$t = \frac{s}{s_1}, \qquad \text{and} \qquad \widehat{q}(t) = s_1\, \widehat{q}_n(s).$$

Equations (1) and (2) become

$$\begin{aligned} t\,x'(t) &= z(t) + t^2\,\hat{q}(t)\, s\big(x(t), y(t)\big), && (5)\\ t\,y'(t) &= -t^2\,\hat{q}(t)\, c\big(x(t), y(t)\big), && (6)\\ t\,z'(t) &= x(t) + C\,t\left(-1 + c\big(x(t), y(t)\big)\right), \quad t \in (0,1), && (7) \end{aligned}$$

where $'$ stands for the derivative with respect to t, and

$$c(x, y) = \frac{x}{\sqrt{x^2 + y^2}}, \quad s(x, y) = \frac{y}{\sqrt{x^2 + y^2}}, \quad (x^2 + y^2 > 0).$$

Observe that the angle Φ as well as the horizontal and vertical displacements of the membrane can be recovered from x, y and z as follows

$$\widehat{w}\,(t\,s_1) = \widehat{w}(s_0) + \frac{s_1}{C} \int_{t_0}^{t} \frac{s\,(x\,(\xi)\,,\,y\,(\xi))}{\xi} \left(C\,\xi + \sqrt{x^2(\xi) + y^2(\xi)} - \nu\, z(\xi) \right) d\xi,$$

$$\cos \Phi(t\,s_1) = c\big(x(t), y(t)\big), \quad \sin \Phi(t\,s_1) = s\big(x(t), y(t)\big),$$

$$\widehat{u}\,(t\,s_1) = \frac{s_1}{C} \left(z(t) - \nu \sqrt{x^2(t) + y^2(t)} \right),$$

where $t_0 \equiv \frac{s_0}{s_1}$.

We recall that tensile stresses ($0 < S_\Phi$, $0 < S_\Theta$) are of special interest, because they grant stably equilibrated states and prevent the membrane from wrinkling. Within the framework of the new variables x, y and z the condition of tensile stresses reads $y \geq 0$ and $z \geq 0$. If the rotation angle Φ is restricted to $0 \leq \Phi \leq \frac{\pi}{2}$, then stably equilibrated states are guaranteed if x, y and z are nonnegative.

As the example of inflation of a membrane against a rigid floor shows, it is of interest to allow external loads modeled by:

The function $\widehat{q}(t)$ *vanishes on* $(0, t_0)$, *is negative and smooth on* $(t_0, 1)$ *and satisfies*

$$-\frac{d}{dt}\,(t\,q(t)) \geq 0,$$

where $q \equiv \widehat{q}\,\Big|_{(t_0,1)}$.

A triplet $(x, y, z) \in C\left([0, 1], R^3\right)$ is called a *regular mild tensile solution, rmt-solution* for short, of equations (5), (6) and (7), if

1. $(x, y, z)\,\Big|_{(0,t_0)}$ and $(x, y, z)\,\Big|_{(t_0,1)}$ satisfies equations (5), (6) and (7) on $(0, t_0)$ and $(t_0, 1)$ respectively,

2. $x(t) \geq 0$ and $x^2(t) + y^2(t) > 0$ for all $t \in (0, 1)$.

A rmt-solution (x, y, z) fulfilling the additional condition $z(t) \geq 0$ is called a *tensile or wrinkle free solution.* In the sequel, interest will be focused on rmt-solutions which are preconditions to stably equilibrated states on the deformed membrane.

In order to determine the stress state of the membrane we need to prescribe either a horizontal displacement, an angle, or a stress component at the outer edge. This prescription leads to the boundary value problems defined by the differential equations (5), (6) and (7) and edge conditions of the following type:

- *Angular data: BVP-A*

$$c(x(1), y(1)) = c_1,$$

where the prescribed c_1 is the cosine of the meridional angle at the outer edge.

- *Radial displacement data: BVP-R*

$$z(1) - \nu \sqrt{x^2(1) + y^2(1)} = u_1,$$

where the prescribed u_1 is related to the radial displacement of the membrane $\widehat{u}_1$ by $u_1 = \frac{C}{s_1}\widehat{u}_1$.

- *Vertical and radial displacement data: BVP-RV*

$$\begin{aligned} z(1) - \nu \sqrt{x^2(1) + y^2(1)} &= u_1, \\ \int_{t_0}^{1} \frac{s(x(\xi), y(\xi))}{\xi} \left(C\,\xi + \sqrt{x^2(\xi) + y^2(\xi)} - \nu\, z(\xi) \right) d\xi &= \frac{C}{s_1} (\widehat{w}_1 - \widehat{w}_0), \end{aligned}$$

where $\widehat{w}_1$ and $\widehat{w}_0$ stand for the vertical displacement of the membrane at the outer edge and at the apex respectively. In general, $\widehat{w}_0$ is another unknown of the problem.

The purpose of this article is to investigate the existence and uniqueness of rmt-solutions of the boundary value problems stated above, but first we derive some general properties of rmt-solutions of equations (5), (6) and (7).

Since $\hat{q}(t)$ and $\Phi(t)$ both vanish on $0 \leq t \leq t_0$, equations (5), (6) and (7) restricted to $(0, t_0)$ simplify to

$$t\, x'(t) = z, \quad y'(t) = 0, \quad t\, z'(t) = x, \quad t \in (0, t_0).$$

The components of their regular solutions $X_r(t)$, which depend on a real parameter r, are

$$x_r(t) = r\, t, \;\; y_r(t) = 0, \;\; z_r(t) = r\, t, \quad t \in (0, t_0). \tag{8}$$

Notice that $x_r(0) = 0$, $x_r'(0) = r$, $z_r(0) = 0$, $z_r'(0) = r$. Since we are only interested in rmt-solutions we impose the condition $r > 0$.

On the other hand, the deformation of the loaded part of the membrane is modeled by

$$X'(t) = \frac{1}{t} M\, X(t) + F\,(t, X(t))\,, \quad t \in (t_0, 1), \tag{9}$$

where $X(t)$ is the transposed of $\big(x(t), y(t), z(t)\big)$, and where $F\big(t, X\big)$ is the transposed of (f_1, f_2, f_3) defined by

$$\begin{aligned} f_1(t, x, y) &:= t\, q(t)\, s(x, y), \\ f_2(t, x, y) &:= -t\, q(t)\, c(x, y) \\ f_3(t, x, y) &:= C\,(c(x, y) - 1), \end{aligned}$$

and M is a constant (3×3)–matrix such that equation (9) is the vector form of equations (5), (6) and (7) restricted to $(t_0, 1)$.

$X \in C^1\left((t_0, 1), R^3\right) \cap C\left([t_0, 1], R^3\right)$ is called a *regular mild tensile solution, rmt-solution* for short, of equations (9) if X solves (9) on $(t_0, 1)$ and its first two components x and y satisfy, $x(t) \geq 0$ and $x^2(t) + y^2(t) > 0$ for all $t \in (0, 1)$.

Theorem 1. *Any rmt-solution (x, y, z) of (5), (6) and (7) satisfies*

1. *$z(t), x(t)$ and $k(t) \equiv \sqrt{x^2(t) + y^2(t)}$ are smooth functions on $(0, 1)$,*
2. *$0 \leq k(t) - z(t) \leq C\,t$ for all $t \in (0, 1]$,*
3. *$0 < y(t) < -t^2 q(t)$ on $(t_0, 1)$, and $x(t) > 0$ on $(0, 1)$. If $z(1) \geq 0$, then $z(t) > 0$ for all $t \in (0, 1)$. Moreover, if $u(t) \equiv \frac{C}{s_1} \widehat{u}(t\, s_1)$ satisfies $u(1) \geq 0$, then $u(t) \geq 0$ on $(0, 1)$.*

Proof. Any rmt-solution (x, y, z) of (5), (6) and (7), restricted to $(0, t_0)$, coincides with $X_r(t)$ for an uniquely determined $r > 0$. Moreover, $k(t) > 0$ on $(0, 1]$, hence $c(x(t), y(t))$ and $s(x(t), y(t))$ are continuous on $(0, 1)$ and

$$0 \leq c(x(t), y(t)) \leq 1, \quad 0 \leq s(x(t), y(t)) \leq 1.$$

The smoothness of $z(t)$ and $x(t)$ at $t = t_0$ is easily checked in equations (5) and (7) noticing that $x(t_0) = z(t_0) = r\, t_0$, $c(x(t_0), y(t_0)) = 1$, and $s(x(t_0), y(t_0)) = 0$. Regarding $k(t)$, observe that

$$k'(t) = \frac{z(t)\, x(t)}{t\, k(t)}, \tag{10}$$

and thus $\lim_{t\to t_0^+} k(t) = \lim_{t\to t_0^-} k(t) = r$.

To prove the second claim, we use (10) together with (7) to obtain

$$\frac{d}{dt}(k(t) - z(t)) = \frac{c(x(t), y(t))}{t}(z(t) - k(t)) + C\,(1 - c(x(t), y(t))).$$

This can be seen as a first order linear differential equation in $k - z$ with a nonnegative independent term. Since $k(t_0) - z(_t0) = 0$, we conclude after applying the formula of variation of parameter that $z(t) - k(t) > 0$, or equivalently, $1 \geq \frac{z(t)}{k(t)}$ on $(0, 1)$. Now again from (10) and (7) we obtain

$$k'(t) \leq \frac{x(t)}{t} = z'(t) + C\,(1 - c(x(t), y(t))),$$

and $k(t) - z(t) \leq C\,t$ on $(0, 1)$ after integrating on both side of the above inequality.

As for third claim, we start with

$$0 \leq y(t) \leq \int_{t_0}^{t} -\xi\, q(\xi)\, d\xi,$$

and observe that $t\,\frac{d}{dt}\,(-t\,q(t)) \geq 0$ implies $-t\,q(t) \leq \frac{d}{dt}\left(-t^2\,q(t)\right)$, which gives $0 \leq y(t) \leq -t^2\,q(t)$.

Now, if $x(t)$ vanishes at some interior point $\xi \in (t_0, 1)$, then $x'(\xi) = 0$ and $x''(\xi) \geq 0$ (recall that $x(t) \geq 0$ on $(0, 1)$). Yet if we evaluate the expression

$$\begin{aligned} t\,x''(t) + x'(t) \quad &= \quad \frac{x(t)}{t} + C\,(1 - c(x(t), y(t))) + \\ &\quad t^2\,q(t)\,\frac{d}{dt}s(x(t), y(t)) + s(x(t), y(t))\,\frac{d}{dt}\left(-t^2\,q(t)\right) \end{aligned}$$

at $t = \xi$, and notice that

$$\frac{d}{dt}s(x(t), y(t)) = c(x(t), y(t))\,\frac{-t^2\,q(t)\,k(t) - y(t)\,z(t)}{t\,k^2(t)}, \tag{11}$$

we reach a contradiction since

$$\xi\,x''(\xi) = -C + \frac{d}{dt}\left(t^2\,q(t)\right)\Big|_{t=\xi} < 0.$$

The last part of the third claim is proved in a similar way: we first get an expression for $x''(t)$ (respectively $u''(t)$), and then rule out the existence of a negative minimum of z (respectively u) at some interior point $\xi \in (t_0, 1)$. We recall that q is negative on $(t_0, 1)$ and that

$$u(t) = z(t) - \nu k(t). \tag{12}$$

□

Theorem 1, apart from being an important tool for further results in this paper, reveals some interesting facts of relevance from the mechanical point of view. Observe that claim *1*, despite of an eventually non continuous load, guarantees the smoothness of both the stresses, S_Φ, S_Θ and the displacements (radial and vertical), the continuity of angle of deflection $\Phi(t\,s_1)$, and thus the smoothness everywhere of the shape of the deformed membrane.

Claim *2* implies

$$S_\Theta < S_\Phi \leq S_\Theta + E\,h,$$

which is by no means obvious from the statement of the problem. Moreover, since

$$\frac{d}{dt}\left(\frac{k(t)}{t}\right) = \frac{z(t)\,x(t) - k^2(t)}{t^2\,k(t)} \leq 0,$$

we can see that the stress S_Φ decreases along the meridional curve.

The first inequalities in claim *3* assures, in view of (11), that the angle of deflection increases along the meridional curve of the deformed membrane.

4. The shooting method

For $r > 0$ let us denote by $X(t, r)$ the unique solution of the initial value problem IVP defined by equation (9) and initial data $X_r(t_0)$ at $t = t_0$, where $X_r(t_0)$ is given by (8). It is seen, that any rmt-solution of (5), (6) and (7), restricted to $[t_0, 1]$, coincides with an uniquely determined $X(t, r)$. Therefore, for a given edge condition $B(X) = B_1$, the problem of existence and uniqueness of the correspondingly boundary value problem, within the class of rm-t solutions, reduces to the existence and uniqueness of an $r > 0$, such that $X(t, r)$ is rmt-solution of (9) and satisfies the equation in r given by $B(X(t, r)) = B_1$. This shooting approach is natural, yet it hides some difficulties, since we have no hints about neither the nature of the set

$$I_P = \left\{ r > 0, \;\middle|\; X(t, r) \text{ is a rmt solution of (9)} \right\},$$

nor the dependence of $X(t, r)$ on the parameter r.

Lemma 2. *The set I_P contains an unbounded interval of positive real numbers. Moreover, if $\widehat{r} \in I_P$, then either $x(1, \widehat{r}) = 0$ or there exists $\delta > 0$ such that $(\widehat{r} - \delta, \widehat{r} + \delta) \subset I_P$.* **Proof.** For all $r > 0$ Piccard's Theorem assures the existence of a non-void (maximal) interval $I_r \subset [t_0, 1]$ and a unique solution $X(t,r), t \in I_r$, of IVP ($X(t,r)$ is not necessarily a rmt-solution). Standard results assure also that the solutions $X(t,r)$ depend smoothly on r, and either the interval of definition is $I_r = [t_0, 1]$ or $I_r = [t_0, t_r)$, with $\liminf_{t\to t_r^-} k^2(t,r) = 0$, where $k^2(t,r) \equiv x^2(t,r) + y^2(t,r)$. Moreover, if for a particular $\widehat{r}$ we have $I_{\widehat{r}} = [t_0, 1]$ and $\liminf_{t\to 1^-} k^2(t,\widehat{r}) > 0$, then there exists $\delta > 0$ such that *i)* $I_r = [t_0, 1]$ for all $r \in (\widehat{r} - \delta, \widehat{r} + \delta)$, and *ii)* the solutions $X(t,r), t \in [t_0, 1]$ depend smoothly on $r \in (\widehat{r} - \delta, \widehat{r} + \delta)$.

For a given $r > 0$ suppose that the corresponding solution $X(t,r)$ satisfies $x(t,r) \geq 0$ on I_r. We are going to show that $I_r = [0, 1]$. Indeed, from equation (6) we obtain that $y(t,r)$ is a strictly increasing function of t on I_r, thus $\liminf_{t\to t_r^-} k^2(t,r) > 0$, which in turn implies that $I_r = [0, 1]$.

The formula of variation of parameters applied to equation (9) with initial data $x(t_0) = r\,t_0$, $y(t_0) = 0$ and $z(t_0) = r\,t_0$ yields (for $t \in I_r$),

$$x(t,r) = r\,t + \big(J_+ f_1\big)(t) + \big(J_- f_3\big)(t), \tag{13}$$

$$y(t,r) = \int_{t_0}^{t} f_2(\xi, r)\,d\xi, \tag{14}$$

$$z(t,r) = r\,t + \big(J_+ f_3\big)(t) + \big(J_- f_1\big)(t), \tag{15}$$

where

$$\big(J_\pm f_j\big)(t) := \int_{t_0}^{t} \frac{t^2 \pm \xi^2}{2t\xi} f_j(\xi, r)\,d\xi,$$

and where for short we use the notation $f_j(\xi,r) \equiv f_j\big(\xi, x(\xi,r), y(\xi,r)\big)$, $(j = 1, 2, 3)$. Since

$$\big|f_1(\xi,r)\big| \leq -\xi\,q(\xi), \qquad \big|f_2(\xi,r)\big| \leq -\xi\,q(\xi), \qquad \big|f_3(\xi,r)\big| \leq 2C.$$

we can readily conclude that $x(t,r) > 0$ on I_r provided,

$$r \geq \int_{t_0}^{1} \left(-\xi\,q(\xi)\,\frac{1+\xi^2}{2} + C\,\frac{1-\xi^2}{2\xi}\right) d\xi$$

and as we have already remarked, for such r we have $I_r = [0, 1]$.

Now suppose that $\widehat{r} \in I_P$ and that $x(1,\widehat{r}) > 0$. According to claim *3* of Theorem 1, there exists ϵ_0 such that $x(t,r) \geq \epsilon_0 > 0$ on $[t_0, 1]$. Since solutions $X(t,r)$ depend continuously on the parameter r, there exists $\delta > 0$ such that $x(t,r) \geq 0$ on $[t_0, 1]$ for all $r \in (\widehat{r} - \delta, \widehat{r} + \delta)$, thus $(\widehat{r} - \delta, \widehat{r} + \delta) \subset I_P$. □

Let J be any interval (not necessary included in I_P) in which we can guarantee the smoothly dependence on r and $I_r = [0, 1]$, for $r \in J$. Recall

$$x(t_0, r) = r\, t_0, \qquad y(t_0, r) = 0, \qquad z(t_0, r) = r\, t_0,$$
$$\frac{\partial}{\partial r} x(t_0, r) = t_0, \quad \frac{\partial}{\partial r} y(t_0, r) = 0, \quad \frac{\partial}{\partial r} y(t_0, r) = t_0.$$

Differentiation of equations (5), (6) and (7) with respect to r yields

$$\begin{aligned} t\Big(\frac{\partial}{\partial r} x(t, r)\Big)' &= \frac{\partial}{\partial r} z + t^2 q(t) \frac{\partial}{\partial r} s(t, r), && && (16)\\ t\Big(\frac{\partial}{\partial r} y(t, r)\Big)' &= -t^2 q(t) \frac{\partial}{\partial r} c(t, r), && t \in (t_0, 1), && (17)\\ t\Big(\frac{\partial}{\partial r} z(t, r)\Big)' &= \frac{\partial}{\partial r} x + C\, t \frac{\partial}{\partial r} c(t, r), && && (18)\end{aligned}$$

where on the right hand side we write for short

$$\begin{aligned} x &= x(t, r),\\ y &= y(t, r),\\ z &= z(t, r),\\ s(t, r) &= s\big(x(t, r), y(t, r)\big);\\ c(t, r) &= c\big(x(t, r), y(t, r)\big).\end{aligned}$$

A straightforward computation yields

$$\left.\begin{aligned} \frac{\partial}{\partial r} c(t, r) &= s(t, r)\, \frac{y \frac{\partial}{\partial r} x - x \frac{\partial}{\partial r} y}{x^2 + y^2},\\ \frac{\partial}{\partial r} s(t, r) &= -c(t, r)\, \frac{y \frac{\partial}{\partial r} x - x \frac{\partial}{\partial r} y}{x^2 + y^2}.\end{aligned}\right\} \qquad (19)$$

It is easily seen that equations (16), (17) and (18) are a linear ODE system for the unknowns $\frac{\partial}{\partial r} x(t, r)$, $\frac{\partial}{\partial r} y(t, r)$ and $\frac{\partial}{\partial r} z(t, r)$, provided that $x = x(t, r)$ and $y = y(t, r)$ are considered as known functions.

The sign of the quantity g,

$$g(t, r) \equiv y\, \frac{\partial}{\partial r} x - x\, \frac{\partial}{\partial r} y,$$

is crucial for further analysis.

Lemma 3. *For all $r \in J$ there exists $\delta > 0$ such that $g(t, r) > 0$ for $t \in (t_0, t_0 + \delta)$.*

Proof. From $c(t_0, r) = 1$ for all $r \in J$, it is readily concluded that

$$c(\xi, r) = 1 + O(\xi - t_0), \quad \frac{\partial}{\partial r} c(\xi, r) = O(\xi - t_0).$$

Equations (6) and (17), along with $y(t_0, r) = 0$ and $\frac{\partial}{\partial r} y(t_0, r) = 0$ respectively yield

$$y(t, r) = -\int_{t_0}^{t} \xi\, q(\xi)\, c(\xi, r)\, d\xi, \quad \frac{\partial}{\partial r} y(t, r) = -\int_{t_0}^{t} \xi\, q(\xi) \frac{\partial}{\partial r}\, c(\xi, r)\, d\xi,$$

therefore

$$g(t, r) = \int_{t_0}^{t} -\xi\, q(\xi) \left(c(\xi, r) \frac{\partial}{\partial r} x(t, r) - x(t, r) \frac{\partial}{\partial r} c(\xi, r) \right) d\xi.$$

Next, observe that the expression $c(\xi, r)\frac{\partial}{\partial r} x(t, r) - x(t, r)\frac{\partial}{\partial r} c(\xi, r)$ is positive on a non-void interval $(t_0, t_0 + \delta) \subset (t_0, 1]$. The claim follows now from the integral expression of $g(t, r)$ and from the assumption that q is negative on $(t_0, 1)$. $\square$

Lemma 4. *Given the matrix* $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and for some integrable right hand side* $R(s)$ *the vector initial value problem*

$$X'(s) - \frac{1}{s}\, AX(s) = R(s) \geq 0, \quad s \in (t_0, 1), \quad X(t_0) = \big(x(t_0), z(t_0)\big)^T \geq 0,$$

with $x(t_0) + z(t_0) > 0$. *Then any regular solution satisfies* $X(t) > 0, \ \ t \in (t_0, 1)$, *where the orderings "*$\geq$*" and "*$>$*" in* $\mathbb{R}^2$ *have to be understood component by component.*

Proof. For $t > t_0$ and any $s \in (t_0, t)$ an application of $\frac{1}{k!}\left(\ln \frac{t}{s} A\right)^k$ on both sides of the differential equation does not influence the above ordering because we have $A^{2j} = Id$ and $A^{2j+1} = A$ for any $j = 0, 1, \ldots$. Summation over k leads to

$$\begin{aligned} e^{A \ln \frac{t}{s}} \left(X'(s) - \frac{1}{s}\, AX(s) \right) &= e^{A \ln t} \frac{d}{ds} \left(e^{-A \ln s} X(s) \right) \\ &= e^{A \ln \frac{t}{s}} R(s) \geq 0, \quad s \in (t_0, t). \end{aligned}$$

Integrating over s this finally gives

$$X(t) \geq e^{A \ln \frac{t}{t_0}} X(t_0) =$$
$$= \sum_{j=0}^{\infty} \frac{1}{(2j)!} \left(\ln \frac{t}{t_0} \right)^{2j} \left\{ \left[\begin{array}{c} x(t_0) \\ z(t_0) \end{array} \right] + \frac{1}{2j+1} \ln \frac{t}{t_0} \left[\begin{array}{c} z(t_0) \\ x(t_0) \end{array} \right] \right\} > 0$$

for all $t \in (t_0, 1)$. □

Lemma 5. *If $r \in J$ and $X(t, r)$ is an rmt-solution of (9), then, for all t in $(t_0, 1]$, $g(t, r)$ is positive and*

$$\begin{aligned} &\frac{\partial}{\partial r} x(t, r) > 0, \quad \frac{\partial}{\partial r} z(t, r) > 0, \\ &\frac{\partial}{\partial r} k(t, r) > 0, \\ &\frac{\partial}{\partial r} c\left(x\left(t, r\right), y\left(t, r\right) r\right) > 0. \end{aligned}$$

Proof. Replacing x', y', $\left(\frac{\partial}{\partial r} y\right)'$, and $\left(\frac{\partial}{\partial r} x\right)'$ in

$$\frac{d}{dt} g(t, r) = y \left(\frac{\partial}{\partial r} x \right)' + y' \frac{\partial}{\partial r} x - x \left(\frac{\partial}{\partial r} y \right)' - x' \frac{\partial}{\partial r} y$$

by their respective expressions (16), (6), (17) and (5), we obtain

$$t \frac{d}{dt} g(t, r) = -t^2 q(t) \frac{\partial}{\partial r} \sqrt{x^2 + y^2} + y \frac{\partial}{\partial r} z - z \frac{\partial}{\partial r} y.$$

It is convenient to define the auxiliary expressions b and k as follows:

$$b(t, r) \equiv -2t\, q(t) \frac{\partial}{\partial r} z - C \frac{\partial}{\partial r} y, \quad k(t, r) \equiv \sqrt{x^2 + y^2}.$$

Again, we differentiate with respect to t in order to obtain

$$\begin{aligned} \frac{d}{dt}\left(t \frac{d}{dt} g(t, r) \right) &= -t^2 q(t) \frac{d}{dt} \left(\frac{\partial}{\partial r} k \right) - \frac{\partial}{\partial r} k \frac{d}{dt} \left(t^2 q(t) \right) \\ &\quad + \frac{d}{dt} \left(y \frac{\partial}{\partial r} z - z \frac{\partial}{\partial r} y \right). \end{aligned}$$

Combined with

$$\frac{d}{dt}\Big(y\,\frac{\partial}{\partial r}z - z\,\frac{\partial}{\partial r}y\Big) = \frac{1}{t}\,\Big(y\,\frac{\partial}{\partial r}x - x\,\frac{\partial}{\partial r}y\Big) + t\,q(t)\,z\,\frac{\partial}{\partial r}c(t,r) + C\,\frac{\partial}{\partial r}y$$
$$- c(t,r)\Big(C\,\frac{\partial}{\partial r}y + t\,q(t)\,\frac{\partial}{\partial r}z\Big)y + C\,y\,\frac{\partial}{\partial r}c(t,r)$$

and

$$\begin{aligned}\frac{d}{dt}\Big(\frac{\partial}{\partial r}k\Big) &= \frac{\partial}{\partial r}\Big(\frac{d}{dt}k\Big) = \frac{\partial}{\partial r}\Big(\frac{c(t,r)z}{t}\Big) = \frac{z}{t}\frac{\partial}{\partial r}c(t,r) + \frac{c(t,r)}{t}\frac{\partial}{\partial r}z,\\ \frac{\partial}{\partial r}k(t,r) &= s(t,r)\frac{\partial}{\partial r}y + c(t,r)\frac{\partial}{\partial r}x,\end{aligned}$$

we get

$$\begin{aligned}\frac{d}{dt}\Big(t\frac{d}{dt}g(t,r)\Big) &= b(t,r)\,c(t,r) + \frac{g(t,r)}{t} - \big(t^2 q(t)\big)'\frac{\partial}{\partial r}k(t,r)\\ &+ C\left(y\,\frac{\partial}{\partial r}\,c(t,r) + \frac{\partial}{\partial r}y\right).\end{aligned}$$

Analogously we have

$$\frac{d}{dt}b(t,r) = -2q(t)\,\frac{\partial}{\partial r}x - C\,t\,q(t)\frac{\partial}{\partial r}c(t,r) - 2\big(t\,q(t)\big)'\frac{\partial}{\partial r}z.$$

After these computations we are ready to proceed with the proof. From the assumption on q and the fact that $X(t,r)$ is rmt-solution is clear that

$$x(t,r) \geq 0, \quad c(t,r) \geq 0, \quad y(t,r) \geq 0, \quad s(t,r) \geq 0.$$

Observe that for any interval $(t_0, t_1) \subset (0,1]$ on which $g(t,r)$ is positive we have

$$\frac{\partial}{\partial r}c(t,r) \geq 0, \quad -\frac{\partial}{\partial r}s(t,r) \geq 0,$$

and consequently $\frac{\partial}{\partial r}y(t,r) > 0$. This, together with Lemma 4 applied to equations (16) and (18), implies

$$\frac{\partial}{\partial r}x(t,r) > 0, \quad \frac{\partial}{\partial r}z(t,r) > 0.$$

Everything considered, we see that $b(t,r) \geq 0$ and

$$\frac{d}{dt}\Big(t\frac{d}{dt}g(t,r)\Big) > 0, \quad t \in (t_0, t_1). \tag{20}$$

Now, let $\tilde{t}$ be the largest t_1 such that $g(t,r) > 0$ for all t in $(t_0,t_1) \subset (0,1]$. Lemma 3 assures there is such a non-void interval. We claim that $\tilde{t} = 1$. If not, then $g(t,r)$ has to be positive and to have a maximum in $(t_0,\tilde{t})$. As inequality (20) remains valid for $t_1 = \tilde{t}$, we have reached a contradiction. □

Lemma 6. *$I_P = [\widehat{r}, \infty)$ with $\widehat{r} > 0$ and $x(1, \widehat{r}) = 0$.*

Proof. If I_P were not an interval, there would exist an $\dot{r} > 0$ and two sequences (r_n) and $(\tilde{r}_n)$ such that

$$r_n \in I_P,\ r_n \to \dot{r},\ \tilde{r}_n \notin I_p,\ \tilde{r}_n \to \dot{r}, \qquad r_n \leq \dot{r} \leq \tilde{r}_n.$$

It is not difficult to prove (by a reasoning analogous to that in the proof of Lemma 2) the existence of a $\delta > 0$ such that $I_r = [t_0, 1]$ and such that the solutions $X(t, r)$ depends smoothly on r, $r \in (\dot{r} - \delta, \dot{r} + \delta)$ according to equations (16), (17) and (18).

Since $x(t, r_n) \geq 0$ for all $t \in [t_0, 1]$, we conclude that $x(t, \dot{r}) \geq 0$ and hence $\dot{r} \in I_P$. Moreover, from Lemma 5 we know that $\frac{\partial}{\partial r} x(t, \dot{r}) > 0$ for $t \in [t_0, 1]$. From this, there exists $\epsilon_0 > 0$ such that

$$x(t, \dot{r} + \epsilon) > x(t, \dot{r}) \geq 0,$$

for any $0 < \epsilon < \epsilon_0$ and for any $t \in [t_0, 1]$. Yet this implies $\dot{r} + \epsilon \in I_P$ for any $0 < \epsilon < \epsilon_0$, which in turn contradicts $\tilde{r}_n \notin I_p$. This proves that I_P is an interval.

Let us denote $\widehat{r} = \inf I_P$. As we did for $\dot{r}$ we can easily show that $x(1, \widehat{r}) = 0$ if $\widehat{r} > 0$. Suppose $\widehat{r} = 0$ and let $r_n \to 0$ be a decreasing sequence in I_P. From Lemma 5 we know that $X(\cdot, r_n)$ is a decreasing sequence of functions (the ordering in $\mathbb{R}^3$ has to be understood component by component); moreover $c(\cdot, r_n)$ (respectively $s(\cdot, r_n)$) is a decreasing (respectively increasing) sequence of continuous functions. Because of the integral representation of $x(t, r)$, $y(t, r)$ and $z(t, r)$ in (13), (14) and (15) and well known results of the Lebesgue theory we have

$$\begin{aligned}
\widehat{x}(t) &= \int_{t_0}^{t} \frac{t^2 + \xi^2}{2t} q(\xi) \widehat{s}(\xi)\, d\xi + C \int_{t_0}^{t} \frac{t^2 - \xi^2}{2t\xi} (-1 + \widehat{c}(\xi))\, d\xi, \\
\widehat{y}(t) &= \int_{t_0}^{t} -\xi\, q(\xi) \widehat{c}(\xi)\, d\xi, \\
\widehat{z}(t) &= \int_{t_0}^{t} \frac{t^2 - \xi^2}{2t} q(\xi) \widehat{s}(\xi)\, d\xi + C \int_{t_0}^{t} \frac{t^2 + \xi^2}{2t\xi} (-1 + \widehat{c}(\xi))\, d\xi,
\end{aligned}$$

where $\widehat{x}, \widehat{y}, \ldots$ are the corresponding limits of $x(t, r_n)$, $y(t, r_n)$, ... as n goes to ∞. Now, since $x(t, r_n) \geq 0$ on $[t_0, 1]$, we obtain $\widehat{x} \geq 0$ on $[t_0, 1]$, and from the above integral representation of $\widehat{x}$ that (recall $q \leq 0$)

$$\int_{t_0}^{t} \frac{t^2 + \xi^2}{2t} q(\xi) \widehat{s}(\xi)\, d\xi = 0, \quad C \int_{t_0}^{t} \frac{t^2 - \xi^2}{2t\xi} (-1 + \widehat{c}(\xi))\, d\xi = 0.$$

It follows now that $\widehat{c}(\xi) = 1$ and $\widehat{s}(\xi) = 0$ a.e. on $[t_0, 1]$, which contradicts the fact that $\widehat{c}(t) < c(t, r) < 1$ for all $r > 0$ and all $t \in (t_0, 1]$. □

5. Existence and uniqueness results

So far we have seen that rmt-solutions of the boundary value problems stated in Section 3 are to be sought in the set of functions

$$X(\cdot, r), \ r \in [\widehat{r}, \infty].$$

Let us estimate $X(1, r)$ for $r \geq \widehat{r}$. From the integral representation of $x(t, r)$, $y(t, r)$ and $z(t, r)$ in (13), (14) and (15) we obtain for $r \geq \widehat{r}$

$$r\, t_0 - const \leq x(1, r), \quad 0 \leq y(1, r) \leq const, \qquad r\, t_0 - const \leq z(1, r),$$

and from this follows

$$\lim_{r \to \infty} x(1, r) = \lim_{r \to \infty} z(1, r) = \infty, \quad \lim_{r \to \infty} \frac{z(1, r)}{x(1, r)} = 1.$$

From $x(1, \widehat{r}) = 0$, (13) and (15) we have $\widehat{r} = -(J_+ f_1)(1) - (J_- f_3)(1)$, and $z(1, \widehat{r}) = \widehat{r} + (J_+ f_3)(1) + (J_- f_1)(1)$. The equation for $z(1, \widehat{r})$ is equivalent to

$$z(1, \widehat{r}) = \int_{t_0}^{1} \xi\,(-\xi\, q(\xi)\, s(\xi, \widehat{r}) - C\,(1 - c(\xi, \widehat{r})))\, d\xi. \tag{21}$$

Theorem 7. *For any $u_1 \geq \tilde{u}_1 \equiv (1 - \nu) \int_{t_0}^{1} -\xi\, q(\xi)\, d\xi$ the boundary value problem BVP-R possesses exactly one tensile solution.*
For any $u_1 \geq \tilde{u}_1$ and any w_1 the boundary value problem BVP-RV possesses exactly one tensile solution.

Proof. The existence and uniqueness of a rmt-solution of BVP-R is equivalent to existence and uniqueness of a zero of the application

$$r \to u(1, r) - u_1, \quad r \geq \widehat{r},$$

where $u(t, r) \equiv z(t, r) - \nu\, k(t, r)$. Moreover, any rmt-solution (x, y, z) of BVP-R with $u_1 \geq 0$, must fulfill $z(1) > 0$, and according to Theorem 1 $z(t) > 0$ for all $t \in [t_0, 1]$; hence any rmt-solution of BVP-R with $u_1 \geq 0$ has to be a tensile (wrinkle free). Regarding the BVP-RV problem the vertical displacement w_1

can be arbitrary prescribed; however this prescription determines the maximal deflection of the membrane at apex $w(0)$.

The uniqueness of the zero obviously follows from $\frac{\partial}{\partial r}u(1,r) > 0$, for all $r > \widehat{r}$ and for all $t \in [t_0, 1]$. Let $r > \widehat{r}$ given. Observe that $\frac{\partial}{\partial r}u(t,r) > 0$ on some interval $[t_0, t_0+\delta]$ with $\delta > 0$. We use equations (5), (6), (7), (16), (17) and (18) to obtain after a some computation that

$$\frac{d}{dt}\Big(\frac{\partial}{\partial r}u\Big) = \frac{1}{t}\frac{\partial}{\partial r}x + \Big(C - \frac{\nu z}{t}\Big)\frac{\partial}{\partial r}c - \frac{\nu}{t}c\frac{\partial}{\partial r}z.$$

Now suppose $\frac{\partial}{\partial r}u = 0$ at some interior point $\xi \in [t_0, 1]$. Then, at $t = \xi$ we have

$$\frac{\partial}{\partial r}z = \nu\frac{\partial}{\partial r}k = \nu c\frac{\partial}{\partial r}x + \nu s\frac{\partial}{\partial r}y.$$

Replacing this in the expression for $\frac{d}{dt}\Big(\frac{\partial}{\partial r}u\Big)$ yields at $t = \xi$

$$\begin{aligned}
\frac{d}{dt}\Big(\frac{\partial}{\partial r}u\Big) &= \Big(C - \frac{\nu z}{t}\Big)\frac{\partial}{\partial r}c + \big(1-\nu^2\big)c^2\frac{\partial}{\partial r}x + y\,\frac{y\frac{\partial}{\partial r}x - \nu^2 x\frac{\partial}{\partial r}y}{t\,k^2} \\
&\geq \Big(C - \frac{\nu z}{t}\Big)\frac{\partial}{\partial r}c + \big(1-\nu^2\big)c^2\frac{\partial}{\partial r}x + \frac{k}{t}\frac{\partial}{\partial r}c \\
&= \big(1-\nu^2\big)c^2\frac{\partial}{\partial r}x + \left(\frac{C + t\,(k-z)}{t}\right)\frac{\partial}{\partial r}c > 0,
\end{aligned}$$

where use have been made of Theorem 1 and equation (19). The inequality $\frac{d}{dt}\Big(\frac{\partial}{\partial r}u\Big) > 0$ wherever $\frac{\partial}{\partial r}u = 0$ and the fact that $\frac{\partial}{\partial r}u > 0$ on some interval $[t_0, t_0+\delta]$ with $\delta > 0$ obviously imply $\frac{\partial}{\partial r}u > 0$ on $[t_0, 1]$.

Now the problem is to estimate $\inf_{r>\widehat{r}} u(1,r) = u(1,\widehat{r})$ and $\lim_{r\to\infty} u(1,r)$. First, from the definition of u and the fact that $\lim_{r\to\infty}\frac{z(1,r)}{x(1,r)} = 1$ we easily get $\lim_{r\to\infty} u(1,r) = \infty$. Second, notice that

$$u(1,\widehat{r}) \leq (1-\nu)\,k(1,\widehat{r}) \leq (1-\nu)\int_{t_0}^{1} -\xi\, q(\xi)\, d\xi.$$

□

A slightly different existence result for BVP-R and BVP-RV can be obtained from expression (21) :

$$u(1,\widehat{r}) \leq z(1,\widehat{r}) \leq \int_{t_0}^{1} -\xi^2\, q(\xi)\, d\xi.$$

From this inequality we have existence for any $u_1 \geq \int_{t_0}^{1} -\xi^2\, q(\xi)\, d\xi$.

Theorem 8. *For any $0 \leq c_1 < 1$ the boundary value problem BVP-A possesses exactly one rmt-solution. Moreover, the solution is tensile if $\frac{-t\,q(t)}{E\,h} \geq 1$*

Proof. We see that the application $r \to c(1, r)$, $r > \widehat{r}$, is smooth, strictly increasing, and satisfies $c(1, \widehat{r}) = 0$, $\lim_{r \to \infty} c(1, r) = 1$.

Regarding the tensile solutions, expression (21) is the key to determine whether a solution is tensile (recall again Theorem 1). Observe that $s(\xi, \widehat{r})$ is exactly the sine of the angle of inclination $\Phi(\xi\, s_1)$, of the deformed membrane determined uniquely by the boundary condition $x(1) = 0$. Thus

$$-\xi\, q(\xi)\, s(\xi, r) - C(1 - c(\xi, r)) = -\xi\, q(\xi)\, \sin \Phi(\xi\, s_1) - E\, h\, (1 - \cos \Phi(\xi\, s_1)).$$

Since $0 \leq \Phi(\xi\, s_1) \leq \frac{\pi}{2}$ and $1 - \cos \Phi(\xi\, s_1) \leq \sin \Phi(\xi\, s_1)$, we get

$$z(1, \widehat{r}) \geq \int_{t_0}^{1} s(\xi)\, (-\xi\, q(\xi) - E\, h) > 0,$$

provided $\frac{-t\, q(t)}{E\, h} \geq 1$ for all $t \in [t_0, 1]$. □

6. Concluding remarks

We have established existence and uniqueness of solutions of some boundary value problems which model the stress state and deformation of a rotation membrane under partially vanishing normal load. This kind of loading is by no means artificial; it is not far from applications and appears in real settings like the inflation of membrane against a rigid floor.

Our uniqueness results are quite sharp. Yet the same cannot be said about the existence results for the radial displacement, for our estimative concerning them do not depend on the elasticity module. Moreover, the criterium we developed to guarantee tensile solutions (wrinkling free solutions) of the boundary value problems–the important ones from the mechanical point of view–is very restrictive. A better criterium as well as a better estimative for the radial displacements are still open problems.

Our analysis set up the necessary tools to solve the more difficult problem of finding the contact area of the inflated membrane against a rigid floor as function of the pressure and the material constant.

Acknowledgments. This investigation was largely carried out while the first author was on leave at Institut für Angewandte Mathematik der Universität Erlangen-Nürnberg (IAM I). He thanks the Deutsche Akademischer Austauschdienst and Colciencias for their support, and the IAM I for its hospitality.

References

[1] J. Arango, *Existenz und Eindeutigkeitsaussagen über Lösungen von Randwertproblemen der nichtlinearen Membrantheorie*, Dissertation, Faculty of Science I, Univ. Erlangen–Nürnberg, July 1989, 121 pp.

[2] R.A. Clark and O.S. Narayanaswamy, *Nonlinear membrane problems for elastic shells of revolution*, Proc. Sympos. Theory of Shells (L.H. Donnell Anniversary Volume; D. Muster, ed.), Univ. of Houston Press, 1967, pp. 81-110.

[3] J. Arango, H. Grabmüller and A. Meisner, *Curved annular elastic membranes under partially vanishing surface load*, J. Appl. Math. Phys. (ZAMP) **49** (1998), 683-709.

[4] J. Arango and H. Grabmüller, *Wrinkle-free deformation of curved circular elastic membranes with an unloaded apex*, Comm. Appl. Analysis. **1** (1997), 435-441.

[5] A. Libai and J.G. Simmonds, *The nonlinear theory of elastic shells, One spatial dimension*, Academic Press, Boston, 1988.

[6] A. Meisner, *Nichtflache Kreisringmembranen unter teilweise verschwindender, axialsymmetrischer Vertikallast*, Diploma–Thesis, Faculty of Science I, Univ. Erlangen–Nürnberg, March 1995, 70 pp.

[7] E. Reissner, *On axisymmetrical deformation of thin shells of revolution*, Proc. Sympos. Appl. Math., Vol III, Amer. Math. Soc., New York, 1950, 27-52.

[8] D.J. Steigmann, *Proof of a conjecture in elastic membrane theory*, ASME J. Appl. Mech. **53** (1986), 955-956.

[9] J. J. Stoker, *Nonlinear Elasticity*, Gordon and Breach, New York, 1968.

Jaime Arango
Departamento de Matemáticas
Universidad del Valle, AA 25360
Cali, Colombia
E-mail: jarango@mafalda.univalle.edu.co

Hans Grabmüller
Institut für Angewandte Mathematik
Friedrich-Alexander-Universität Erlangen-Nürnberg
D-91058 Erlangen, Germany
E-mail: grabmue@am.uni-erlangen.de