Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol.31, No. 2, 225–239

L^2 harmonic forms and stability of hypersurfaces with constant mean curvature

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Abstract. We prove that a complete noncompact oriented strongly stable hypersurface M^n with cmc (constant mean curvature) H in a complete oriented manifold N^{n+1} with bi-Ricci curvature, satisfying b-$\tilde{\text{Ric}}(u, v) \geq \frac{n^2(n-5)}{4} H^2$ along M, admits no nontrivial L^2 harmonic 1-forms. This implies if M^n $(2 \leq n \leq 4)$ is a complete noncompact strongly stable hypersurface in hyperbolic space $H^{n+1}(-1)$ with cmc H $(H^2 \geq \frac{4(2n-1)}{(5-n)n^2})$, there exist no nontrivial L^2 harmonic 1-forms on M. We also classify complete oriented strongly stable surfaces with cmc H in a complete oriented manifold N^3 with scalar curvature $\tilde{S}$ satisfying $\inf_M \tilde{S} \geq -3H^2$.

Keywords: Riemannian manifold, Strongly stable hypersurface, Constant mean curvature, L^2 harmonic form.

Mathematical subject classification: 53C20, 58E15.

0. Introduction

The Bernstein conjecture states that any complete minimal graph in R^{n+1} is a hyperplane. It is known to be true for $n \leq 7$ and false for $n \geq 8$. In [Si], Simons studied the stability of minimal hypersurfaces and concluded the nonexistence of stable compact oriented minimal hypersurfaces in a space of positive Ricci curvature. Since then, there have been a lot of work in the stability of minimal and constant mean curvature hypersurfaces. For example, in [dCP] and [FS], do Carmo and Peng, and Fischer-Colbrie and Schoen independently proved that complete oriented stable minimal surfaces in R^3 are planes. But this result for higher dimensions is still not known. In [P], Palmer considered L^2 harmonic

Received 11 February 2000.

forms on a complete noncompact oriented stable minimal hypersurface M in R^{n+1} and proved that there exist no nontrivial L^2 harmonic 1-forms on such an M. According to Corollary 1 in [D](p.293), nonexistence of nontrivial L^2 harmonic 1-forms on M implies any codimension one cycle on M must disconnect M. Hence, Palmer's result gave some topological obstruction for the stability of M. This result has been recently generalized by Miyaoka ([M]) and Tanno([T]). In [M], Miyaoka obtained that there exist no nontrivial L^2-harmonic 1-forms on a complete noncompact oriented stable minimal hypersurface in a complete oriented manifold N^{n+1} with nonnegative sectional curvature. In [T], this result was shown to hold for minimal hypersurfaces in an ambient manifold N^{n+1} with nonnegative bi-Ricci curvature (See the definition of bi-Ricci curvature in §1). Also, in [L] , Li considered the case that M^n $(2 \leq n \leq 5)$ is a hypersurface with constant mean curvature. He proved that a complete noncompact oriented strongly stable hypersurface M^n $(2 \leq n \leq 5)$ with constant mean curvature in a complete oriented manifold N^{n+1} of non-negative bi-Ricci curvature admits no nontrivial L^2 harmonic 1-forms. On the other hand, Anderson ([A]) proved that there is a rich class of complete area-minimizing graphs in hyperbolic space $H^{n+1}(-1)$ with certain (allowable) prescribed asymptotic behavior and hence the classical Bernstein theorem fails in $H^{n+1}(-1)$. Thus, it is natural to consider complete stable hypersurfaces with nonzero constant mean curvature in $H^{n+1}(-1)$. For example, da Silveira ([S]) obtained a result, similar to that in[dCP] and [FS], on complete noncompact stable surfaces with constant mean curvature in $H^3(-1)$.

In this paper, we consider the relation between strong stability of hypersurfaces with constant mean curvature and existence of L^2 harmonic 1-forms on them. In Theorem 1, we prove that an n-dimensional complete noncompact oriented strongly stable hypersurface M^n with constant mean curvature H in a complete oriented manifold N^{n+1} with bi-Ricci curvature b-$\tilde{\text{Ric}}$, satisfying along M

$$\text{b-}\tilde{\text{Ric}}(u, v) \geq \frac{(n-5)n^2}{4} H^2 \tag{0.1}$$

admits no nontrivial L^2 harmonic 1-forms on M. In particular we obtain the result corresponding to Palmer's result in hyperbolic space $H^{n+1}(-1)$ for $2 \leq n \leq 4$ (Corollary 1). In theorem 2, we show that M^n has some geometric properties if M is a compact oriented strongly stable hypersurface with constant mean curvature H in a complete oriented manifold N^{n+1} with bi-Ricci curvature b-$\tilde{\text{Ric}}$ satisfying (0.1) and if M admits a nontrivial harmonic 1-form (i.e. the first Betti number $\beta_1(M) \neq 0$, by Hodge's theorem). Since not much is known about the stability of complete hypersurfaces with $H \neq 0$ in a general ambient manifold when $n \geq 3$, our results in theorem 1 and 2 give some topological obstruction to it. Theorem 3 is a generalized version of Fischer-Colbrie and Schoen's theorem on complete oriented stable minimal surfaces in a complete oriented 3-manifold of non-negative scalar curvature. In this theorem, we give the classification of complete strongly stable oriented surfaces with constant mean curvature H in a complete oriented manifold N^3 with scalar curvature $\tilde{S}$ satisfying $\inf_M \tilde{S} \geq -3H^2$. This theorem is also related to the result on complete weakly stable oriented surfaces with constant mean curvature H in a complete oriented manifold N^3 in [F]. In [F], Frensel proved the genus of M satisfies $g \leq 3$ when M is a compact oriented weakly stable surface with constant mean curvature H in a 3-dimensional complete oriented manifold N with Ricci curvature satisfying $\inf_M \tilde{\mathrm{Ric}}_N > -2H^2$. By comparing this result with our theorem 3 (i), we obtain that if $\inf_M \tilde{\mathrm{Ric}}_N > -2H^2$, then $g \leq 3$ when M is weakly stable; and $g \leq 1$ when M is strongly stable. Both results are sharp.

§1. Notations and statements of theorems

Let N^{n+1} be a complete oriented $(n+1)$-dimensional Riemannian manifold. Let $i: \quad M^n \to N^{n+1}$ be a complete oriented isometric immersion of a connected manifold M. Denote by $\tilde{\nabla}$ and ∇ the Levi-Civita connection of N and M respectively. Fix a point $p \in M$ and a local orthonormal frame field $\{e_1, e_2, \cdots, e_n, \mathcal{N}\}$ at p on N such that $\{e_1, e_2, \cdots, e_n\}$ are tangent fields and $\mathcal{N}$ is a unit normal vector field at p on M. Define a linear map $\mathcal{A} : T_pM \to T_pM$ by

$$\langle \mathcal{A}X, Y \rangle = \left\langle \tilde{\nabla}_X Y, \mathcal{N} \right\rangle,$$

where X, Y are tangent fields. Define mean curvature of M as

$$H = \frac{1}{n}(\mathrm{Tr}\,\mathcal{A}).$$

Recall that M is said to be strongly stable if

$$I(h) = \int_M \{|\nabla h|^2 - (\tilde{\mathrm{Ric}}(\mathcal{N}, \mathcal{N}) + \|A\|^2)h^2\} dv \geq 0, \tag{1.1}$$

for every C^∞ function $h : M \to R$ with compact support. Here ∇h is the gradient of h, and dv is the volume form.

M is said to be weakly stable if (1.1) is true for every C^∞ function $h : M \to R$ with compact support satisfying $\int_M f dv = 0$.

To state our result we need to recall the definitions of L^2 harmonic 1-form and bi-Ricci curvature for a Riemannian manifold.

Definition 1.1. *A 1-form ω on an n-dimensional complete oriented Riemannian manifold M is said to be L^2 harmonic if it satisfies*

$$\int_M \omega \wedge \star\omega < +\infty, \quad \Delta\omega = 0,$$

where $\Delta = d\delta + \delta d$ is the Hodge-Laplace operator on M.

By Proposition 1 in [Y], a 1-form ω is L^2 harmonic if and only if

$$\int_M \omega \wedge \star\omega < +\infty, \quad d\omega = 0, \quad \delta\omega = 0.$$

In a local orthonormal frame field $\{e_1, e_2, \cdots, e_n\}$ at $p \in M, d\omega = 0, \quad \delta\omega = 0$ are equivalent respectively to

$$(\nabla_i\omega_j)(p) = (\nabla_j\omega_i)(p), \quad i, j = 1, \cdots, n; \quad \sum_{i=1}^{n}(\nabla_i\omega_i)(p) = 0.$$

where

$$\nabla_i\omega_j = \nabla_{e_i}\omega_j, \ \omega = \sum_{i=1}^{n}\omega_j\varphi^j,$$

and $\{\varphi^1, \varphi^2, \cdots, \varphi^n\}$ is the coframe field dual to $\{e_1, e_2, \cdots, e_n\}$ (See [W], p.302).

Definition 1.2. *Given N^{n+1} an $(n+1)$-dimensional Riemannian manifold, and u, v two orthonormal tangent vectors, the bi-Ricci curvature in the directions u, v is defined as*

$$b\text{-}\tilde{\mathrm{Ric}}(u, v) = \tilde{\mathrm{Ric}}(u) + \tilde{\mathrm{Ric}}(v) - \tilde{K}(u, v).$$

Remark 1. From this definition we see that the nonnegativity of the sectional curvature of N^{n+1} implies the nonnegativity of the bi-Ricci curvature of N^{n+1}. If the dimension of N is 3, the bi-Ricci is equal to the scalar curvature $\tilde{S}$, where

$$\tilde{S} = \tilde{K}(e_1, e_2) + \tilde{K}(e_1, e_3) + \tilde{K}(e_2, e_3)$$

for an orthonormal base $\{e_1, e_2, e_3\}$ in T_pN. The concept of bi-Ricci curvature was introduced in [ShY]. In their paper, they gave an estimate of the diameter of a closed stable minimal hypersurface in $N^{n+1}(2 \leq n \leq 4)$, with b-$\tilde{\mathrm{Ric}}$ strictly positive. This is the generalization of a result of Schoen and Yau [ScY] (that is valid for $n = 2$) when scalar curvature is replaced by bi-Ricci curvature.

In our paper, we prove that

Theorem 1. *Let M^n be a complete noncompact oriented strongly stable hypersurface with constant mean curvature H in a manifold N^{n+1} with bi-Ricci curvature b-$\tilde{\mathrm{Ric}}$, satisfying along M*

$$b\text{-}\tilde{\mathrm{Ric}}(u, v) \geq \frac{(n-5)n^2}{4} H^2.$$

Then there exist no nontrivial L^2 harmonic 1-forms on M. In particular, any codimension one cycle on M disconnects M.

From this theorem, we have directly

Corollary 1. *Let M^n $(2 \leq n \leq 4)$ be a complete noncompact strong stable hypersurface with constant mean curvature H in hyperbolic space $H^{n+1}(-1)$. If*

$$H^2 \geq \frac{4(2n-1)}{(5-n)n^2},$$

there exist no nontrivial L^2 harmonic 1-forms on M.

Remark 2. The hypersurfaces satisfying the condition of theorem 1 indeed exist. For example the horospheres (with constant mean curvature $H = 1$) in hyperbolic space $H^3(-1)$ satisfy the condition of theorem 1.

Remark 3. Theorem 1 implies the conclusion in [L]. But in the case that $2 \leq n \leq 5$, b-$\tilde{\mathrm{Ric}}(u, v)$ is allowed to be nonpositive in our theorem, which results in corollary 1. Also, the result for $H = 0$ (i.e. M^n is a complete noncompact stable minimal surface) in theorem 1 was proved in [T].

Theorem 2. *Let M^n be a compact oriented strongly stable hypersurface with constant mean curvature H in a manifold N^{n+1} with bi-Ricci curvature b-$\tilde{\mathrm{Ric}}$ satisfying along M*

$$b\text{-}\tilde{\mathrm{Ric}}(u, v) \geq \frac{(n-5)n^2}{4} H^2.$$

If M^n admits a nontrivial harmonic 1-form ω, then ω is parallel, and

(1) *When $n = 2$, M is umbilic, and the scalar curvature of N^3 is a constant $\tilde{S} = -3H^2$ along M. If $H = 0$, M is totally geodesic.*

(2) *When $n \geq 3$, M has $n-1$ principal curvatures which are equal and the other one is different if $H \neq 0$. If $H = 0$, M is totally geodesic.*

Remark 4. When $n = 2$, the condition on the b-$\tilde{\mathrm{Ric}}$ curvature becomes $\tilde{S} \geq -3H^2$. For $H = 0$, the result in Theorem 2 was obtained in [T].

Theorem 3. *Let M^2 be a complete strongly stable oriented surface with constant mean curvature H in a 3- dimensional manifold N with scalar curvature $\tilde{S}$ satisfying on M $\inf_M \tilde{S} \geq -3H^2$. Then there are two possibilities*

(i) *M is compact. Then M is conformally equivalent to the sphere S^2 or the torus T^2. If M is conformally equivalent to T^2, M is umbilic, flat and $\tilde{S} = -3H^2$ along M. If $\tilde{S} > -3H^2$ along M, M is conformally equivalent to S^2.*

(ii) *M is noncompact. Then M is conformally equivalent to the complex plane C or the cylinder $C\backslash\{0\}$.*

Remark 5. When $H = 0$, Theorem 3 was proved in [FS]. In [F], Frensel obtained a result related to (ii) when M^2 is a complete noncompact weakly stable surface with constant mean curvature in a manifold N^3 with bounded geometry under the condition that $\inf_M \tilde{\mathrm{Ric}}_N \geq -2H^2$, where $\tilde{\mathrm{Ric}}_N(u) = \tilde{K}(v_1, u) + \tilde{K}(v_2, u)$, $v_1, v_2 \in T_pM$, u, v_1, v_2 orthonormal in T_pN. Also, in [M], Miyaoka gave a proof of Fischer-Colbrie and Schoen's result using harmonic 1-forms.

§2. Proofs of the theorems

First we prove an algebra lemma.

Lemma 2.1. *Let A be an $n \times n$ real symmetric matrix with* $\mathrm{Tr}\, A = nH$. *Then*

$$\|A\|^2\|X\|^2 - \|AX\|^2 + nH\langle AX, X\rangle \geq -\frac{n^2(n-5)H^2}{4}\|X\|^2, \tag{2.1}$$

for any n-vector $X \in R^n$. Equality holds if and only if $X = 0$ or $A = 0$ or the following case occurs:

(1) *When $n = 2$, $\lambda_1 = \lambda_2 = H$;*

(2) *When $n \geq 3$, there exists a unique $j \in \{1, 2, \cdots, n\}$ such that $\lambda_j = -\frac{n(n-3)}{2}H$, $|x_j| = \|X\| \neq 0$, and $\lambda_i = \frac{n}{2}H$, $x_i = 0$ for the other $i \neq j$, where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A, $X = \sum_{i=1}^n x_i\xi_i$, and $\xi_1, \xi_2, \cdots, \xi_n$ are the orthonormal eigenvectors of A corresponding to $\lambda_1, \cdots, \lambda_n$.*

Proof. Denote $F(A, X) = \|A\|^2\|X\|^2 - \|AX\|^2 + nH\langle AX, X\rangle$. We can choose an orthonormal basis $\xi_1, \cdots, \xi_n$ of R^n such that $A\xi_i = \lambda_i\xi_i$, $i = 1, \cdots, n$. Then we can express

$$X = \sum_{i=1}^{n} x_i\xi_i, \quad AX = \sum_{i=1}^{n}\lambda_i x_i\xi_i, \quad \langle Ax, x\rangle = \sum_{i=1}^{n}\lambda_i x_i^2,$$

and

$$F(A, X) = (\sum_{i=1}^{n}\lambda_i^2)\|X\|^2 - \sum_{i=1}^{n}\lambda_i^2 x_i^2 + nH\sum_{i=1}^{n}\lambda_i x_i^2.$$

Denote $y_i^2 = \|X\|^2 - x_i^2$, $1 \le i \le n$. Then $\sum_{i=1}^{n} y_i^2 = (n-1)\|X\|^2$. We have

$$\begin{aligned} F(A, X) &= \sum_{i=1}^{n}\lambda_i^2(\|X\|^2 - x_i^2) + nH\sum_{i=1}^{n}\lambda_i x_i^2 \\ &= \sum_{i=1}^{n}\lambda_i^2 y_i^2 + nH\sum_{i=1}^{n}\lambda_i(\|X\|^2 - y_i^2) \\ &= \sum_{i=1}^{n}(\lambda_i^2 - nH\lambda_i)y_i^2 + n^2H^2\|X\|^2 \\ &= \sum_{i=1}^{n}(\lambda_i - \frac{nH}{2})^2 y_i^2 - \frac{n^2H^2}{4}\sum_{i=1}^{n} y_i^2 + n^2H^2\|X\|^2 \qquad (2.2) \\ &= \sum_{i=1}^{n}(\lambda_i - \frac{nH}{2})^2 y_i^2 - \frac{n^2(n-5)H^2}{4}\|X\|^2 \\ &= \sum_{i=1}^{n}(\lambda_i - \frac{nH}{2})^2(\|X\|^2 - x_i^2) - \frac{n^2(n-5)H^2}{4}\|X\|^2 \\ &\ge -\frac{n^2(n-5)H^2}{4}\|X\|^2, \end{aligned}$$

It is easily seen that equality holds if and only if $(\lambda_i - \frac{nH}{2})^2(\|X\|^2 - x_i^2) = 0$, $i = 1, 2, \cdots, n$. Then equality holds if and only if either $\|X\| = 0$ or there are the other two possibilities,

(i) If $x_i^2 \neq \|X\|^2$, for all i, then $\lambda_1 = \cdots\lambda_n = \frac{nH}{2}$, it follows from $\sum_{i=1}^{n}\lambda_i = nH$ that n has to be 2 when $H \neq 0$; $\lambda_1 = \cdots\lambda_n = 0$ $(A = 0)$ when $H = 0$. This implies $A = 0$ or (1) is true in the lemma 2.1.

(ii) If for some j, $x_j^2 = \|X\|^2 \neq 0$, then $x_i = 0$ for the other $i \neq j$. Hence $\lambda_i = \frac{nH}{2}$, $i \neq j$, and $\lambda_j = -\frac{n(n-3)}{2}H$.

From the above, we see that the lemma holds true. □

Lemma 2.2 below might be known. Since we have not found a proper reference, we give here a proof for the sake of completeness.

Lemma 2.2. *Let ω be a 1-form on M^n. Then Kato's inequality holds on M in sense of distributions, i.e.,*

$$\|\nabla\|\omega\|\|^2 \leq \|\nabla\omega\|^2. \tag{2.3}$$

where, $\nabla\omega$ is covariant differential of ω and $\nabla\|\omega\|$ is the gradient of $\|\omega\|$. Moreover, equality holds if and only if $\nabla_i\omega_j(p) = \lambda_i(p)\omega_j(p)$, for all $p \in M$, where $\lambda_i(p)$ is a constant depending only on i and p. In addition if ω is a closed and co-closed 1-form, then equality implies that ω is parallel and $\|\omega\| \equiv$ constant.

Proof.

$$\|\nabla\|\omega\|\|^2(p) = \frac{1}{\|\omega\|^2}\sum_{i=1}^{n}(\sum_{j=1}^{n}\omega_j\nabla_i\omega_j)^2(p), \quad \|\nabla\omega\|^2 = \sum_{i,j=1}^{n}(\nabla_i\omega_j)^2(p).$$

It follows from Cauchy-Schwartz inequality that

$$\begin{aligned}(\sum_{j=1}^{n}\omega_j\nabla_i\omega_j)^2(p) &\leq (\sum_{j=1}^{n}\omega_j^2)(p)[\sum_{j=1}^{n}(\nabla_i\omega_j)^2](p) \\ &= \|\omega\|^2(p)[\sum_{j=1}^{n}(\nabla_i\omega_j)^2](p), \qquad \text{for all } i = 1, \cdots, n.\end{aligned} \tag{2.4}$$

Then

$$\sum_{i=1}^{n}(\sum_{j=1}^{n}\omega_j\nabla_i\omega_j)^2(p) \leq \|\omega\|^2(p)[\sum_{i,j=1}^{n}(\nabla_i\omega_j)^2](p), \tag{2.5}$$

namely

$$\|\nabla\|\omega\|\|^2(p) \leq \|\nabla\omega\|^2(p). \tag{2.6}$$

Observe that equality in (2.6) holds if and only if the equalities hold in (2.4) for all $i = 1, \cdots, n$. Then $\nabla_i\omega_j(p) = \lambda_i(p)\omega_j(p)$, where $\lambda_i(p)$ depends only on i and p.

In the following, suppose ω is closed and coclosed. Then

$$\sum_{i=1}^{n}(\nabla_i\omega_i)(p)=0,\quad (\nabla_i\omega_j)(p)=(\nabla_j\omega_i)(p),\ \forall i,j=1,\cdots,n. \tag{2.7}$$

We will prove $\nabla\omega=0$ if equality holds in (2.6). If $\lambda_i(p)=0$, for some i, then $\nabla_i\omega_j(p)=0$, for all j. If $\lambda_i(p)\neq 0$, for some i, it follows from the above, that $\lambda_i(p)\omega_j(p)=\lambda_j(p)\omega_i(p)$, and

$$\begin{aligned}
0&=\sum_{j=1}^{n}\nabla_j\omega_j(p)\\
&=\sum_{j=1}^{n}\lambda_j(p)\omega_j(p)\\
&=\sum_{j=1}^{n}\lambda_j(p)\cdot\frac{\lambda_j(p)}{\lambda_i(p)}\omega_i(p)\\
&=\frac{\sum_{j=1}^{n}\lambda_j^2(p)}{\lambda_i(p)}\omega_i(p).
\end{aligned}$$

Then $\omega_i(p)=0,\quad$ and $\nabla_j\omega_i(p)=\lambda_j(p)\omega_i(p)=0,$ for all j,

Thus $\nabla_i\omega_j(p)=\nabla_j\omega_i(p)=0,$ for all j. We conclude that $\nabla_i\omega_j(p)=0$, for all i, j. i.e. $(\nabla\omega)(p)=0$. This means that ω is parallel, and since in the sense of distribution,

$$\|\nabla\|\omega\|\|^2\leq\|\nabla\omega\|^2=0.$$

Thus, $\|\omega\|\equiv$constant. □

Let ω be a nontrivial L^2 harmonic 1-form on M. Suppose X is the vector field dual to ω. Then X is a nontrivial L^2-harmonic vector field on M. It is well known that

$$(-\Delta)\|\omega\|^2=2(\|\omega\|(-\Delta)\|\omega\|+\|\nabla\|\omega\|\|^2)$$

holds on M (In the sense of distributions at the zeros of ω), where Δ denotes Hodge-Laplace operator on M. Since ω is a harmonic 1-form, the Weitzenböck's formula yields (see [W], p.307),

$$(-\Delta)\|\omega\|^2=2(\mathrm{Ric}(X,X)+\|\nabla\omega\|^2).$$

Then

$$\|\omega\|(-\Delta)\|\omega\|=\mathrm{Ric}(X,X)+\|\nabla\omega\|^2-\|\nabla\|\omega\|\|^2=\mathrm{Ric}(X,X)+P(\omega), \tag{2.8}$$

where $P(\omega) = \|\nabla\omega\|^2 - \|\nabla\|\omega\|\|^2$. For any function $f \in C_o^\infty(M)$, we choose the test function $h = f\|\omega\|$ in (1.1). Then we have

$$
\begin{aligned}
I(h) &= \int_M -f\|\omega\|(-\Delta)(f\|\omega\|) - \tilde{\mathrm{Ric}}(\mathcal{N},\mathcal{N})f^2\|\omega\|^2 - \|\mathcal{A}\|^2 f^2\|\omega\|^2 \\
&= -\int_M f^2\{\|\omega\|(-\Delta)\|\omega\| + \tilde{\mathrm{Ric}}(\mathcal{N},\mathcal{N})\|\omega\|^2 + \|\mathcal{A}\|^2\|\omega\|^2\} \\
&\qquad - \int_M 2f\|\omega\|\langle\nabla f, \nabla\|\omega\|\rangle - \int_M f\|\omega\|^2(-\Delta)f, \\
&= -\int_M f^2\{\|\omega\|(-\Delta)\|\omega\| + \tilde{\mathrm{Ric}}(\mathcal{N},\mathcal{N})\|\omega\|^2 + \|\mathcal{A}\|^2\|\omega\|^2)\} \qquad (2.9) \\
&\qquad - \frac{1}{2}\int_M \langle\nabla f^2, \nabla\|\omega\|^2\rangle - \frac{1}{2}\int_M \|\omega\|^2\{(-\Delta)f^2 - 2\|\nabla f\|^2\}, \\
&= -\int_M f^2\{\|\omega\|(-\Delta)\|\omega\| + \tilde{\mathrm{Ric}}(\mathcal{N},\mathcal{N})\|\omega\|^2 + \|\mathcal{A}\|^2\|X\|^2\}+ \\
&\qquad + \int_M \|\omega\|^2\|\nabla f\|^2.
\end{aligned}
$$

It follows from (2.8) that (2.9) becomes

$$
\begin{aligned}
I(h) = -\int_M f^2\{&\mathrm{Ric}(X,X) + \|\nabla\omega\|^2 - \|\nabla\|\omega\|\|^2 + \tilde{\mathrm{Ric}}(\mathcal{N},\mathcal{N})\|\omega\|^2+ \\
&+ \|\mathcal{A}\|^2\|X\|^2\} + \int_M \|\omega\|^2\|\nabla f\|^2.
\end{aligned} \qquad (2.10)
$$

By the Gauss equation

$$
\mathrm{Ric}(X,X) = \tilde{\mathrm{Ric}}(X,X) - \langle\mathcal{A}X, \mathcal{A}X\rangle - \left\langle\tilde{\mathcal{R}}(X,\mathcal{N})X, \mathcal{N}\right\rangle + nH\langle\mathcal{A}X, X\rangle, \qquad (2.11)
$$

(2.10) becomes

$$
\begin{aligned}
I(h) &= -\int_M f^2\{\tilde{\mathrm{Ric}}(X,X) - \left\langle\tilde{\mathcal{R}}(X,\mathcal{N})X, \mathcal{N}\right\rangle - \|\mathcal{A}X\|^2 \\
&\qquad + nH\langle\mathcal{A}X, X\rangle + \tilde{\mathrm{Ric}}(\mathcal{N},\mathcal{N})\|X\|^2 + \|\mathcal{A}\|^2\|X\|^2 + \|\nabla\omega\|^2 \\
&\qquad - \|\nabla\|\omega\|\|^2\} + \int_M \|\omega\|^2\|\nabla f\|^2 \qquad (2.12) \\
&= -\int_M f^2\{\text{b-}\tilde{\mathrm{Ric}}(X,\mathcal{N}) + P(\omega) + \|A\|^2\|X\|^2 - \|AX\|^2+ \\
&\qquad nH\langle AX, X\rangle\} + \int_M \|\omega\|^2\|\nabla f\|^2,
\end{aligned}
$$

where

$$
\text{b-}\tilde{\mathrm{Ric}}(X,\mathcal{N}) = \tilde{\mathrm{Ric}}(X,X) + \tilde{\mathrm{Ric}}(\mathcal{N},\mathcal{N})\|X\|^2 - \left\langle\tilde{R}(X,\mathcal{N})X, \mathcal{N}\right\rangle.
$$

From Lemma 2.1, we have

$$I(h) \leq -\int_M f^2\{\text{b-}\tilde{\text{Ric}}(X, \mathcal{N}) + P(\omega) - \frac{n^2(n-5)H^2}{4}\|X\|^2\} + \\ + \int_M \|\omega\|^2 \|\nabla f\|^2. \tag{2.13}$$

We are now ready to prove our theorems.

Proof of Theorem 1. Assume for the sake of contradiction that there exists a nontrivial L^2 harmonic 1-form ω on M. Suppose X is the vector field dual to ω. We choose the C^∞ function f satisfying:

(1) $0 \leq f \leq 1$,

(2) $f \equiv 1$ on $B(\frac{r}{2})$, and $f \equiv 0$ outside $B(r)$,

(3) $\|\nabla f\| \leq \frac{C}{r}$, where C is a positive constant.

Then,

$$0 \leq I(h) \\ \leq -\int_{B(\frac{r}{2})} \{\text{b-}\tilde{\text{Ric}}(X, \mathcal{N}) + P(\omega) - \frac{n^2(n-5)H^2}{4}\|X\|^2\} + \frac{C}{r^2}\int_{B(r)} \|\omega\|^2, \tag{2.14}$$

where, by Kato's inequality, $P(\omega) = \|\nabla\omega\|^2 - \|\nabla\|\omega\|\|^2 \geq 0$. By letting $r \to \infty$, the second term of (2.14) tends to zero because of L^2 integrability of ω. By hypothesis, along M

$$\text{b-}\tilde{\text{Ric}}(u, v) \geq \frac{n^2(n-5)}{4}H^2.$$

Hence the integrand of the first term of (2.14) must be identically to zero and equalities must hold in all inequalities we have used. Thus,

$$P(\omega) = 0, \tag{2.15}$$

$$\text{b-}\tilde{\text{Ric}}(X, \mathcal{N}) - \frac{n^2(n-5)H^2}{4}\|X\|^2 = 0, \tag{2.16}$$

$$\|A\|^2\|X\|^2 - \|AX\|^2 + nH\langle AX, X\rangle = -\frac{n^2(n-5)}{4}H^2\|X\|^2. \tag{2.17}$$

From $P(\omega)=0$ and Lemma 2.2, it follows that $\|\omega\|$ =constant and ω is parallel. Hence, by (2.8), $\operatorname{Ric}(X,X)=0$. By Gauss equation (2.11), we have

$$\tilde{\operatorname{Ric}}(X,X)-\|AX\|^2-\left\langle \tilde{R}(X,\mathcal{N})X,\mathcal{N}\right\rangle+nH\langle AX,X\rangle=0. \tag{2.18}$$

Then, by the definition of bi-Ricci curvature, (2.18) becomes

$$\begin{aligned} &\text{b-}\tilde{\operatorname{Ric}}(X,\mathcal{N})+(\|A\|^2\|X\|^2-\|AX\|^2+nH\langle AX,X\rangle)\\ &\quad-\tilde{\operatorname{Ric}}(\mathcal{N},\mathcal{N})\|X\|^2-\|A\|^2\|X\|^2=0, \end{aligned} \tag{2.19}$$

By (2.16) and (2.17), we obtain

$$\tilde{\operatorname{Ric}}(\mathcal{N},\mathcal{N})\|X\|^2+\|A\|^2\|X\|^2=0,$$

By $\|X\|^2=\|\omega\|^2=Constant\neq 0$, we have

$$\|A\|^2+\tilde{\operatorname{Ric}}(\mathcal{N},\mathcal{N})=0. \tag{2.20}$$

For any tangent vector ξ on M, from Gauss equation (2.11) which holds also for any ξ,

$$\begin{aligned} \operatorname{Ric}(\xi,\xi)&=\tilde{\operatorname{Ric}}(\xi,\xi)-\|A\xi\|^2-(\tilde{R}(\xi,\mathcal{N})\xi,\mathcal{N})+nH\langle A\xi,\xi\rangle\\ &=nH\langle A\xi,\xi\rangle-\|A\xi\|^2+\|A\|^2\|\xi\|^2+\\ &\qquad+\text{b-}\tilde{\operatorname{Ric}}(\xi,\mathcal{N})-\tilde{\operatorname{Ric}}(\mathcal{N},\mathcal{N})\|\xi\|^2-\|A\|^2\|\xi\|^2 \end{aligned}$$

By Lemma 2.1,

$$\operatorname{Ric}(\xi,\xi)\geq-\frac{n^2(n-5)}{4}H^2\|\xi\|^2+\text{b-}\tilde{\operatorname{Ric}}(\xi,\mathcal{N})-\tilde{\operatorname{Ric}}(\mathcal{N},\mathcal{N})\|\xi\|^2-\|A\|^2\|\xi\|^2$$

By hypothesis and (2.20), we obtain

$$\operatorname{Ric}(\xi,\xi)\geq-\tilde{\operatorname{Ric}}(\mathcal{N},\mathcal{N})\|\xi\|^2-\|A\|^2\|\xi\|^2=0.$$

We conclude from [Y] that the volume of M is infinite because M is complete noncompact with nonnegative Ricci curvature. Since ω is an L^2 1-form, $\|\omega\|$ =constant and $\operatorname{vol}(M)=\infty$, we have $\|\omega\|$ has to be zero which is a contradiction. □

Proof of Theorem 2. Suppose that ω is a nontrivial harmonic 1-form on M^n and X is the vector field dual to ω. We can choose $f \equiv 1$ in (2.13). Similar to the proof of Theorem 1, the strong stability of M implies :

$$\|\nabla\omega\|^2 = \|\nabla\|\omega\|\|^2,$$

$$\text{b-}\tilde{\text{Ric}}(X, N) - \frac{n^2(n-5)}{4}H^2\|X\|^2 = 0,$$

$$\|A\|^2\|X\|^2 - \|AX\|^2 + nH\langle AX, X\rangle = -\frac{n^2(n-5)}{4}H^2\|X\|^2.$$

Then the conclusion can be obtained from Lemma 2.1 and 2.2. Observe that when $n = 2$, b-$\tilde{\text{Ric}}$ of N is equal to the scalar curvature of N. □

Proof of Theorem 3. Suppose that $\{e_1, e_2, \mathcal{N}\}$ is an orthonormal frame field of T_pN at $p \in M$, where $\{e_1, e_2, \}$ is an orthonormal frame field of T_pM and $\mathcal{N}$ is a normal vector field at $p \in M$. Since b-$\tilde{\text{Ric}}(e_1, e_2) = \tilde{S}$, then (2.13) becomes

$$0 \le I(h) \le -\int_M f^2\{\tilde{S}\|X\|^2 + P(\omega) + 3H^2\|X\|^2\} + \int_M \|\omega\|^2\|\nabla f\|^2. \tag{2.21}$$

(i)When M is compact, choose $f \equiv 1$ in (2.21)

$$0 \le I(h) \le -\int_M (\tilde{S}\|X\|^2 + P(\omega) + 3H^2\|X\|^2). \tag{2.22}$$

If $\|\omega\| \equiv 0$, i.e. there exists no nontrivial harmonic 1-form on M, then the first Betti number $\beta_1(M) = 0$. This implies M must be conformally equivalent to a sphere ([FK], p.73, Corollary 1). Otherwise, i.e. there exists a nontrivial harmonic 1-form on M, then it follows that, from Theorem 2, M is umbilic, $\tilde{S}$ is a constant $\tilde{S} = -3H^2$ along M, and ω is parallel. Parallelity of ω implies $K \equiv 0$, i.e. M is flat. By the Gauss-Bonnet formula, $\mathcal{X}(M) = 0$. Thus M has to be conformally equivalent to a torus. ([FK], p.90, Corollary 1).

(ii) When M is noncompact, choose f as in (2.14). Then (2.21) becomes

$$0 \le I(h) \le -\int_{B(\frac{r}{2})} \{\tilde{S}\|X\|^2 + P(\omega) + 3H^2\|X\|^2\} + \frac{c}{r^2}\int_{B(r)} \|\omega\|^2. \tag{2.23}$$

Let $\tilde{M}$ be the universal covering of M. Then $\tilde{M}$ is conformally equivalent to the complex plane C or the disk D. Since the strongly stability of surfaces with constant mean curvature is defined by compactly supported variation, $\tilde{M}$ is still a complete noncompact strongly stable surface in N (The argument is similar to that in [dCP]). Hence by Theorem 1, there exist no nontrivial L^2 harmonic 1-forms on $\tilde{M}$. But we know there exist nontrivial L^2 harmonic 1-forms on disk D ([D]), thus $\tilde{M}$ must be conformally equivalent to C. Hence M is conformally equivalent to either C or $C\backslash\{0\}$. ([FK], p.193). □

Acknowledgment. The author wishes to thank Professor Manfredo do Carmo for his encouragement and orientation.

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