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Asymptotic behavior of a tagged particle in simple exclusion processes

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Abstract. We review in this article central limit theorems for a tagged particle in the simple exclusion process. In the first two sections we present a general method to prove central limit theorems for additive functional of Markov processes. These results are then applied to the case of a tagged particle in the exclusion process. Related questions, such as smoothness of the diffusion coefficient and finite dimensional approximations, are considered in the last section.

Keywords: central limit theorem, interacting particles systems, tagged particle.

1 Introduction

In the early 80's Kipnis and Varadhan [6] proved an invariance principle for the position of a tagged particle in a symmetric simple exclusion process in equilibrium. Their proof relies on a general central limit theorem for additive functionals of reversible Markov processes. Time reversibility and translation invariance of the system are the basic ingredients of this method, which in principle can be applied to any system with these two symmetries. Later it has been extended to non-reversible processes that satisfy a sector condition, [21], or a graded sector condition, [19].

The effective diffusion matrix of the limiting Brownian Motion is a function $D(\alpha)$ of the density α of the particles. These diffusion coefficients are usually expressed in terms of integrals of time correlation functions (Green-Kubo formulas), or as infinite dimensional variational formulas. They also appear in the diffusive equations that govern the non-equilibrium evolution of the conserved quantities of the system. In order to have regular strong solution to these diffusive

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equations it is important to establish the regularity of these diffusion coefficients as functions of the conserved quantities.

In this article we review some recent results on the central limit theorem for a tagged particle in the simple exclusion process and on the smoothness of the diffusion coefficient.

In the next section, we present a general method to prove a central limit theorem for an additive functional of Markov ergodic processes and show that the proof is reduced to the verification of a bound in $\mathcal{H}_{-1}$ for the solution of the resolvent equation. In the third section we show that this estimate can be deduced if the generator of the Markov process has some properties, called the sector and the graded sector condition. In the fourth section, we apply these results to prove a central limit theorem for the tagged particle for mean zero exclusion processes and for asymmetric exclusion processes in dimension $d \geq 3$. In the last section we show that the covariance matrix depends smoothly on the density of particles and we present some extensions.

2 Central limit theorem for Markov processes

The purpose of this section is to find conditions which guarantee a central limit theorem for an additive functional of a Markov process. The idea is to represent the additive functional as the sum of a martingale with a small term, that vanishes in the limit, and to use the well known central limit theorem for martingales that we now recall.

On a probability space $(\Omega, P, \mathcal{F})$, consider a square-integrable martingale $\{M_t : t \geq 0\}$ which vanishes at time 0 and denote by $< M, M >_t$ its quadratic variation.

Lemma 2.1. *Assume that the increments of the martingale M_t are stationary and that its quadratic variation converges in $L^1(P)$ to some positive constant σ^2: for every $t \geq 0$, $n \geq 1$ and $0 \leq s_0 < \cdots < s_n$,*

$$(M_{s_1} - M_{s_0}, \ldots, M_{s_n} - M_{s_{n-1}}) \;=\; (M_{t+s_1} - M_{t+s_0}, \ldots, M_{t+s_n} - M_{t+s_{n-1}})$$

in distribution and

$$\lim_{t\to\infty} E\Big[\Big|\frac{< M, M >_t}{t} - \sigma^2\Big|\Big] \;=\; 0\,.$$

Then, $M_t/\sqrt{t}$ converges in distribution to a mean zero Gaussian law with variance σ^2.

In this statement E stands for the expectation with respect to P. The proof of this result is a simple consequence of the well known central limit theorem for sums of stationary and ergodic square integrable martingale differences.

It follows from the stationarity assumption that $\sigma^2 = E[M_1^2]$ because

$$\begin{aligned} E[< M, M >_n] &= \sum_{0\le j<n} E[< M, M >_{j+1} - < M, M >_j] \\ &= \sum_{0\le j<n} E[(M_{j+1} - M_j)^2] \;=\; nE[M_1^2]\,. \end{aligned}$$

Consider now a Markov process X_t taking values in a complete separable metric space E endowed with its Borel σ-algebra $\mathcal{E}$. Assume that there exists a stationary ergodic state π. Denote by L the generator of the Markov process in $L^2(\pi)$ and by $\mathcal{D}(L)$ its domain. Let L^* be the adjoint of L in $L^2(\pi)$. Since π is stationary, L^* is itself the generator of a Markov process. Assume that there exists a core $C \subset \mathcal{D}(L) \cap \mathcal{D}(L^*)$ for both generators L and L^*. We denote by $\mathbb{P}_\pi$ the measure on the path space $D(\mathbb{R}_+, E)$ induced by the Markov process X_t starting from π and by $\mathbb{E}_\pi$ expectation with respect to $\mathbb{P}_\pi$.

Fix a function $V: E \to \mathbb{R}$ in $L^2(\pi)$. The object of this section is to find conditions on V which guarantee a central limit theorem for

$$\frac{1}{\sqrt{t}} \int_0^t V(X_s)\, ds\,.$$

Assume first that there exists a solution f in $\mathcal{D}(L)$ of the Poisson equation

$$-Lf \;=\; V\,. \tag{2.1}$$

In this case a central limit theorem follows from the central limit theorem for martingales stated in Lemma 2.1. Indeed, since f belongs to the domain of the generator,

$$M_t \;=\; f(X_t) \;-\; f(X_0) \;-\; \int_0^t (Lf)(X_s)\, ds$$

is a martingale with quadratic variation given by

$$< M, M >_t = \int_0^t ds \left\{ (Lf^2(X_s) - 2f(X_s)(Lf)(X_s) \right\}\,.$$

Since f is the solution of the Poisson equation (2.1), we may write the additive functional in terms of the martingale M_t:

$$\frac{1}{\sqrt{t}} \int_0^t V(X_s)\, ds \;=\; \frac{M_t}{\sqrt{t}} \;+\; \frac{f(X_0) - f(X_t)}{\sqrt{t}}\,.$$

Since f belongs to $L^2(\pi)$ and the measure is stationary, $[f(X_0) - f(X_t)]/\sqrt{t}$ vanishes in $L^2(\mathbb{P}_\pi)$ as $t \uparrow \infty$. It remains to check that the martingale M_t satisfies the assumptions of Lemma 2.1.

The increments of the martingale M_t are stationary because X_t under $\mathbb{P}_\pi$ is itself a stationary Markov process. On the other hand, since π is ergodic, in view of the formula for the quadratic variation of the martingale in terms of f, $t^{-1} < M, M >_t$ converges in $L^1(\mathbb{P}_\pi)$, as $t \uparrow \infty$, to $E_\pi[Lf^2 - 2f(Lf)] = 2 < f, (-L)f >_\pi$, where $< \cdot, \cdot >_\pi$ stands for the inner product in $L^2(\pi)$. Since the martingale M_t vanishes at time 0, by Lemma 2.1, $t^{-1/2} M_t$, and therefore $t^{-1/2} \int_0^t V(X_s)\, ds$, converges in distribution to a Gaussian law with mean zero and variance $\sigma^2 = 2 < f, (-L)f >_\pi$.

Of course the existence of a solution f in $L^2(\pi)$ of the Poisson equation (2.1) is too strong and should be weakened. To state the main result of this section, we need to introduce the Sobolev spaces $\mathcal{H}_1$, $\mathcal{H}_{-1}$ associated to the generator L.

2.1 The Sobolev spaces $\mathcal{H}_1$, $\mathcal{H}_{-1}$

Consider the semi-norm $\| \cdot \|_1$ defined in the domain of the generator $\mathcal{D}(L)$ by

$$\|f\|_1^2 = < f, (-L)f >_\pi \ . \tag{2.2}$$

Let $\sim_1$ be the equivalence relation in $\mathcal{D}(L)$ defined by $f \sim_1 g$ if $\|f - g\|_1 = 0$ and denote by $\mathcal{G}_1$ the normed space $(\mathcal{D}(L)|_{\sim_1}, \| \cdot \|_1)$. It is easy to see from definition (2.2) that the norm $\| \cdot \|_1$ satisfies the parallelogram law so that $\mathcal{H}_1$, the completion of $\mathcal{G}_1$ with respect to the norm $\| \cdot \|_1$, is a Hilbert space with inner product $< \cdot, \cdot >_1$ given by polarization:

$$< f, g >_1 = \frac{1}{4}\Big\{\|f + g\|_1^2 - \|f - g\|_1^2\Big\} .$$

Notice that in this definition only the symmetric part of the generator, $S = (1/2)(L + L^*)$, plays a role because

$$\|f\|_1^2 = < f, (-L)f >_\pi = < f, (-S)f >_\pi \ .$$

It is also easy to check that for any f, g in the domain of L,

$$< f, g >_1 = < f, (-S)g >_\pi$$

and that $\|c\|_1 = 0$ for any constant c.

Associated to the Hilbert space $\mathcal{H}_1$, is the dual space $\mathcal{H}_{-1}$ defined as follows. For f in $L^2(\pi)$, let

$$\|f\|_{-1}^2 \;=\; \sup_{g\in\mathcal{D}(L)} \left\{2 < f, g >_\pi - \|g\|_1^2\right\} .$$

Denote by $\mathcal{G}_{-1}^0$ the subspace of $L^2(\pi)$ of all functions with finite $\|\cdot\|_{-1}$ norm. Introduce in $\mathcal{G}_{-1}^0$ the equivalence relation $\sim_{-1}$ by stating that $f \sim_{-1} g$ if $\|f - g\|_{-1} = 0$ and denote by $\mathcal{G}_{-1}$ the normed space $(\mathcal{G}_{-1}^0|_{\sim_{-1}}, \|\cdot\|_{-1})$. The completion of $\mathcal{G}_{-1}$ with respect to the norm $\|\cdot\|_{-1}$, denoted by $\mathcal{H}_{-1}$, is again a Hilbert space with inner product defined through polarization.

Before we state the main result of this section, we summarize some properties of the spaces $\mathcal{H}_1$, $\mathcal{H}_{-1}$ that will be repeatedly used. It is easy to check from the variational formula for the $\mathcal{H}_{-1}$ norm that for every function f in $\mathcal{D}(L)$ and every functions g in $L^2(\pi) \cap \mathcal{H}_{-1}$

$$\Big| < f, g >_\pi \Big| \;\le\; \|f\|_1 \, \|g\|_{-1} \, . \tag{2.3}$$

The same variational formula permits to show that a function in $\mathcal{D}(L)$ belongs to $\mathcal{H}_{-1}$ if and only if there exists a finite constant C_0 such that

$$< f, g >_\pi \;\le\; C_0 \|g\|_1 \tag{2.4}$$

for every g in $\mathcal{D}(L)$. In this case, $\|f\|_{-1} \le C_0$. Finally, it is not difficult to show (cf. [15], Lemma 2.5) that a function f belongs to $\mathcal{H}_{-1}$ if and only if there exists h in the domain of $\sqrt{-S}$ such that $\sqrt{-S}h = f$. In this case $\|f\|_{-1} = \|h\|_0$, which means that

$$\|f\|_{-1}^2 \;= < h, h >_\pi = < (-S)^{-1/2} f, (-S)^{-1/2} f >_\pi = < (-S)^{-1} f, f >_\pi$$

because S is symmetric.

We may now state the main result of this section. Fix a function V in $L^2(\pi) \cap \mathcal{H}_{-1}$, $\lambda > 0$ and consider the resolvent equation

$$\lambda f_\lambda \;-\; L f_\lambda \;=\; V \, . \tag{2.5}$$

Theorem 2.2. *Suppose that*

$$\sup_{0<\lambda\le 1} \|L f_\lambda\|_{-1} \;<\; \infty \, . \tag{2.6}$$

Then, $t^{-1/2}\int_0^t V(X_s)\,ds$ converges in law to a mean zero Gaussian distribution with variance

$$\sigma^2 = \lim_{\lambda\to 0} \|f_\lambda\|_1^2 \, .$$

Notice that

$$\sup_{0<\lambda\le 1} \|Lf_\lambda\|_{-1} \;<\; \infty \quad \text{if and only if} \quad \sup_{0<\lambda\le 1} \|\lambda f_\lambda\|_{-1} \;<\; \infty \tag{2.7}$$

because V belongs to $\mathcal{H}_{-1}$.

The proof of this theorem is divided into two parts. We first compute in the next subsection the limiting variance of $t^{-1/2}\int_0^t V(X_s)\,ds$ and show that it is bounded by a multiple of the $\mathcal{H}_{-1}$ norm of V. If the generator is symmetric, i.e., if π is a reversible measure for the Markov process, the limiting variance is equal to $2\|V\|_{-1}^2$. In subsection 2.3, we prove that a central limit theorem holds for $t^{-1/2}\int_0^t V(X_s)\,ds$, V satisfying the assumptions of Theorem 2.2, provided the following two conditions are satisfied:

$$\lim_{\lambda\to 0} \lambda\,\|f_\lambda\|_0^2 \;=\; 0 \quad \text{and} \quad \lim_{\lambda\to 0} \|f_\lambda - f\|_1 \;=\; 0$$

for some f in $\mathcal{H}_1$. Finally, in subsection 2.4, we show that the bound (2.6) implies the previous two conditions.

2.2 The limiting variance

We estimate in this subsection the limiting variance of the integral $\int_0^t W(X_s)ds$ for a mean zero function W in $L^2(\pi)$. Let

$$\sigma(W)^2 \;=\; \limsup_{t\to\infty} \mathbb{E}_\pi\Big[\Big(\frac{1}{\sqrt{t}}\int_0^t W(X_s)\,ds\Big)^2\Big]\,.$$

Denote by P_t the semi-group of the Markov process X_t. Since π is invariant, a change of variables shows that for each fixed t the expectation on the right hand side of the previous formula is equal to

$$\begin{aligned}\frac{1}{t}\int_0^t ds\int_0^t dr\,\mathbb{E}_\pi[W(X_{|s-r|})W(X_0)] &= \frac{1}{t}\int_0^t ds\int_0^t dr\; < P_{|s-r|}W, W >_\pi \\ &= 2\int_0^t ds\,[1-(s/t)] < P_s W, W >_\pi \, .\end{aligned}$$

Denote by a^+ the positive part of a: $a^+ = \max\{0, a\}$, so that

$$\sigma(W)^2 \;=\; 2\limsup_{t\to\infty}\int_0^\infty ds\,[1-(s/t)]^+ < P_s W, W >_\pi \, . \tag{2.8}$$

In the general case, it is not clear whether this limsup is in fact a limit or wheter it is finite without some restrictions on W. However, in the reversible case, the sequence is increasing in t because $< P_s W, W >_\pi = < P_{s/2} W, P_{s/2} W >_\pi$ is a positive function. Hence, in the reversible case, by the monotone convergence theorem,

$$\begin{aligned}\sigma(W)^2 &= 2 \lim_{t\to\infty} \int_0^\infty ds\, [1-(s/t)]^+ < P_s W, W >_\pi \\ &= 2 \int_0^\infty ds \ < P_s W, W >_\pi \ .\end{aligned}$$

In the general case, one can show that $\sigma(W)^2$ defined in (2.8) is finite provided the function W belongs to the Sobolev space $\mathcal{H}_{-1}$: there exists a universal constant C_0 such that

$$\sigma(W)^2 \ \le \ C_0 \, \| W \|_{-1}^2 \tag{2.9}$$

for all functions W in $\mathcal{H}_{-1} \cap L^2(\pi)$. The main difference between $\sigma(W)^2$ and $\| W \|_{-1}^2$ is that while in the second term only the symmetric part of the generator is involved, in the first term the full generator appears. Formally,

$$\begin{aligned}\sigma(W)^2 &= 2 \int_0^\infty < P_s W, W >_\pi \ = \ 2 < W, (-L)^{-1} W >_\pi \\ &= 2 < W, [(-L)^{-1}]^s W >_\pi \ , \\ \text{and} \quad \| W \|_{-1}^2 &= < W, (-S)^{-1} W >_\pi \ .\end{aligned}$$

In this formula and below, M^s represents the symmetric part of the operator M and $A = (1/2)(L - L^*)$ is the asymmetric part of the generator L. Since,

$$\Big\{ [(-L)^{-1}]^s \Big\}^{-1} \ = \ -S \ + \ A^*(-S)^{-1} A \ \ge \ -S \ ,$$

we have that $[(-L)^{-1}]^s \le (-S)^{-1}$, from what it follows that $\sigma(W)^2 \le 2\| W \|_{-1}^2$. We now present a rigorous proof of this informal argument.

Lemma 2.3. *Fix $T > 0$ and a function $f \colon [0,T] \times E \to \mathbb{R}$. Assume that $f(t,\cdot)$ belongs to $L^2(\pi)$ for each $0 \le t \le T$ and that $f(\cdot, x)$ is smooth for each x in E. There exists a finite universal constant C_0 such that,*

$$\mathbb{E}_\pi \Big[\sup_{0 \le t \le T} \Big(\int_0^t f(s, X_s)\, ds \Big)^2 \Big] \ \le \ C_0 \int_0^T ds \, \| f(s,\cdot) \|_{-1}^2 \ .$$

The proof of this lemma relies on a representation formula for an additive functional of a Markov process by forward and backward martingales. The proof can be found in [3], Lemma 4.3 or in [19], Theorem 2.2. This lemma applied to the function $f(s,\cdot) = W(\cdot)$ proves (2.9).

Remark 2.4. *We just proved the existence of a universal finite constant C_0 such that $\sigma(W)^2 \le C_0 \|W\|_{-1}^2$. It has been proved ([18], Theorem 1.1) that for the asymmetric simple exclusion process in dimensions* 1 *and* 2*, there are local functions W* **not** *in $\mathcal{H}_{-1}$ and for which $\sigma(W)^2 < \infty$ We suspect that a central limit theorem holds for these functions with the usual scaling $t^{-1/2}$.*

2.3 The resolvent equation

We assume from now on that V belongs to $\mathcal{H}_{-1} \cap L^2(\pi)$. Taking inner product with respect to f_λ on both sides of the resolvent equation (2.5), we get that

$$\lambda < f_\lambda, f_\lambda >_\pi + \|f_\lambda\|_1^2 = < V, f_\lambda > . \tag{2.10}$$

Since f_λ belongs to $\mathcal{D}(L)$ and V belongs to $L^2(\pi) \cap \mathcal{H}_{-1}$, by Schwarz inequality (2.3), the right hand side is bounded above by $\|V\|_{-1}\|f_\lambda\|_1$. In particular, $\|f_\lambda\|_1 \le \|V\|_{-1}$ so that $\|V\|_{-1}\|f_\lambda\|_1 \le \|V\|_{-1}^2$ and

$$\lambda < f_\lambda, f_\lambda >_\pi + \|f_\lambda\|_1^2 \le \|V\|_{-1}^2 . \tag{2.11}$$

Therefore,

$$\lim_{\lambda\to 0} \lambda f_\lambda = 0 \text{ in } L^2(\pi) \quad \text{and} \quad \sup_{0<\lambda\le 1} \|f_\lambda\|_1 < \infty . \tag{2.12}$$

The purpose of this subsection is to show that a central limit theorem for $t^{-1/2}\int_0^t V(X_s)ds$ holds provided we can prove the following stronger statements:

$$\lim_{\lambda\to 0} \lambda\, \|f_\lambda\|_0^2 = 0 \quad \text{and} \quad \lim_{\lambda\to 0} \|f_\lambda - f\|_1 = 0 \tag{2.13}$$

for some f in $\mathcal{H}_1$.

Proposition 2.5. *Fix a function V in $L^2(\pi) \cap \mathcal{H}_{-1}$ and assume (2.13). Then, $t^{-1/2}\int_0^t V(X_s)\,ds$ converges in law to a mean zero Gaussian distribution with variance*

$$\sigma(V)^2 = 2 \lim_{\lambda\to 0} \|f_\lambda\|_1^2 .$$

It follows from (2.13) and (2.10) that

$$\sigma(V)^2 = 2 \lim_{\lambda \to 0} \| f_\lambda \|_1^2 = 2 \lim_{\lambda \to 0} < V, f_\lambda >_\pi . \tag{2.14}$$

The idea of the proof of Proposition 2.5 is again to express $\int_0^t V(X_s)\, ds$ as the sum of a martingale and a term which vanishes in the limit. This is proved in (2.16) and Lemma 2.7 below. We start taking advantage of the resolvent equation (2.5) to build up a martingale closely related to $\int_0^t V(X_s)\, ds$.

For each fixed $\lambda > 0$, let M_t^λ be the martingale defined by

$$M_t^\lambda = f_\lambda(X_t) - f_\lambda(X_0) - \int_0^t (L f_\lambda)(X_s)\, ds$$

so that

$$\int_0^t V(X_s)\, ds = M_t^\lambda + f_\lambda(X_0) - f_\lambda(X_t) + \lambda \int_0^t f_\lambda(X_s)\, ds . \tag{2.15}$$

Lemma 2.6. *The martingale M_t^λ converges in $L^2(\pi)$, as $\lambda \downarrow 0$, to a martingale M_t and $\lambda \int_0^t f_\lambda(X_s)\, ds$ vanishes.*

Proof. We prove first that M_t^λ is a Cauchy sequence in $L^2(\pi)$. Indeed, for λ, $\lambda' > 0$, since π is an invariant state, the expectation of the quadratic variation of the martingale $M_t^\lambda - M_t^{\lambda'}$ is

$$\begin{aligned} &\mathbb{E}_\pi\Big[\int_0^t ds \Big\{ L f_{\lambda,\lambda'}(X_s)^2 - 2 f_{\lambda,\lambda'}(X_s) L f_{\lambda,\lambda'}(X_s) \Big\} \Big] \\ &\quad = 2t < f_\lambda - f_{\lambda'}, (-L) f_\lambda(X_s) - f_{\lambda'} >_\pi = 2t \, \| f_\lambda - f_{\lambda'} \|_1^2 . \end{aligned}$$

In this formula, $f_{\lambda,\lambda'} = f_\lambda - f_{\lambda'}$. By assumption (2.13), f_λ converges in $\mathcal{H}_1$. In particular, M_t^λ is a Cauchy sequence in $L^2(\pi)$ and converges to a martingale M_t. This proves the first statement. The second assertion of the lemma follows from (2.12) and Schwarz inequality. □

It follows from this result and from identity (2.15) that $f_\lambda(X_t) - f_\lambda(X_0)$ also converges in $L^2(\pi)$ as $\lambda \downarrow 0$. Denote this limit by R_t so that

$$\int_0^t V(X_s)\, ds = M_t + R_t . \tag{2.16}$$

Lemma 2.7. *$t^{-1/2}R_t$ vanishes in $L^2(\pi)$ as $t \uparrow \infty$.*

Proof. Putting together equation (2.15) with (2.16), we get that

$$\frac{R_t}{\sqrt{t}} = \frac{1}{\sqrt{t}}\Big\{M_t^\lambda - M_t + f_\lambda(X_0) - f_\lambda(X_t) + \lambda \int_0^t f_\lambda(X_s)\, ds\Big\} . \tag{2.17}$$

We consider separately each term on the right hand side of this expression.

Since M_t^λ converges in $L^2(\pi)$ to M_t,

$$\frac{1}{t}\,\mathbb{E}_\pi\Big[\Big(M_t^\lambda - M_t\Big)^2\Big] = \frac{1}{t}\lim_{\lambda'\to 0}\mathbb{E}_\pi\Big[\Big(M_t^\lambda - M_t^{\lambda'}\Big)^2\Big] .$$

In the previous Lemma, we computed the expectation of the quadratic variation of the martingale $M_t^\lambda - M_t^{\lambda'}$. This calculation shows that the previous expression is equal to

$$\lim_{\lambda'\to 0}\|f_\lambda - f_{\lambda'}\|_1^2 = \|f_\lambda - f\|_1^2 .$$

In the last step we used assumption (2.13) which states that f_λ converges in $\mathcal{H}_1$ to some f.

We now turn to the second term in (2.17). Since π is invariant, the expectation of its square is bounded by

$$2t^{-1}\mathbb{E}_\pi\Big[f_\lambda(X_t)^2\Big] + 2t^{-1}\mathbb{E}_\pi\Big[f_\lambda(X_0)^2\Big] = 4t^{-1}\|f_\lambda\|_0^2 .$$

On the other hand, by Schwarz inequality, the expectation of the square of the third term in (2.17) is bounded by $t\lambda^2\|f_\lambda\|_0^2$.

Putting together all previous estimates, we obtain that

$$\frac{1}{t}\,\mathbb{E}_\pi[R_t^2] \le 3\,\|f_\lambda - f\|_1^2 + 3(4t^{-1} + \lambda^2)\,\|f_\lambda\|_0^2$$

for all $\lambda > 0$. Set $\lambda = t^{-1}$ to conclude the proof of the lemma in view of hypotheses (2.13). □

We may now prove Proposition 2.5. Recall equation (2.16). By the previous lemma the second term on the right hand side divided by $\sqrt{t}$ vanishes in $L^2(\pi)$ (and therefore in probability) as $t \uparrow \infty$. On the other hand, by the martingale convergence theorem, $M_t/\sqrt{t}$ converges in law to a mean zero Gaussian distribution with variance

$$\begin{aligned}\sigma^2 &= \mathbb{E}_\pi[M_1^2] = \lim_{\lambda\to 0}\mathbb{E}_\pi[(M_1^\lambda)^2] \\ &= \lim_{\lambda\to 0}\mathbb{E}_\pi[< M^\lambda, M^\lambda >_1] = 2\lim_{\lambda\to 0}\|f_\lambda\|_1^2 .\end{aligned}$$

The last identity follows from the computation of the expectation of the quadratic variation of the martingale M_t^λ performed in the proof of Lemma 2.6.

2.4 An $\mathcal{H}_{-1}$ estimate

In the previous subsection we showed that the central limit theorem for the additive functional $t^{-1/2}\int_0^t V(X_s)ds$ follows from conditions (2.13) if V belongs to $L^2(\pi)\cap\mathcal{H}_{-1}$. In the present section we prove that (2.13) follows from the bound (2.6) on the solution of the resolvent equation (2.5).

Lemma 2.8. *Fix a function V in $\mathcal{H}_{-1}\cap L^2(\pi)$ and denote by $\{f_\lambda,\ \lambda>0\}$ the solution of the resolvent equation (2.5). Assume that* $\sup_{\lambda>0}\|Lf_\lambda\|_{-1}\le C_0$ *for some finite constant C_0. Then, there exists f in $\mathcal{H}_1$ such that*

$$\lim_{\lambda\to 0}\lambda < f_\lambda, f_\lambda > \;=\; 0 \quad \textit{and} \quad \lim_{\lambda\to 0} f_\lambda \;=\; f$$

strongly in $\mathcal{H}_1$.

Proof. We already proved in (2.11) that

$$\sup_{0<\lambda\le 1}\|f_\lambda\|_1 \;\le\; \|V\|_{-1} \quad \text{and} \quad \sup_{0<\lambda\le 1}\lambda < f_\lambda, f_\lambda > \;\le\; \|V\|_{-1}^2\,.$$

In particular, λf_λ converges to 0 in $L^2(\pi)$, as $\lambda\downarrow 0$.

Since $\sup_{\lambda>0}\|Lf_\lambda\|_{-1}$ is bounded, for any sequence $\lambda_n\downarrow 0$, there exists a subsequence, still denoted by λ_n, such that Lf_{λ_n} converges weakly in $\mathcal{H}_{-1}$ to some function U. We claim that the weak limit is unique and equal to $-V$. Indeed, fix a function g in $L^2(\pi)\cap\mathcal{H}_1$. Since g belongs to $\mathcal{H}_1$ and Lf_{λ_n} converges weakly to U, $< U, g >_\pi = \lim_n < Lf_{\lambda_n}, g >_\pi$. On the other hand, since f_λ is the solution of the resolvent equation, $\lim_n < Lf_{\lambda_n}, g > = - < V, g > + \lim_n < \lambda_n f_{\lambda_n}, g >$. This latter expression is equal to $- < V, g >$ because g belongs to $L^2(\pi)$ and λf_λ converges strongly to 0 in $L^2(\pi)$, as $\lambda\downarrow 0$. Thus, $< U, g > = - < V, g >$ for all functions g in $L^2(\pi)\cap\mathcal{H}_1$. Since this set is dense in $\mathcal{H}_1$, $U=-V$, proving the claim.

In the same way, since $\sup_{\lambda>0}\|f_\lambda\|_1$ is bounded, each sequence $\lambda_n\downarrow 0$ has a subsequence still denoted by λ_n, for which f_{λ_n} converges weakly in $\mathcal{H}_1$ to some function, denoted by W. We claim that any such limit W satisfies the relation $\|W\|_1^2 = < W, V >$. To check this identity, through convex combinations of the sequences f_{λ_n}, Lf_{λ_n}, we obtain sequences v_n, Lv_n which converge strongly to W, $-V$, respectively. On the one hand, since v_n (resp. Lv_n) converges strongly in $\mathcal{H}_1$ (resp. $\mathcal{H}_{-1}$) to W (resp. $-V$), $< v_n, Lv_n >$ converges to $- < W, V >$. On the other hand, since $- < v_n, Lv_n > = \|v_n\|_1^2$, it converges to $\|W\|_1^2$. Therefore, $\|W\|_1^2 = < W, V >$.

We have now all elements to prove the first part of the lemma. Suppose by contradiction that $\lambda < f_\lambda, f_\lambda >$ does not converge to 0 as $\lambda \downarrow 0$. In this case there exists $\varepsilon > 0$ and a subsequence $\lambda_n \downarrow 0$ such that $\lambda_n < f_{\lambda_n}, f_{\lambda_n} > \geq \varepsilon$ for all n. We have just shown the existence of a sub-subsequence $\lambda_{n'}$ for which $f_{\lambda_{n'}}$ converges weakly in $\mathcal{H}_1$ to some W satisfying the relation $< W, V >= \|W\|_1^2$. Since f_λ is solution of the resolvent equation,

$$\begin{aligned} \limsup_{n'\to\infty} \|f_{\lambda_{n'}}\|_1^2 \;&\leq\; \limsup_{n'\to\infty} \left\{\lambda_{n'}\|f_{\lambda_{n'}}\|^2 \;+\; \|f_{\lambda_{n'}}\|_1^2\right\} \\ &=\; \limsup_{n'\to\infty} < f_{\lambda_{n'}}, V > \;=\; < W, V > \;=\; \|W\|_1 \;\leq\; \limsup_{n'\to\infty} \|f_{\lambda_{n'}}\|_1^2 \,. \end{aligned}$$

This contradicts the fact that $\lambda_n < f_{\lambda_n}, f_{\lambda_n} > \geq \varepsilon$ for all n, so that $\lim_{\lambda\to 0}\lambda < f_\lambda, f_\lambda >= 0$.

It follows also from the previous argument that $f_{\lambda_{n'}}$ converges to W strongly in $\mathcal{H}_1$. In particular, all sequences λ_n have subsequences $\lambda_{n'}$ for which $f_{\lambda_{n'}}$ converges strongly in $\mathcal{H}_1$. To show that f_λ converges strongly, it remains to check uniqueness of the limit.

Consider two decreasing sequences λ_n, μ_n, vanishing as $n \uparrow \infty$. Denote by W_1, W_2 the strong limit in $\mathcal{H}_1$ of f_{λ_n}, f_{μ_n}, respectively. Since f_λ is the solution of the resolvent equation,

$$< \lambda_n f_{\lambda_n} - \mu_n f_{\mu_n}, f_{\lambda_n} - f_{\mu_n} >_\pi \;+ \|f_{\lambda_n} - f_{\mu_n}\|_1^2 \;=\; 0$$

for all n. Since f_{λ_n}, f_{μ_n} converges strongly to W_1, W_2 in $\mathcal{H}_1$,

$$\lim_{n\to\infty} \|f_{\lambda_n} - f_{\mu_n}\|_1^2 \;=\; \|W_1 - W_2\|_1^2 \,.$$

On the other hand, since $\lambda\|f_\lambda\|^2$ vanishes as $\lambda \downarrow 0$,

$$\begin{aligned} &\lim_{n\to\infty} < \lambda_n f_{\lambda_n} - \mu_n f_{\mu_n}, f_{\lambda_n} - f_{\mu_n} >_\pi \\ &\quad= -\lim_{n\to\infty} \left\{ < \lambda_n f_{\lambda_n}, f_{\mu_n} >_\pi \;+\; < \mu_n f_{\mu_n}, f_{\lambda_n} >_\pi \right\} . \end{aligned}$$

Each of these terms vanish as $n \uparrow \infty$. Indeed,

$$\lambda_n < f_{\lambda_n}, f_{\mu_n} >_\pi \;=\; \lambda_n < f_{\lambda_n}, f_{\mu_n} - W_2 >_\pi \;+ \lambda_n < f_{\lambda_n}, W_2 >_\pi$$

By Schwarz inequality (2.3), the first term on the right hand side is bounded above by $\|\lambda_n f_{\lambda_n}\|_{-1}\|f_{\mu_n} - W_2\|_1$, which vanishes because λf_λ is bounded in $\mathcal{H}_{-1}$ and f_{μ_n} converges to W_2 in $\mathcal{H}_1$. The second term of the previous formula also vanishes in the limit because W_2 belongs to $\mathcal{H}_1$ and λf_λ converges weakly to 0 in $\mathcal{H}_{-1}$. This concludes the proof of the lemma. □

Theorem 2.2 follows from this lemma and Proposition 2.5.

3 Some examples

Fix a function V in $L^2(\pi) \cap \mathcal{H}_{-1}$. In this section we present three conditions which guarantee that the solution f_λ of the resolvent equation (2.5) satisfies the bound (2.6).

3.1 Reversibility

Assume that the generator L is self-adjoint in $L^2(\pi)$. In this case, by Schwarz inequality,

$$\Big| < Lf, g >_\pi \Big| \;\le\; \|f\|_1 \, \|g\|_1 \, .$$

Therefore, in view of the variational formula (2.4) for the $\mathcal{H}_{-1}$ norm, for any f in $L^2(\pi) \cap \mathcal{H}_1$, Lf belongs to $\mathcal{H}_{-1}$ and

$$\|Lf\|_{-1} \;\le\; \|f\|_1 \, .$$

In particular, in the reversible case (2.6) follows from the elementary estimate (2.12).

3.2 Sector condition

Assume now that the generator L satisfies the sector condition

$$< f, Lg >_\pi^2 \;\le\; C_0 < f, (-L)f >_\pi < g, (-L)g >_\pi \tag{3.1}$$

for some finite constant C_0 and every functions f, g in the domain of the generator. In view of (2.4), for any function g in $\mathcal{D}(L)$,

$$\|Lg\|_{-1} \;\le\; C_0 \|g\|_1$$

and condition (2.6) follows from estimate (2.12).

The previous inequality states that the generator L is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$. Since S, the symmetric part of the generator, has certainly this property, L is bounded if and only if A, the asymmetric part of the generator, is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$, i.e., if

$$< g, A^*(-S)^{-1}Ag >_\pi \;=\; \|Ag\|_{-1}^2 \;\le\; C_0 \|g\|_1^2 \;=\; C_0 < g, (-S)g >_\pi$$

for all functions g in $\mathcal{D}(L)$. Hence, the sector condition requires that

$$A^*(-S)^{-1}A \;\le\; C_0(-S)$$

for some finite constant C_0. This inequality states that the asymmetric part of the generator can be estimated by the symmetric part. Furthermore, in this case, in view of the computations performed just after (2.9),

$$(-S) \le (-S) + A^*(-S)^{-1}A \le (1+C_0)(-S)$$

so that

$$C_1^{-1}\sigma(V)^2 \le \|V\|_{-1}^2 \le C_1\sigma(V)^2$$

for some finite constant C_1. This means that under the sector condition, the limiting variance is finite if and only if the function belongs to $\mathcal{H}_{-1}$.

3.3 Graded sector condition

Now, instead of assuming that the generator satisfies a sector condition on the all space, we decompose $L^2(\pi)$ as a direct sum of orthogonal spaces $\mathcal{A}_n$ and assume that on each subspace $\mathcal{A}_n$, the generator satisfies a sector condition with a constant which may be different on each $\mathcal{A}_n$.

Assume that $L^2(\pi)$ can be decomposed as a direct sum $\oplus_{n\ge 0}\mathcal{A}_n$ of orthogonal spaces. Functions in $\mathcal{A}_n$ are said to have degree n. For $n \ge 0$, denote by π_n the orthogonal projection on $\mathcal{A}_n$ so that

$$f = \sum_{n\ge 0}\pi_n f \quad \text{and} \quad \pi_n f \in \mathcal{A}_n$$

for all $n \ge 0$, f in $L^2(\pi)$.

Suppose that that the generator L keeps the degree of a function or changes it by one: $L\colon \mathcal{D}(L)\cap\mathcal{A}_n \to \mathcal{A}_{n-1}\cup\mathcal{A}_n\cup\mathcal{A}_{n+1}$. Denote by L_- (resp. L_+ and L_0) the piece of the generator that decreases (resp. increases and keeps) the degree of a function. Assume that L_0 can be decomposed as $R_0 + B_0$, where $-R_0$ is a positive operator bounded by $-C_0L$ for some positive constant C_0:

$$0 \le < f, (-R_0)f >_\pi \le C_0 < f, (-L)f >_\pi \tag{3.2}$$

for all functions f in $\mathcal{D}(L)$.

Since $-R_0$ is a positive operator, repeating the steps of Subsection 2.1 with R_0 in place of L, we define the Sobolev spaces $\mathcal{H}_{0,1}$, $\mathcal{H}_{0,-1}$ and the norms $\|\cdot\|_{0,1}$, $\|\cdot\|_{0,-1}$ associated to R_0. Since R_0 keeps the degree of a function,

$$\begin{aligned}\|f\|_{0,1}^2 &= < f, (-R_0)f >_\pi = < \sum_{n\ge 0}\pi_n f, (-R_0)\sum_{n\ge 0}\pi_n f >_\pi \\ &= \sum_{n\ge 0} < \pi_n f, (-R_0)\pi_n f >_\pi = \sum_{n\ge 0}\|\pi_n f\|_{0,1}^2 .\end{aligned}$$

for all functions f in the domain of the generator. By the same reasons, for a function f in $L^2(\pi)$,

$$\|f\|_{0,-1}^2 = \sup_{g\in\mathcal{D}(L)} \{2 < f, g >_\pi - \|g\|_{0,1}^2\} = \sum_{n\geq 0} \|\pi_n f\|_{0,-1}^2 .$$

In terms of the new norm $\|\cdot\|_{0,1}$, (3.2) translates to

$$\|f\|_{0,1} \leq \sqrt{C_0}\, \|f\|_1$$

for all function f in the domain of the generator and some finite constant C_0. It follows from this inequality and from the variational formula for the $\mathcal{H}_{-1}$, $\mathcal{H}_{0,-1}$ norms that

$$\|f\|_{-1} \leq \sqrt{C_0}\, \|f\|_{0,-1} \tag{3.3}$$

for all function f in $L^2(\pi)$ and the same finite constant C_0.

Suppose now that a sector condition holds on each subspace $\mathcal{A}_n$ with a constant which depends on n: there exists $\beta < 1$ and a finite constant C_0 such that

$$\begin{aligned} < f, (-L_+)g >_\pi^2 &\leq C_0 n^{2\beta} < f, (-R_0)f >_\pi < g, (-R_0)g >_\pi , \\ < g, (-L_-)f >_\pi^2 &\leq C_0 n^{2\beta} < f, (-R_0)f >_\pi < g, (-R_0)g >_\pi \end{aligned} \tag{3.4}$$

for all g in $\mathcal{D}(L) \cap \mathcal{A}_n$ and f in $\mathcal{D}(L) \cap \mathcal{A}_{n+1}$. It follows from the previous assumptions and from the variational formula for the $\|\cdot\|_{-1,0}$ norm that

$$\|L_+ g\|_{0,-1} \leq \sqrt{C_0}\, n^\beta \|g\|_{0,1} , \quad \|L_- f\|_{0,-1} \leq \sqrt{C_0}\, n^\beta \|f\|_{0,1} \tag{3.5}$$

for all g in $\mathcal{D}(L) \cap \mathcal{A}_n$ and f in $\mathcal{D}(L) \cap \mathcal{A}_{n+1}$. The proof of Lemma 3.1 below, due to [14], [19], shows that the restriction $\beta < 1$ is crucial.

Lemma 3.1 *Let V be a function in $L^2(\pi)$ such that*

$$\sum_{n\geq 0} n^{2k} \|\pi_n V\|_{0,-1}^2 < \infty .$$

Denote by f_λ the solution of the resolvent equation (2.5). There exists a finite constant C_1 depending only on β, k and C_0 such that

$$\sum_{n\geq 0} n^{2k} \|\pi_n f_\lambda\|_{0,1}^2 \leq C_1 \sum_{n\geq 0} n^{2k} \|\pi_n V\|_{0,-1}^2 .$$

Proof. Consider an increasing sequence $\{t_n : n \geq 0\}$, to be fixed later, and denote by $T\colon L^2(\pi) \to L^2(\pi)$ the operator which is a multiple of the identity on each subspace $\mathcal{A}_n$:

$$Tf = \sum_{n\geq 0} t_n \pi_n f \ .$$

Apply T to both sides of the resolvent equation and take the inner product with respect to Tf_λ on both sides of the identity to obtain that

$$\begin{aligned}\lambda < Tf_\lambda, Tf_\lambda >_\pi \; &- \; < Tf_\lambda, LTf_\lambda >_\pi \\ =< Tf_\lambda, TV > \; &- \; < Tf_\lambda, [L,T] f_\lambda >_\pi \ .\end{aligned}$$

In this formula, $[L,T]$ stands for the comutator of L and T and is given by $LT - TL$. By assumption (3.2), the left hand side is bounded below by

$$C_0^{-1} < Tf_\lambda, (-R_0)Tf_\lambda >_\pi = C_0^{-1} \|Tf_\lambda\|_{0,1}^2 = C_0^{-1} \sum_{n\geq 0} t_n^2 \, \|\pi_n f_\lambda\|_{0,1}^2 \ .$$

Let $\delta > 0$. We now estimate the scalar product $< Tf_\lambda, [L,T] f_\lambda >_\pi$ in terms of $\|Tf_\lambda\|_{0,1}^2$. Since T commutes with any operator that keeps the degree, $[L,T] = [L_+ + L_-, T]$. To fix ideas, consider the operator $[L_+, T]$, the other expression being estimated in a similar way. Since L_+ increases the degree by one, by definition of the comutator,

$$\pi_n [L_+, T] f = L_+ T \pi_{n-1} f - T L_+ \pi_{n-1} f = (t_{n-1} - t_n) L_+ \pi_{n-1} f$$

for all functions f in $\mathcal{D}(L)$. Therefore,

$$\begin{aligned} < Tf_\lambda, [L_+, T] f_\lambda >_\pi &= \sum_{n\geq 0} < \pi_n T f_\lambda, \pi_n [L_+, T] f_\lambda >_\pi \\ &= \sum_{n\geq 0} (t_{n-1} - t_n) t_n < \pi_n f_\lambda, L_+ \pi_{n-1} f_\lambda >_\pi \ .\end{aligned}$$

By (3.4) and since the sequence t_n is increasing, the previous expression is bounded below by

$$\begin{aligned} &\sum_{n\geq 0} (t_n - t_{n-1}) t_n C_0 n^\beta \|\pi_n f_\lambda\|_{0,1} \|\pi_{n-1} f_\lambda\|_{0,1} \\ &\quad \leq \frac{1}{2} \sum_{n\geq 0} (t_n - t_{n-1}) t_n C_0 n^\beta \|\pi_n f_\lambda\|_{0,1}^2 \\ &\quad + \frac{1}{2} \sum_{n\geq 0} (t_n - t_{n-1}) t_n C_0 n^\beta \|\pi_{n-1} f_\lambda\|_{0,1}^2 \ .\end{aligned}$$

Since $\beta < 1$, there exists $n_1 = n_1(C_0, \beta, \delta, k)$ such that

$$C_0 n^\beta \Big\{ 1 - \frac{(n-1)^{2k}}{n^{2k}} \Big\} \;\le\; \delta \,, \quad C_0 n^\beta \Big\{ \frac{n^{2k}}{(n-1)^{2k}} - 1 \Big\} \frac{n^{2k}}{(n-1)^{2k}} \;\le\; \delta$$

for all $n \ge n_1$. Fix $n_2 > n_1$ and set $t_n = n_1^{2k} \mathbf{1}\{n < n_1\} + n^{2k} \mathbf{1}\{n_1 \le n \le n_2\} + n^{2k} \mathbf{1}\{n > n_2\}$. With this definition, we obtain that the previous expression is bounded by

$$\delta \sum_{n \ge 0} t_n^2 \| \pi_n f_\lambda \|_{0,1}^2 \;=\; \delta \, \| T f_\lambda \|_{0,1}^2 \;.$$

It remains to estimate $< T f_\lambda, T V >_\pi$. By (2.3), and since $2ab \le A^{-1} a^2 + A b^2$ for every $A > 0$,

$$\begin{aligned} < T f_\lambda, T V >_\pi \;&=\; \sum_{n \ge 0} t_n^2 < \pi_n f_\lambda, \pi_n V >_\pi \;\le\; \sum_{n \ge 0} t_n^2 \, \| \pi_n f_\lambda \|_{0,1} \, \| \pi_n V \|_{0,-1} \\ &\le\; \delta \sum_{n \ge 0} t_n^2 \, \| \pi_n f_\lambda \|_{0,1}^2 \;+\; \delta^{-1} \sum_{n \ge 0} t_n^2 \, \| \pi_n V \|_{0,-1}^2 \\ &=\; \delta \| T f_\lambda \|_{0,1}^2 \;+\; \delta^{-1} \| T V \|_{0,-1}^2 \;. \end{aligned}$$

Putting together the previous three estimates, we obtain that

$$C_0^{-1} \| T f_\lambda \|_{0,1}^2 \;\le\; 3\delta \, \| T f_\lambda \|_{0,1}^2 \;+\; \delta^{-1} \| T V \|_{0,-1}^2$$

so that

$$\| T f_\lambda \|_{0,1}^2 \;\le\; 16 C_0^2 \, \| T V \|_{0,-1}^2$$

if we choose $\delta = 1/4C_0$. Recall the definition of the sequence t_n. This estimate holds uniformly in n_2. Let $n_2 \uparrow \infty$ and definite T' as the operator associated to the sequence t'_n, where $t'_n = n^{2k} \mathbf{1}\{n \ge n_1\} + n_1^{2k} \mathbf{1}\{n < n_1\}$, to deduce that

$$\begin{aligned} \sum_{n \ge 0} n^{2k} \| \pi_n f_\lambda \|_{0,1}^2 \;&\le\; \sum_{n \ge 0} (t'_n)^2 \| \pi_n f_\lambda \|_{0,1}^2 \\ &\le\; 16 C_0^2 \sum_{n \ge 0} (t'_n)^2 \| \pi_n V \|_{0,-1}^2 \\ &\le\; 16 C_0^2 n_1^{2k} \sum_{n \ge 0} n^{2k} \| \pi_n V \|_{0,-1}^2 \;. \end{aligned}$$

To conclude the proof of the lemma, it remains to recall that we fixed $\delta = 1/4C_0$ and that $n_1 = n_1(C_0, k, \beta, \delta)$. □

Assume now that $L_0 = R_0 + B_0$ satisfies a sector condition on each subset $\mathcal{A}_n$:

$$< g, (-L_0) f >_\pi^2 \le C_0 n^{2\gamma} < f, (-R_0) f >_\pi < g, (-R_0) g >_\pi \tag{3.6}$$

for some $\gamma > 0$ and all functions f, g in $\mathcal{D}(L) \cap \mathcal{A}_n$. Notice that we do not impose any condition on γ. By the variational formula for the norm $\| \cdot \|_{0,-1}$,

$$\|L_0 f\|_{0,-1} \le \sqrt{C_0}\, n^\gamma \|f\|_{0,1} \tag{3.7}$$

for all functions f in $\mathcal{D}(L) \cap \mathcal{A}_n$.

Lemma 3.2. *Suppose that the generator L satisfies (3.2), (3.4) and (3.6). Fix a function V such that*

$$\sum_{n\ge 0} n^{2k} \|\pi_n V\|_{0,-1}^2 < \infty$$

for some $k \ge (\beta \vee \gamma)$. Let f_λ be the solution of the resolvent equation (2.5). Then,

$$\sup_{0<\lambda\le 1} \|L f_\lambda\|_{0,-1} < \infty .$$

Proof. It follows from (3.3) that

$$\|L f_\lambda\|_{-1}^2 \le \|L f_\lambda\|_{0,-1}^2 = \sum_{n\ge 0} \|\pi_n L f_\lambda\|_{0,-1}^2 . \tag{3.8}$$

Fix $n \ge 0$. Since $\pi_n L f_\lambda = L_- \pi_{n+1} f_\lambda + L_0 \pi_n f_\lambda + L_+ \pi_{n-1} f_\lambda$, by (3.5), (3.7),

$$\begin{aligned} \|\pi_n L f_\lambda\|_{0,-1} &\le \|L_-\pi_{n+1} f_\lambda\|_{0,-1} + \|L_0 \pi_n f_\lambda\|_{0,-1} + \|L_+ \pi_{n-1} f_\lambda\|_{0,-1} \\ &\le C_0 n^\beta \|\pi_{n+1} f_\lambda\|_{0,1} + C_0 n^\gamma \|\pi_n f_\lambda\|_{0,1} + C_0 n^\beta \|\pi_{n-1} f_\lambda\|_{0,1} . \end{aligned}$$

In particular, by Schwarz inequality, by Lemma 3.1 and since $k \ge (\beta \vee \gamma)$, the right hand side of (3.8) is bounded above by

$$C_1 \sum_{n\ge 0} n^{2k} \|\pi_n f_\lambda\|_{0,1}^2 \le C_1 \sum_{n\ge 0} n^{2k} \|\pi_n V\|_{0,-1}^2$$

for some finite constant C_2 depending only on C_0, β and γ. This proves the lemma. □

Therefore, to prove a central limit theorem for an additive functional of a Markov process, it is enough to check whether its generator satisfies the graded sector conditions (3.2), (3.4) and (3.6).

4 Tagged particle in simple exclusion process

We prove in this section a central limit theorem for the position of a tagged particle in the simple exclusion process with the method presented in the previous section.

Among the simplest and most widely studied interacting particle systems is the simple exclusion process. It represents the evolution of random walks on the lattice $\mathbb{Z}^d$ with a hard-core interaction that prevents more than one particle per site and may be described as follows. Fix a probability measure $p(\cdot)$ on $\mathbb{Z}^d$ and distribute particles on the lattice in such a way that each site is occupied by at most one particle. Particles evolve on $\mathbb{Z}^d$ as random walks with translation–invariant transition probability $p(x, y) = p(y - x)$. Each time a particle tries to jump over a site already occupied, the jump is suppressed to respect the exclusion rule.

This informal description corresponds to a Markov process on $\mathcal{X}_d = \{0, 1\}^{\mathbb{Z}^d}$ whose generator L is given by

$$(Lf)(\eta) \;=\; \sum_{x,z\in\mathbb{Z}^d} \eta(x)[1 - \eta(x + z)]\, p(z)\, [f(\sigma^{x,x+z}\eta) - f(\eta)]\,. \qquad (4.1)$$

Here, η stands for a configuration of $\mathcal{X}_d$ so that $\eta(x)$ is equal to 1 (resp. 0) if the site x is occupied (resp. unoccupied) for the configuration η. f is a cylinder function, which means that it depends on η only through a finite number of coordinates, and $\sigma^{x,y}\eta$ is the configuration obtained from η by interchanging the occupation variables $\eta(x)$, $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) \;=\; \begin{cases} \eta(z) & \text{if } z \neq x, y\,, \\ \eta(y) & \text{if } z = x\,, \\ \eta(x) & \text{if } z = y\,. \end{cases}$$

The simple exclusion process is said to be symmetric if the transition probability is symmetric ($p(z) = p(-z)$) and to be mean zero if the transition probability is not symmetric but has zero average: $\sum_z z p(z) = 0$. All other cases are said to be asymmetric

To avoid degeneracies, we assume that the transition probability $p(\cdot)$ is irreducible in the sense that the set $\{x : p(\pm x) > 0\}$ generates $\mathbb{Z}^d$, i.e., that for any pair of sites x, y in $\mathbb{Z}^d$, there exists $M \geq 1$ and a sequence $x = x_0, \dots, x_M = y$ such that $p(x_{i+1} - x_i) + p(x_i - x_{i+1}) > 0$ for $0 \leq i \leq M - 1$. We also suppose that the transition probability is of finite range: there exists A_0 in $\mathbb{N}$ such that $p(z) = 0$ for all sites z outside the cube $[-A_0, A_0]^d$.

Notice that the total number of particles is conserved by the dynamics. This conservation is reflected in the existence of a one parameter family of invariant measures. For $0 \le \alpha \le 1$, denote by ν_α the Bernoulli product measure of parameter α. This means that under ν_α the variables $\{\eta(x),\ x \in \mathbb{Z}^d\}$ are independent with marginals given by

$$\nu_\alpha\{\eta(x) = 1\} \;=\; \alpha \;=\; 1 - \nu_\alpha\{\eta(x) = 0\}\,.$$

An elementary computation shows that the Bernoulli measures $\{\nu_\alpha\,, 0 \le \alpha \le 1\}$ are invariant for simple exclusion processes. Denote by L^* the generator defined by (4.1) associated to the transition probability $p^*(x) = p(-x)$. L^* is the adjoint of L in $L^2(\nu_\alpha)$. In particular, symmetric simple exclusion processes are self-adjoint with respect to each ν_α.

For $t \ge 0$, denote by η_t the state at time t of the Markov process. Among all particles, tag one of them and denote by X_t its position at time t. X_t by itself is not a Markov process because its evolution depends on the position of the other particles. However, (η_t, X_t) is a Markov process on $\mathcal{X}_d \times \mathbb{Z}^d$.

Denote by $\{\tau_x,\ x \in \mathbb{Z}^d\}$ the group of translations on $\mathcal{X}_d$. For x, y in $\mathbb{Z}^d$ and a configuration η in $\mathcal{X}_d$, $(\tau_x\eta)(y) = \eta(x+y)$. The action of the translation group is naturally extended to functions and measures.

Denote by ξ_t the state of the process at time t as seen form the tagged particle: $\xi_t = \tau_{X_t}\eta_t$. Notice that the origin is always occupied because $\xi_t(0) = (\tau_{X_t}\eta_t)(0) = \eta_t(X_t) = 1$. In particular, we can consider either ξ as a configuration of $\mathcal{X}_d$ with a particle at the origin or ξ as a configuration of $\mathcal{X}_d^* = \{0, 1\}^{\mathbb{Z}^d_*}$, where $\mathbb{Z}^d_* = \mathbb{Z}^d - \{0\}$. We adopt here the latter convention. It is also not difficult to show that ξ_t is a Markov process on $\mathcal{X}_d^*$ with generator $\mathcal{L}$ given by

$$\begin{aligned}(\mathcal{L}f)(\xi) = &\sum_{x,y\in\mathbb{Z}^d_*} p(y-x)\xi(x)[1-\xi(y)][f(\sigma^{x,y}\xi) - f(\xi)] \\ &+ \sum_{z\in\mathbb{Z}^d_*} p(z)[1-\xi(z)][f(\theta_z\xi) - f(\xi)]\,.\end{aligned} \tag{4.2}$$

The first part of the generator takes into account the jumps of the environment, while the second one corresponds to jumps of the tagged particle. In the above formula, $\theta_y\xi$ stands for the configuration where the tagged particle, sitting at the origin, is first transferred to site z and then all the configuration is translated by $-z$: for all y in $\mathbb{Z}^d_*$

$$(\theta_z\xi)(y) = \begin{cases} \xi(z) & \text{if } y = -z, \\ \xi(y+z) & \text{for } y \ne -z. \end{cases}$$

For $0 \le \alpha \le 1$, denote by ν_α^* the Bernoulli product measure on $\mathcal{X}_d^*$, by $\mathcal{L}^*$ the generator defined by (4.2) with the transition probability $p^*(y) = p(-y)$ in place of p and by $< \cdot, \cdot >_\mu$ the inner product in $L^2(\mu)$, for a probability measure μ. An elementary computation shows that ν_α^* is an invariant state for the Markov process ξ_t and that its adjoint in $L^2(\nu_\alpha^*)$ is $\mathcal{L}^*$. In particular, $\mathcal{L}$ is self-adjoint in the symmetric case.

We have seen that the $\mathcal{H}_1$ plays an important role in the investigation of the central limit theorem. For the simple exclusion process as seen from a tagged particle, a simple computation shows that for any function f in the domain of the generator,

$$\begin{aligned}
&\langle f, (-\mathcal{L}) f\rangle_{\nu_\alpha^*} \\
&= (1/2) \sum_{x,y\in\mathbb{Z}_d^*} p(y-x) \int \xi(x)[1-\xi(y)]\,[f(\sigma^{x,y}\xi) - f(\xi)]^2\, d\nu_\alpha^* \\
&+ \; (1/2) \sum_{z\in\mathbb{Z}_d^*} p(z) \int [1-\xi(z)]\,[f(\theta_z\xi) - f(\xi)]^2\, d\nu_\alpha^* \,.
\end{aligned} \tag{4.3}$$

The first question on the asymptotic behavior of the tagged particle concerns the law of large numbers. For $0 < \alpha < 1$, denote by $\mathbb{P}_{\nu_\alpha^*}$ the measure on the path space $D(\mathbb{R}_+, \mathcal{X}_d^*)$ induced by the Markov process with generator $\mathcal{L}$ starting from ν_α^*. Saada proved in [17] the following result.

Theorem 4.1. *For every* $0 \le \alpha \le 1$

$$\lim_{t\to\infty} \frac{X_t}{t} \;=\; [1-\alpha]\,\gamma$$

in $\mathbb{P}_{\nu_\alpha^*}$ *probability, where* $\gamma = \sum_{z\in\mathbb{Z}_*^d} z p(z)$.

To investigate the central limit theorem, denote by Z_t the re-scaled position of the tagged particle:

$$Z_t \;=\; \frac{X_t - t\,\gamma\,(1-\alpha)}{\sqrt{t}} \,.$$

For each z such that $p(z) > 0$ and for $s < t$, denote by $N_{[s,t]}^z$ the total number of jumps of the tagged particle from the origin to z in the time interval $[s, t]$. Let $N_t^z = N_{[0,t]}^z$. It is not difficult to check that that

$$M_t^z \;=\; N_t^z \;-\; \int_0^t p(z)[1-\xi_s(z)]\,ds \quad \text{and} \quad (M_t^z)^2 \;-\; \int_0^t p(z)[1-\xi_s(z)]\,ds$$

are martingales vanishing at $t = 0$. In the same way, for y, z in $\mathbb{Z}_*^d$ such that $p(z-y) > 0$, $s < t$, denote by $N_{[s,t]}^{y,z}$ the number of jumps of a particle from y

to z in the interval $[s, t]$. Let $N_t^{y,z} = N_{[0,t]}^{y,z}$. As before

$$M_t^{y,z} = N_t^{y,z} - \int_0^t p(z-y)\xi_s(y)[1-\xi_s(z)]\,ds$$
$$\text{and} \quad (M_t^{y,z})^2 - \int_0^t p(z-y)\xi_s(y)[1-\xi_s(z)]\,ds$$

are martingales.

Since $\{M_t^z,\ p(z) > 0\}$, $\{M_t^{y,y+z},\ y \in \mathbb{Z}_*^d,\ p(z) > 0\}$ are pure jumps martingales and do not have common jumps, they are orthogonal in the sense that the product of two such martingales is still a martingale.

To obtain the position at time t of the tagged particle, we just need to sum the number of jumps multiplied by the size of the jumps: $X_t = \sum_z zN_t^z$ so that

$$X_t = \sum_{z\in\mathbb{Z}^d} zN_t^z = \sum_{z\in\mathbb{Z}^d} zM_t^z + \sum_{z\in\mathbb{Z}^d} \int_0^t ds\, z\, p(z)[1-\xi_s(z)]\,.$$

In particular, for any vector a in $\mathbb{R}^d$,

$$(a\cdot Z_t) = \frac{M_t^a}{\sqrt{t}} + \frac{1}{\sqrt{t}}\int_0^t V_a(\xi_s)\,ds\,,$$

where M_t^a is the one-dimensional martingale defined by

$$M_t^a = \sum_{z\in\mathbb{Z}_*^d} (a\cdot z) M_t^z$$

and V_a is the mean-zero cylinder function

$$V_a(\xi) = \sum_{z\in\mathbb{Z}_*^d} (a\cdot z)p(z)[\alpha - \xi(z)]\,. \tag{4.4}$$

In these formulas and below $(a\cdot b)$ stands for the inner product in $\mathbb{R}^d$.

To prove a central limit theorem for the tagged particle, we need to represent $\int_0^t V_a(\xi_s)\,ds$ as a martingale $\tilde{M}_t$ plus a small term and to compute the limiting variance of $M_t^a + \tilde{M}_t$. We have seen in Theorem 2.2 that such a representation is possible provide the solution f_λ of the resolvent equation

$$\lambda f_\lambda - \mathcal{L} f_\lambda = V_a \tag{4.5}$$

satisfies (2.6). In this case $\tilde{M}_t$ is the limit, as $\lambda \downarrow 0$ of the martingale M_t^λ given by

$$M_t^\lambda \;=\; f_\lambda(\xi_t) \;-\; f_\lambda(\xi_0) \;-\; \int_0^t (\mathcal{L} f_\lambda)(\xi_s)\, ds\;.$$

This martingale can be expressed in terms of the elementary martingales M_t^z, $M_t^{x,y}$ introduced above. Since

$$\begin{aligned} f_\lambda(\xi_t) \;-\; f_\lambda(\xi_0) \;&=\; \sum_{x,y\in\mathbb{Z}_*^d} \int_0^t [f_\lambda(\sigma^{x,y}\xi_{s-}) - f_\lambda(\xi_{s-})]\, dN_s^{x,y} \\ &\quad+\; \sum_{z\in\mathbb{Z}_*^d} \int_0^t [f_\lambda(\theta_z\xi_{s-}) - f_\lambda(\xi_{s-})]\, dN_s^{z}\;, \end{aligned}$$

an elementary computation shows that

$$\begin{aligned} M_t^\lambda \;&=\; \sum_{x,y\in\mathbb{Z}_*^d} \int_0^t [f_\lambda(\sigma^{x,y}\xi_{s-}) - f_\lambda(\xi_{s-})]\, dM_s^{x,y} \\ &\quad+\; \sum_{z\in\mathbb{Z}_*^d} \int_0^t [f_\lambda(\theta_z\xi_{s-}) - f_\lambda(\xi_{s-})]\, dM_s^{z}\;. \end{aligned}$$

By Theorem 2.2, if (2.6) holds for the solution of the resolvent equation,

$$(a\cdot Z_t) \;=\; \frac{1}{\sqrt{t}} M_t^a \;+\; \frac{1}{\sqrt{t}} M_t^\lambda \;+\; R_t^\lambda\;,$$

where

$$\lim_{t\to\infty}\lim_{\lambda\to 0} R_t^\lambda = 0 \quad\text{in}\quad L^2(\nu_\alpha^*)\,.$$

Since the martingale $M_t^a + \tilde{M}_t$ satisfies the assumptions of Lemma 2.1 for every a in $\mathbb{R}^d$, under assumption (2.6), Z_t converges in law to a mean zero Gaussian distribution with co-variance $D(\alpha)$ given by

$$\begin{aligned} a\cdot D(\alpha)a \;&=\; E_{\nu_\alpha^*}[(M_1^a + \tilde{M}_1)^2] \;=\; \lim_{\lambda\to 0} E_{\nu_\alpha^*}[(M_1^a + M_1^\lambda)^2] \\ &= \lim_{\lambda\to 0}\Big\{ E_{\nu_\alpha^*}\Big[\Big(\sum_{x,y\in\mathbb{Z}_*^d} \int_0^1 [f_\lambda(\sigma^{x,y}\xi_{s-}) - f_\lambda(\xi_{s-})]\, dM_s^{x,y}\Big)^2\Big] \\ &\quad+\; E_{\nu_\alpha^*}\Big[\Big(\sum_{z\in\mathbb{Z}_*^d} \int_0^1 \Big\{(a\cdot z) + [f_\lambda(\theta_z\xi_{s-}') - f_\lambda(\xi_{s-})]\Big\}\, dM_s^{z}\Big)^2\Big]\Big\} \end{aligned}$$

$$= \lim_{\lambda\to 0} \Bigg\{ \sum_{x,y\in\mathbb{Z}^d_*} p(y-x) E_{\nu^*_\alpha}\Big[\xi(x)[1-\xi(y)][f_\lambda(\sigma^{x,y}\xi) - f_\lambda(\xi)]^2\Big]$$

$$+ \sum_{z\in\mathbb{Z}^d_*} p(z) E_{\nu^*_\alpha}\Big[[1-\xi(z)]\Big\{(a\cdot z) + [f_\lambda(\theta_z\xi) - f_\lambda(\xi)]\Big\}^2\Big]\Bigg\} .$$

Here we used extensively the fact that the martingales M_t^z, $M_t^{x,y}$ are orthogonal and the explicit form of their quadratic variation. Developing the square, we get that for each fixed λ the previous expectation is equal to

$$\begin{aligned}
&(1-\alpha)\sum_{z\in\mathbb{Z}^d_*} p(z)\,(a\cdot z)^2 \\
&+ 2\sum_{z\in\mathbb{Z}^d_*} (a\cdot z)\,p(z) E_{\nu^*_\alpha}\Big[[1-\xi(z)]\,[f_\lambda(\theta_z\xi) - f_\lambda(\xi)]\Big] \\
&+ \sum_{x,y\in\mathbb{Z}^d_*} p(y-x) E_{\nu^*_\alpha}\Big[\xi(x)[1-\xi(y)][f_\lambda(\sigma^{x,y}\xi) - f_\lambda(\xi)]^2\Big] \\
&+ \sum_{z\in\mathbb{Z}^d_*} p(z) E_{\nu^*_\alpha}\Big[[1-\xi(z)][f_\lambda(\theta_z\xi) - f_\lambda(\xi)]^2\Big] .
\end{aligned}$$

In view of (4.3), the last two terms are equal to $2\|f_\lambda\|_1^2$. A change of variables $\zeta = \theta_z\xi$ in the the expectation $E_{\nu^*_\alpha}[[1-\xi(z)]f_\lambda(\theta_z\xi)]$ permits to write the second term as $2 < W_a, f_\lambda >_{\nu^*_\alpha}$, where

$$W_a \;=\; \sum_{z\in\mathbb{Z}^d_*} (a\cdot z)p(z)\{\xi(z) - \xi(-z)\} . \tag{4.6}$$

In conclusion,

$$\begin{aligned}
a\cdot D(\alpha)a \;=\;& (1-\alpha)\sum_{z\in\mathbb{Z}^d_*} p(z)\,(a\cdot z)^2 \\
&+ \lim_{\lambda\to 0}\Big\{2 < W_a, f_\lambda >_{\nu^*_\alpha} + 2\,\|f_\lambda\|_1^2\Big\}
\end{aligned} \tag{4.7}$$

for every a in $\mathbb{R}^d$. Recall from (2.14) that $\lim_{\lambda\to 0}\|f_\lambda\|_1^2 = \lim_{\lambda\to 0} < V_a, f_\lambda >$. Since, on the other hand, $V_a + W_a = \sum_z (a\cdot z)p(z)[\alpha - \xi(-z)]$,

$$a\cdot D(\alpha)a \;=\; (1-\alpha)\sum_{z\in\mathbb{Z}^d_*} p(z)\,(a\cdot z)^2 \;+\; 2\lim_{\lambda\to 0} < \tilde{W}_a, f_\lambda >_{\nu^*_\alpha} , \tag{4.8}$$

where

$$\tilde{W}_a = \sum_z (a \cdot z) p(z) [\alpha - \xi(-z)] \, .$$

Up to this point we have shown that a central limit theorem for the tagged particle in the simple exclusion process holds provided that (2.6) is in force for the solution of the resolvent equation (4.5). In this case the limiting variance is given by (4.8). In the next three subsections, we prove condition (2.6) in different contexts.

4.1 Symmetric case

Assume that p is symmetric. In this case the generator $\mathcal{L}$ is self-adjoint. To apply the method presented in sections 2 and 3 and the results proved in subsection 3.1, we first need to examine whether V_a belongs to the Sobolev space $\mathcal{H}_{-1}$ associated to the generator $\mathcal{L}$.

Fix a function f in $L^2(\nu_\alpha^*)$. Since V_a has mean zero, if $< f >_{\nu_\alpha^*}$ stands for the expectation of f with respect to ν_α^*,

$$\begin{aligned} E_{\nu_\alpha^*}[V_a f] \;&=\; \sum_{z \in \mathbb{Z}_*^d} (a \cdot z)\, p(z) \int [\alpha - \xi(z)]\, [f - < f >_{\nu_\alpha^*}]\, d\nu_\alpha^* \\ &=\; \sum_{z \in \mathbb{Z}_*^d} (a \cdot z)\, p(z) \int [1 - \xi(z)]\, [f - < f >_{\nu_\alpha^*}]\, d\nu_\alpha^* \, . \end{aligned}$$

Write this last expression as the sum of two halfs. In one of the sums, perform the change of variables $\zeta = \theta_z \xi$, which is possible because the indicator $[1 - \xi(z)] = \mathbf{1}\{\xi(z) = 0\}$ guarantees that there are no particles at z. After these operations the last term becomes

$$\begin{aligned} &(1/2) \sum_{z \in \mathbb{Z}_*^d} (a \cdot z)\, p(z) \int [1 - \xi(z)]\, [f - < f >_{\nu_\alpha^*}]\, d\nu_\alpha^* \\ &\quad + (1/2) \sum_{z \in \mathbb{Z}_*^d} (a \cdot z)\, p(z) \int [1 - \xi(-z)]\, [f(\theta_{-z}\xi) - < f >_{\nu_\alpha^*}]\, d\nu_\alpha^* \, . \end{aligned}$$

Change variables $z' = -z$ in the second sum, recall that p is symmetric and add the two terms to obtain that the previous sum is equal to

$$(1/2) \sum_{z \in \mathbb{Z}_*^d} (a \cdot z)\, p(z) \int [1 - \xi(z)]\, [f(\xi) - f(\theta_z \xi)]\, d\nu_\alpha^* \, .$$

It remains to apply Schwarz inequality to bound the square of this expression by

$$(1/4)(1-\alpha)\Big(\sum_{z\in\mathbb{Z}^d_*}(a\cdot z)^2 p(z)\Big)\sum_z p(z)\int[1-\xi(z)]\,[f(\xi)-f(\theta_z\xi)]^2\,d\nu^*_\alpha\,.$$

In view of formula (4.3) for the Dirichlet form of f, we have just proved that

$$\Big\{ < V_a, f >_{\nu^*_\alpha} \Big\}^2 \;\le\; (1/2)(1-\alpha)\sum_{z\in\mathbb{Z}^d_*}(a\cdot z)^2 p(z) < f, (-\mathcal{L})f >_{\nu^*_\alpha}\;.$$

This proves not only that V_a belongs to $\mathcal{H}_{-1}$ but gives also the bound

$$\|V_a\|^2_{-1} \;\le\; (1/2)(1-\alpha)\sum_{z\in\mathbb{Z}^d_*}(a\cdot z)^2 p(z) \tag{4.9}$$

for the $\mathcal{H}_{-1}$ norm of V_a.

We have just proved that V_a belongs to $\mathcal{H}_{-1}$. Since, on the other hand, the generator is self-adjoint, in view of subsection 3.1, the assumptions of Theorem 2.2 are in force. This proves a central limit theorem for the tagged particle in the reversible context, originally proved by Kipnis and Varadhan [6]:

Theorem 4.2. *Assume that the transition probability $p(\cdot)$ is symmetric. Then, Z_t converges in distribution, as $t\uparrow\infty$, to a mean zero Gaussian law with matrix co-variance $D(\alpha)$ characterized by (4.8).*

4.2 Mean zero case

Consider now the mean zero case. Varadhan in [21], Theorem 5.1, proved a sector condition for this model. He showed the existence of a finite constant C_0 such that

$$\Big\{ < f, (-\mathcal{L})g >_{\nu^*_\alpha} \Big\}^2 \;\le\; C_0 < f, (-\mathcal{L})f >_{\nu^*_\alpha} < g, (-\mathcal{L})g >_{\nu^*_\alpha}$$

for all functions in the domain of the generator. He proved furthermore that the local function V_a given by (4.4) belongs to $\mathcal{H}_{-1}$. In view of the results presented in subsection 3.2, (2.6) holds for the solution of the resolvent equation (4.5). We have therefore the following theorem due to Varadhan [21]:

Theorem 4.3. *Assume that the transition probability $p(\cdot)$ has mean zero. Then, Z_t converges in distribution, as $t\uparrow\infty$, to a mean zero Gaussian law with co-variance matrix $D(\alpha)$ given by (4.8).*

4.3 Asymmetric exclusion process

Assume now that $p(\cdot)$ is asymmetric. For each $n \geq 0$, denote by $\mathcal{E}_{*,n}$ the subsets of $\mathbb{Z}_*^d$ with n points and let $\mathcal{E}_* = \cup_{n \geq 0} \mathcal{E}_{*,n}$ be the class of finite subsets of $\mathbb{Z}_*^d$. For each A in $\mathcal{E}_*$, let Ψ_A be the local function

$$\Psi_A = \prod_{x \in A} \frac{\xi(x) - \alpha}{\sqrt{\chi(\alpha)}} ,$$

where $\chi(\alpha) = \alpha(1 - \alpha)$. By convention, $\Psi_\phi = 1$. It is easy to check that $\{\Psi_A, \ A \in \mathcal{E}_*\}$ is an orthonormal basis of $L^2(\nu_\alpha^*)$. For each $n \geq 1$, denote by $\mathcal{G}_n$ the subspace of $L^2(\nu_\alpha^*)$ generated by $\{\Psi_A, \ A \in \mathcal{E}_{*,n}\}$, so that $L^2(\nu_\alpha^*) = \oplus_{n \geq 0} \mathcal{G}_n$. Functions of $\mathcal{G}_n$ are said to have degree n.

Consider a local function f. Since $\{\Psi_A : A \in \mathcal{E}_*\}$ is a basis of $L^2(\nu_\alpha^*)$, we may write

$$f = \sum_{n \geq 0} \sum_{A \in \mathcal{E}_{*,n}} \mathfrak{f}(A) \Psi_A = \sum_{n \geq 0} \pi_n f.$$

Here we have denoted by π_n the orthogonal projection on $\mathcal{G}_n$. Notice that the coefficients $\mathfrak{f}(A)$ depend not only on f but also on the density α: $\mathfrak{f}(A) = \mathfrak{f}(\alpha, A)$. Since f is a local function, $\mathfrak{f}\colon \mathcal{E}_* \to \mathbb{R}$ is a function of finite support.

For a subset A of $\mathbb{Z}^d$ and x, y in $\mathbb{Z}^d$, denote by $A_{x,y}$, $S_y A$ the sets defined by

$$A_{x,y} = \begin{cases} (A \backslash \{x\}) \cup \{y\} & \text{if } x \in A, \ y \notin A, \\ (A \backslash \{y\}) \cup \{x\} & \text{if } y \in A, \ x \notin A, \\ A & \text{otherwise} ; \end{cases}$$

$$S_y A = \begin{cases} A - y & \text{if } y \notin A, \\ (A - y)_{0,-y} & \text{if } y \in A . \end{cases} \tag{4.10}$$

In this formula, $B + z$ is the set $\{x + z; \ x \in B\}$. Therefore, to obtain $S_y A$ from A in the case where y belongs to A, we first translate A by $-y$ (getting a new set that contains the origin) and we then remove the origin and add site $-y$.

Denote by $s(\cdot)$ (resp. $a(\cdot)$) the symmetric (resp. asymmetric) part of the transition probability p:

$$s(x) = (1/2)\{p(x) + p(-x)\} , \quad a(x) = (1/2)\{p(x) - p(-x)\} .$$

A simple computation shows that

$$(\mathcal{L} f) = \sum_{A \in \mathcal{E}_*} \Big\{ (\mathfrak{L}_{0,\alpha} \mathfrak{f})(A) + (\mathfrak{L}_{\tau,\alpha} \mathfrak{f})(A) \Big\} \Psi_A ,$$

where $\mathfrak{L}_{0,\alpha}$ is an operator that can be decomposed as $\mathfrak{L}_{0,\alpha} = \mathfrak{L}_0^1 + (1-2\alpha)\mathfrak{L}_0^2 + 2\sqrt{\chi(\alpha)}(\mathfrak{L}_0^+ - \mathfrak{L}_0^-)$, with

$$
\begin{aligned}
(\mathfrak{L}_0^1 \mathfrak{f})(A) &= (1/2) \sum_{x,y \in \mathbb{Z}_*^d} s(y-x)[\mathfrak{f}(A_{x,y}) - \mathfrak{f}(A)] , \\
(\mathfrak{L}_0^2 \mathfrak{f})(A) &= \sum_{\substack{x \in A, y \notin A \\ x \neq 0, y \neq 0}} a(y-x)[\mathfrak{f}(A_{x,y}) - \mathfrak{f}(A)] , \\
(\mathfrak{L}_0^- \mathfrak{f})(A) &= \sum_{\substack{x \notin A, y \notin A \\ x \neq 0, y \neq 0}} a(y-x)\mathfrak{f}(A \cup \{x\}) , \\
(\mathfrak{L}_0^+ \mathfrak{f})(A) &= \sum_{\substack{x \in A, y \in A \\ x \neq 0, y \neq 0}} a(y-x)\mathfrak{f}(A - \{y\}) ;
\end{aligned}
\tag{4.11}
$$

and $\mathfrak{L}_{\tau,\alpha}$ is an operator which can be decomposed as $\alpha\mathfrak{L}_\tau^1 + (1-\alpha)\mathfrak{L}_\tau^2 + \sqrt{\chi(\alpha)}(\mathfrak{L}_\tau^+ + \mathfrak{L}_\tau^-)$, where

$$
\begin{aligned}
(\mathfrak{L}_\tau^1 \mathfrak{f})(A) &= \sum_{y \in A} p(y)[\mathfrak{f}(S_y A) - \mathfrak{f}(A)] , \\
(\mathfrak{L}_\tau^2 \mathfrak{f})(A) &= \sum_{y \notin A} p(y)[\mathfrak{f}(S_y A) - \mathfrak{f}(A)] , \\
(\mathfrak{L}_\tau^+ \mathfrak{f})(A) &= \sum_{y \in A} p(y)[\mathfrak{f}(A - \{y\}) - \mathfrak{f}(S_y A - \{-y\})] , \\
(\mathfrak{L}_\tau^- \mathfrak{f})(A) &= \sum_{y \notin A} p(y)[\mathfrak{f}(A \cup \{y\}) - \mathfrak{f}(S_y A \cup \{-y\})] .
\end{aligned}
\tag{4.12}
$$

We may therefore decompose the generator $\mathcal{L}$ as

$$
\mathcal{L} = \mathcal{L}_- + \mathcal{R}_0 + \mathcal{B}_0 + \mathcal{L}_+ ,
$$

where

$$
\begin{aligned}
\mathcal{L}_- f &= \sqrt{\chi(\alpha)} \sum_{A \in \mathcal{E}_*} \{-2\mathfrak{L}_0^- + \mathfrak{L}_\tau^-\} \mathfrak{f}\,(A)\Psi_A , \\
\mathcal{L}_+ f &= \sqrt{\chi(\alpha)} \sum_{A \in \mathcal{E}_*} \{2\mathfrak{L}_0^+ + \mathfrak{L}_\tau^+\} \mathfrak{f}\,(A)\Psi_A , \\
\mathcal{B}_0 f &= \sum_{A \in \mathcal{E}_*} \{(1-2\alpha)\mathfrak{L}_0^2 + \alpha\mathfrak{L}_\tau^1 + (1-\alpha)\mathfrak{L}_\tau^2\} \mathfrak{f}\,(A)\Psi_A , \\
\mathcal{R}_0 f &= \sum_{A \in \mathcal{E}_*} \mathfrak{L}_0^1 \mathfrak{f}\,(A)\Psi_A .
\end{aligned}
$$

The space $L^2(\nu_\alpha^*)$ and the generator $\mathcal{L}$ have therefore exactly the structure presented in subsection 3.3. Denote by $\mathcal{H}_{0,1}$, $\mathcal{H}_{0,-1}$ the Sobolev spaces induced by the local functions and the symmetric positive operator $\mathcal{R}_0$.

To conclude the proof of the central limit theorem for the tagged particle, we need only to check that the local function V_a given by (4.4) satisfies the assumption of Lemma 3.2 and that the generator satisfies hypotheses (3.2), (3.4), (3.6).

Sethuraman, Varadhan and Yau [19] proved that in dimension $d \geq 3$ all mean zero local functions belong to $\mathcal{H}_{-1}$. In particular, since V_a has mean zero,

$$\left| < V_a, f >_{\nu_\alpha^*} \right| \leq C_0 \| f \|_1$$

for some finite constant C_0 and all functions f in the domain of the generator. Notice that we only need to consider functions f having degree one because V_a has degree one. In this case, by Lemma 4.4 in [8], $\| f \|_1 \leq C_0 \| f \|_{0,1}$ for some finite constant C_0 so that V_a belongs to $\mathcal{H}_{0,-1}$ and the first assumption is fulfilled.

We now turn to the second set of hypotheses of Lemma 3.2. We need to check assumptions (3.2), (3.4) and (3.6). In view of formula (3.5) in [8], (3.2) holds for the asymmetric simple exclusion process. On the other hand, putting together Lemma 4.1 and formula (3.1) in [19] with Lemma 5.1 of [7], we show that (3.4) holds with $\beta = 1/2$ for the asymmetric exclusion process in dimension $d \geq 3$. Finally, in respect to (3.6), observe that $\mathcal{B}_0$ has two pieces. The first one, corresponding to $\alpha \mathfrak{L}_\tau^1 + (1-\alpha)\mathfrak{L}_\tau^2$, satisfies (3.6) with $\gamma = 1$ in view of [8], Lemma 4.2. However, the piece which corresponds to $(1-2\alpha)\mathfrak{L}_0^2$ does not satisfy (3.6). What we can prove instead, [9], is that

Lemma 4.4. *Let Φ be a function such that $\|\pi_n \Phi\|_{0,-1} < \infty$ for each $n \geq 1$. Let h_λ be the solution of the resolvent equation*

$$\lambda h_\lambda - \mathcal{L} h_\lambda = \Phi \ .$$

There exists a finite constant C_0, independent of λ and α, such that

$$\begin{aligned} \| \mathcal{L}_0^2 h_{\lambda,n} \|_{0,-1} \leq & \frac{C_0 \sqrt{n}}{\alpha} \| \pi_n \Phi \|_{0,-1} \\ & + \frac{C_0 n^{3/2}}{\alpha} \sum_{j=n-1}^{n+1} \| \pi_j h_\lambda \|_{0,1} \end{aligned}$$

for all $n \geq 1$, $0 < \alpha < 1$ and $\lambda > 0$.

It very easy to check that this estimate may replace condition (3.6) in the proof of Lemma 3.2. In view of the previous estimates, we have the following result due to Sethuraman, Varadhan and Yau [19]:

Theorem 4.5. *Assume that the transition probability $p(\cdot)$ is asymmetric. Then, in dimension $d \geq 3$, Z_t converges in distribution, as $t \uparrow \infty$, to a mean zero Gaussian law with co-variance matrix $D(\alpha)$ characterized by (4.8).*

5 Comments and extensions

We list in this section some results related to the problem of the central limit theorem for the tagged particle.

5.1 Invariance principle

With a little more effort, one can prove in fact that the tagged particle converges to a d-dimensional Brownian motion with diffusion coefficient characterized by (4.8).

Remark 5.1. *In the conditions of Theorem 4.2, Theorem 4.3 or Theorem 4.5, $Z_t^N = Z_{Nt}$ converges, as $N \uparrow \infty$, to a Brownian motion with diffusion coefficient given by (4.8).*

We never excluded the possibility that the variance vanishes. In fact one can prove that $D(\alpha)$ is strictly positive in all cases but one. In the nearest-neighbor one-dimensional symmetric simple exclusion process $D(\alpha) = 0$. In fact, in this case, $\sqrt{t}$ is not the correct renormalization and this is easy to understand. Since particles cannot jump over the others, if we want the tagged particle to move from the origin up to N, we need also to move all particles which were originally between the origin and N to the right of N. The displacement of the tagged particle is thus much more rigid. Relating the exclusion process to the symmetric nearest-neighbor one-dimensional zero-range process, Arratia [1] and Rost and Vares [16] proved that $X_{tN}/N^{1/4}$ converges in distribution to a fractionary Brownian motion.

In a similar spirit, Landim, Olla and Volchan [10] and Landim and Volchan [13] considered the evolution of an asymmetric tagged particle in $\mathbb{Z}$, jumping to the right with intensity $p > 1/2$ and to the left with intensity $1 - p$, evolving as a random walk with exclusion in a medium of symmetric particles. They proved a law of large numbers and an equilibrium central limit theorem for the position of a tagged particle.

5.2 Smoothness of the diffusion coefficient

Recall from (4.8) the characterization of the self-diffusion coefficient $D(\cdot)$ and notice that it depends on α, the density of particles in the environment. Based on the duality introduced in Subsection 4.3, Landim, Olla and Varadhan [8] proved that this dependence is smooth:

Theorem 5.2. *In the symmetric case the self-diffusion coefficient $D(\cdot)$ is of class C^∞ on $[0, 1]$ and in the asymmetric case, in dimension $d \geq 3$, it is of class C^∞ on $(0, 1]$.*

It is not yet clear whether the lack of smoothness at the origin comes from the method (essentially the factor α^{-1} appearing in the statement of Lemma 4.4) or whether it is intrinsic to the problem.

Here is the idea of the proof. Recall equation (4.8) for the self-diffusion matrix. Let $R_a = \chi(\alpha)^{-1/2} V_a$ be the cylinder function given by

$$R_a(\xi) \;=\; \frac{1}{\sqrt{\alpha(1-\alpha)}} \sum_{y \in \mathbb{Z}^d_*} p(y)(y \cdot a)[\alpha - \xi(y)]\;.$$

With the notation introduced in the previous section, we may write R_a as

$$R_a(\xi) \;=\; -\sum_{y \in \mathbb{Z}^d_*} (y \cdot a) p(y) \Psi_y\;,$$

where $\Psi_z = \Psi_{\{z\}}$ for z in $\mathbb{Z}^d_*$. For $\lambda > 0$, denote by g_λ the solution of the resolvent equation:

$$\lambda g_\lambda - \mathcal{L} g_\lambda \;=\; R_a\;.$$

Of course, $g_\lambda = \chi(\alpha)^{-1/2} f_\lambda$. In view of (4.8),

$$a \cdot D(\alpha) a \;=\; (1-\alpha) \sum_{z \in \mathbb{Z}^d_*} p(z)\,(a \cdot z)^2 \;+\; 2\chi(\alpha) \lim_{\lambda \to 0} < S_a, g_\lambda >_{\nu^*_\alpha}\;, \tag{5.1}$$

where $S_a = -\sum_z (a \cdot z) p(z) \Psi_{-z}$.

Denote by $\mathfrak{g}_\lambda(\alpha, A)$ the coefficients of g_λ on the basis $\{\Psi_A,\ A \in \mathcal{E}_*\}$. Writing both g_λ and R_a on the basis $\{\Psi_A,\ A \in \mathcal{E}_*\}$, we obtain an equation for the coefficients $\mathfrak{g}_\lambda(\alpha, A)$:

$$\lambda \mathfrak{g}_\lambda - \mathfrak{L}(\alpha) \mathfrak{g}_\lambda \;=\; \mathfrak{R}_a\;. \tag{5.2}$$

Here, $\mathfrak{L}(\alpha)$ is the operator $\mathfrak{L}_{0,\alpha} + \mathfrak{L}_{\tau,\alpha}$ defined in (4.11), (4.12) and, for each a in $\mathbb{R}^d$, $\mathfrak{R}_a = \mathfrak{R}_a^{\cdot}$ is the real function defined on $\mathcal{E}_*$ by $\mathfrak{R}_a(\{y\}) = -(y \cdot a)p(y)$, $\mathfrak{R}_a(A) = 0$ for $|A| > 1$ and $A = \phi$.

In view of (5.1), to prove that D is smooth, we just need to show that

$$\lim_{\lambda \to 0} < S_a, g_\lambda >_{\nu_\alpha^*} = - \lim_{\lambda \to 0} \sum_{z \in \mathbb{Z}_*^d} p(z)(a \cdot z)\mathfrak{g}_\lambda(\alpha, \{z\})$$

is a smooth function in the density α. We need thus to show that there exists a subsequence $\lambda_k \downarrow 0$ such that $\{\mathfrak{g}_{\lambda_k}(\alpha, \{y\}) \, : \, k \geq 1\}$ converges uniformly to a smooth function for all y with $p(y) > 0$.

To prove that $\mathfrak{g}_\lambda(\alpha, \{y\})$ is a sequence of smooth functions, observe from equation (5.2) and from the explicit form of the operator $\mathfrak{L}(\alpha)$ given in (4.11), (4.12) that $\mathfrak{g}_\lambda(\alpha, \cdot)$ is the solution of a elliptic equation for each fixed α. The density α is now a parameter of the equation and we want to prove that the solutions depend smoothly on this parameter.

In the case where the operator $\mathfrak{L}(\alpha)$ doesn't change the degree of a function, we would have a one-parameter family of finite dimensional elliptic equation. To show that the solutions depend smoothly on the parameter α, we would first deduce the equations satisfied by the derivatives of $\mathfrak{g}_\lambda$ and then obtain estimates, uniform in λ, on the L^∞ norm of these derivatives to conclude the existence of a subsequence λ_k for which $\mathfrak{g}_{\lambda_k}$ converges to a smooth function.

In our case, the operator $\mathfrak{L}(\alpha)$ changes the degree of a function by at most one. To apply the previous ideas, one need first to show that the solution is such that the high degrees are small in some sense. This is exactly the content of Lemma 3.1. Details of the proof can be found in [8], [9].

Remark 5.3. *The approach just presented to prove smoothness of the self-diffusion coefficient provides Taylor expansions at any order of the co-variance matrix through the inversion of finite-dimensional parabolic operators (cf. [8]).*

5.3 Bulk diffusion

The method presented above is quite general and can be used to prove that the Bulk diffusion coefficient of nongradient [20] interacting particle systems are smooth (cf. [2]).

These results have an important application. There are essentially two general methods to prove the hydrodynamic behavior of an interacting particle system. The first one, introduced by Guo, Papanicolau and Varadhan, [5], requires uniqueness of weak solutions of the partial differential equation which describes the macroscopic behavior of the system. The second one, called the relative entropy method and due to Yau, [22], requires the existence of smooth solutions.

For some nongradient systems, the differential equation is of parabolic type and the diffusion matrix is given by a variational formula, similar to the ones derived in this article. In order to apply the relative entropy method, one needs to check that this diffusion matrix is regular in order to guarantee the existence of smooth solutions. The approach presented here may therefore validate the relative entropy method for nongradient systems. For instance, in the case of the Navier-Stokes correction [4], [11], [12] for the asymmetric simple exclusion process, the same method permits to prove that the bulk diffusion coefficient in dimension $d \geq 3$ is smooth in the interval $[0, 1/2) \cup (1/2, 1]$ (cf. [9]).

5.4 Finite dimensional approximations

Fix $N \geq 1$ and consider a finite dimensional version of the symmetric exclusion process on the torus $\{-N, \dots, N\}^d$ (i.e. with periodic boundary conditions, preserving in this manner the translation symmetry). Since we want to work with an ergodic process, we also fix the total number K of particles. Consider now a tagged particle in this finite system. If N is much larger than the size of a single jump, the motion of the tagged particle has a unique canonical lifting to $\mathbb{Z}^d$. We get in this manner a process $X_N(t)$ with values in $\mathbb{Z}^d$. Let us denote by $D_{[N,K]}$ the variance of the Brownian motion which is the limit of the scaled process $\varepsilon X_N(\varepsilon^{-2}t)$ as $\varepsilon \to 0$. It is proved in [7] that

$$\lim_{\substack{N\to\infty \\ K/(2N)^d \to \alpha}} D_{[N,K]} = D(\alpha) .$$

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