Torsion Decomposition of Finite CW-complexes *

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Introduction

Peter Freyd proved in (3), Theorem 7.2, that in the stable category $\mathscr{S}$ defined there, each object X is equivalent to the cone of some morphism

$$f : S^{q_1} \vee \ldots \vee S^{q_r} \longrightarrow T$$

in $\mathscr{S}$, where the left hand side is the sum of certain spherical objects and the right hand side is a torsion object in $\mathscr{S}$.

With suitable reinterpretations of these concepts the theorem can also be proved in the category $\mathscr{C}_f$ of finite CW-complexes with basepoint, under the additional condition that the object X is first sufficiently often suspended. That is the purpose of this paper.

The material is presented in four chapters:

Chapter I: Identification spaces are introduced as sets of equivalence classes of points of a given space X with the largest topology for which the function sending each point into the class it represents still is continuous.

A particular type of identification spaces is obtained by generating the equivalence relation in the following way: Le A be a closed subspace of the space X and $f : A \longrightarrow Y$ some map. Then the relation $R = \{(a, f(a)) \mid a \in A\}$ on the disjoint union $Y \sqcup X$ of X and Y generates an equivalence relation $\bar{R}$, the smallest one on $Y \sqcup X$ that contains R. Then the identification space of $Y \sqcup X$ with respect to $\bar{R}$ is denoted by $Y_f \sqcup X$ and called the adjunction

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space of X and Y with respect to f. Intuitively one may think of $Y_f \sqcup X$ as being obtained by gluing X along A to Y by means of f. The adjunction operation is compatible with the operation of forming the topological product with a locally compact space and the universal property for identification spaces has the special form of a pushout for adjuntion spaces. The latter property will guarantee a certain amount of freedom in the representation of iterated adjunction spaces (cf. Theorems 6 and 7). Theorem 7 actually makes a statement about mapping cones. special adjunction spaces which are obtained by gluing along the base of the cone of a space.

Using the particular homotopy properties of mapping cones it will be shown that the suspension of any mapping cone has the same homotopy type as some new mapping cone into which an additional map and an additional space are built such that the base space of the new mapping cone is the mapping cone of the additional map (cf. Theorem 10).

Chapter II: Torsion spaces are defined as spaces with base point having a homotopy-commutative coproduct structure for wich the constant map is a homotopy left and right identity, and the class of the identity map has finite order in the semigroup of homotopy classes of maps from the space into itself.

Pseudo-projective spaces prove to be the most elementary non-trivial torsion spaces from which all the others, which have the same homotopy type as a finite CW-complex may be obtained in much the same way as CW-complexes are obtained from spheres. This follows from the Theorems 13 and 14. The reduced integral homology of any torsion space of the same homotopy type as a CW-complex is finite. This homology condition is also sufficient for a simply connected finite CW-complex to be torsion, after it has been twice suspended (cf. Corollary 15).

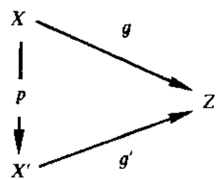
Chapter III: In accordance with the inductive construction of finite CW-complexes the main result of this paper will be obtained by induction. The induction step depends on the right choices of the additional map and the additional space in the reinterpretation of a suspended mapping cone by a new mapping cone (cf. Theorem 10). The choices will be made on the bases of the finiteness of almost all homotopy groups of spheres. The non-finite case $\pi_{2n-1}(S^n)$ is avoided by a sufficient number of suspensions, while the other non-finite case $\pi_n(S^n)$ which cannot be removed by suspensions is further transformed until the top-space of the new mapping cone has the form of a sum of spheres together with possibly one pseudo-projective space.

Chapter IV: All the previous results are organized in the induction step.

Chapter I: Adjunction Spaces

1) IDENTIFICATION SPACES. If X is any topological space and $R \subset X \times X$ some relation on X, then there exists a smallest equivalence relation $\bar{R}$ on X, containing R, called the equivalence relation generated by R.

If $f: X \longrightarrow Y$ is any map such that R-related points of X have the same image under f, then also $\bar{R}$-related points will have the same image under f. Further, there exists a unique space X' and a continuous surjection p: $X \longrightarrow X'$ such that any map g: $X \longrightarrow Z$ which also sends R-related points to the same image point uniquely factors through p giving rise to a new map g': $X' \longrightarrow Z$ making the following triangle of maps commutative:



Under these conditions p will be called the identification map with respect to the equivalence relation $\bar{R}$ generated by the relation R on X.

X' will be called the identification space of X with respect to the equivalence relation $\bar{R}$ generated by the relation R on X.

This general situation may be specialized by defining the relation R as follows:

Let A be a closed subspace of the space X and f: $A \longrightarrow Y$ some map, where $X \cap Y = \varnothing$. Then $R = \{(a, f(a)) \mid \in A\}$ is a relation on the disjoint union $Y \sqcup X$.

2) DEFINITION. The identification space of $Y \sqcup X$ with respect to the equivalence relation $\bar{R}$ generated by the relation $R = \{(a, f(a)) \mid a \in A\}$ is called the *adjunction space* obtained from X and Y by means of gluing X to Y along A using f. It is denoted by $Y_f \sqcup X$ and $\hat{f}: Y \sqcup X \longrightarrow Y_f \sqcup X$ denotes the identification map.

3) OBSERVATIONS. a) $g: Y \sqcup X \longrightarrow Z$ maps R-related points (R as in definition 2) to the same image point iff the following square of maps is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g|X} \\ Y & \xrightarrow[\;g|Y\;]{} & Z \end{array}$$

Here i is the inclusion of the closed subspace A into X.

Hence the following commutative square of maps is a pushout:

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ Y & \xrightarrow[\;\bar{i}\;]{} & Y_f \sqcup X \end{array}$$

Here the following notation is introduced:

$$\bar{f} = \hat{f}|_X$$
$$\bar{i} = \hat{f}|_Y$$

b) If $p: X \longrightarrow X'$ is an identification map and Z a locally-compact space then also $p \times 1_Z : X \times Z \longrightarrow X' \times Z$ is an identification map. Applying this to the special case of an adjunction space it follows that the pushout square of observation a) multiplied by Z anew yields a pushout.

$$
\begin{array}{ccc}
A \times Z & \xrightarrow{\;i \times 1_Z\;} & X \times Z \\
{\scriptstyle f \times 1_Z}\downarrow & & \downarrow{\scriptstyle \bar{f} \times 1_Z} \\
Y \times Z & \xrightarrow[\;\bar{i} \times 1_Z\;]{} & (Y_f \sqcup X) \times Z
\end{array}
$$

Since with i, also $i \times 1_Z$ is the inclusion of a closed subspace there is a pushout determining the adjunction space $(Y \times Z)_{f \times 1_Z} \sqcup (X \times Z)$.

$$
\begin{array}{ccc}
A \times Z & \xrightarrow{\;i \times 1_Z\;} & X \times Z \\
{\scriptstyle f \times 1_Z}\downarrow & & \downarrow{\scriptstyle \overline{f \times 1_Z}} \\
Y \times Z & \xrightarrow[\;\overline{i \times 1_Z}\;]{} & (Y \times Z)_{f \times 1_Z} \sqcup (X \times Z)
\end{array}
$$

From the universal property of pushouts it follows that

$$(Y_f \sqcup X) \times Z \cong (Y \times Z)_{f \times 1_Z} \sqcup (X \times Z)$$

i.e. the topological product with a locally-compact space and adjunction are compatible operations. (cf. (1), Theorem 4.6.6).

This property will be of particular use for $Z = I$ (closed unit interval), to show that cone- and suspension-construction respectively are compatible with the adjunction operation. In view of observation b) it will suffice to show that identification and adjunction are compatible operations. More precisely.

4) LEMMA.

$$
\begin{array}{ccccc}
Y & \xleftarrow{\;f\;} & A & \xrightarrow{\;i\;} & X \\
{\scriptstyle k_Y}\downarrow & & {\scriptstyle k_A}\downarrow & & {\scriptstyle k_X}\downarrow \\
Y' & \xleftarrow[\;f'\;]{} & A' & \xrightarrow[\;i'\;]{} & X'
\end{array}
$$

If this diagram commutes and i, i', k_X, k_A, k_Y are inclusions of closed subspaces, and if $A = A' \cap X \neq \emptyset$ is saturated with respect to f', then

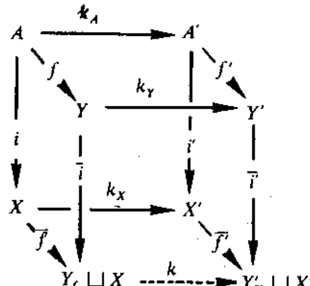
a) *$Y_f \sqcup X$ can be considered as a closed subspace of $Y'_{f'} \sqcup X'$*

b) *A'/A can be considered as a closed subspace of X'/X.*

c) *there exists a unique map $g : A'/A \longrightarrow Y'/Y$ such that*

$$(Y'_{f'} \sqcup X')/(Y_f \sqcup X) \cong (Y'/Y)_g \sqcup (X'/X)$$

PROOF. a)



The solid-arrow subdiagram is commutative by hypothesis. The universal property of pushouts implies the existence of a unique map k such that the whole cube of maps commutes.

It remains to show that $\hat{k} : Y_f \sqcup X \longrightarrow k(Y_f \sqcup X) \subset (Y'_{f'} \sqcup X')$ induced by k is a homeomorphism with respect to the subspace topology on $k(Y_f \sqcup X)$.

a1) $\hat{k}$ *is* 1-1: It suffices to show this set-theoretically. But as sets

$$Y_f \sqcup X = Y \sqcup (X - A)$$

and

$$Y'_{f'} \sqcup X' = Y' \sqcup (X' - A').$$

If $y_1, y_2 \in Y$ such that $\hat{k}(y_1) = \hat{k}(y_2)$ then also $k \circ \bar{i}(y_1) = k \circ \bar{i}(y_2)$. However $k \circ \bar{i} = \bar{i}' \circ k_Y$. Since both k_Y and $\bar{i}'$ are monomorphic, it follows that $y_1 = y_2$.

If $x_1, x_2 \in X - A$ such that $\hat{k}(x_1) = \hat{k}(x_2)$ then also $k \circ \bar{f}(x_1) = k \circ \bar{f}(x_2)$. However $k \circ \bar{f} = \bar{f}' \circ k_X$. Since both k_X and $\bar{f}'|_{X'-A'}$ are monomorphic and $A = A' \cap X$, it follows that $x_1 = x_2$.

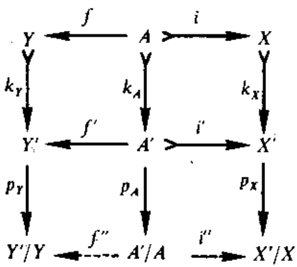
Finally, if $x \in X - A$ and $v \in Y$ such that $\bar{k}(x) = \bar{k}(v)$ then $k \circ \bar{f}(x) = k \circ \bar{i}(v)$ or equivalently $\bar{f}' \circ k_X(x) = \bar{i}' \circ k_Y(y)$ indicating that $k_X(x)$, $k_Y(y)$ are $\bar{R}'$-related (where $R' = \{(a', f'(a')) \mid a' \in A'\}$). Since Y' and X' are disjoint, this implies that $k_X(x) \in A'$ and $f'(k_X(x)) = k_Y(y)$. However k_X maps $X - A$ into $X' - A'$ by hypothesis, so that this case cannot occur. Hence $\bar{k}$ is $1-1$.

a2) $\bar{k}$ *is open*: Since $\bar{k}$ is $1-1$ and onto, being open is equivalent to being closed. Let $C \subset Y_f \sqcup X$ be any closed subset. Hence $\bar{i}^{-1}(C)$ is closed in Y and therefore also in Y', and $\bar{f}^{-1}(C)$ is closed in X and therefore also in X'.

Since the only non-trivial identification of $\hat{f}'$: $Y' \sqcup X' \longrightarrow Y'_{f'} \sqcup X'$ is due to f': $A' \longrightarrow Y'$ and A is saturated with respect to f', it follows that $\bar{i}^{-1}(C) \sqcup \bar{f}^{-1}(C) \subset Y' \sqcup X'$ is saturated with respect to f'. Since it is also closed in $Y' \sqcup X'$, its image under f' which is equal to $k(C)$ is closed in $Y'_{f'} \sqcup X'$ and therefore closed in $k(Y_f \sqcup X)$ as a subspace of $Y'_{f'} \sqcup X'$.

Since $\bar{k}$ is also continuous and onto by construction, it follows from a1) and a2) that it is a homeomorphism.

b)



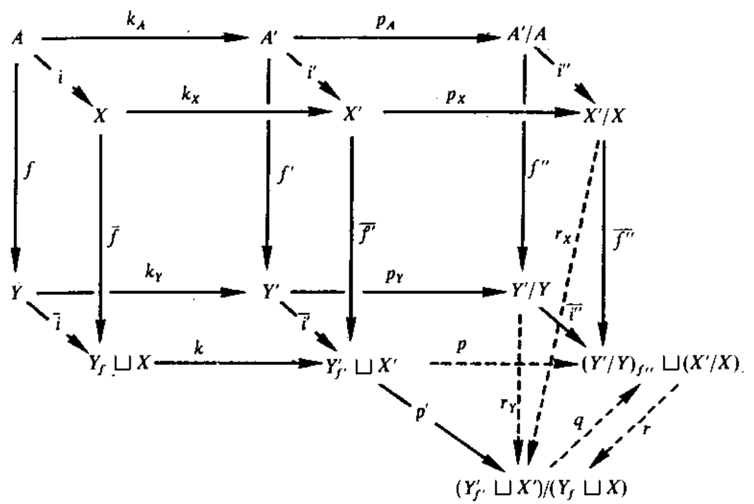
$p_X \circ i'$ maps A to the basepoint of X'/X and so factors uniquely through p_A giving rise to the map i''. (Also $p_Y \circ f'$ maps A to the basepoint of Y'/Y and so factors uniquely through p_A, giving rise to the map f'' which will be the map g of part c) of the lemma). It is claimed that i'' induces a homeomorphism $\bar{i}'' : A'/A \longrightarrow i''(A'/A) \subset X'/X$ with respect to the subspace topology of $i''(A'/A)$.

b1) $\bar{i}''$ *is* $1-1$: Let $a_1, a_2 \in A'/A$ such that $\bar{i}''(a_1) = \bar{i}''(a_2)$. There exist elements $a'_1, a'_2 \in A'$ such that $p_A(a'_1) = a_1$ and $p_A(a'_2) = a_2$. Then also $p_X \circ i'(a'_1) = = p_X \circ i'(a'_2)$. However, the only non-trivial identification effected by p_X is the collapsing of X to one point. Hence $i'(a_1'), i'(a_2') \in X$, which implies that $a_1', a_2' \in A$. Hence $a_1 = a_2$.

b2) i'' *is closed*: For any $C \subset A'/A$ closed, $p_A^{-1}(C)$ is closed in A' and hence in X'. It is p_X-saturated if $p_A^{-1}(C) \cap A = \emptyset$. If $p_A^{-1}(C) \cap A \neq \emptyset$, $p_A^{-1}(C) \cup X$ is p_X-saturated and $p_X(p_A^{-1}(C) \cup X) = p_X(p_A^{-1}(C))$. In both cases $i''(C) = = p_X(p_A^{-1}(C))$ is closed in X'/X and therefore in $i''(A'/A)$ as a subspace of X'/X.

i'' is onto by definition and $1-1$ by b1), hence it is open by b2). By construction it is also continuous and so actually a homeomorphism.

c)



c1) The solid-arrow subdiagram is commutative. The universal property of pushouts implies the existence of a unique map p such that

$$p \circ \bar{i}' = \bar{i}'' \circ p_Y$$

and

$$p \circ \bar{f}' = \bar{f}'' \circ p_X .$$

c2) Also p maps the entire subspace $Y_f \sqcup X$ to the basepoint of $(Y'/Y)_{f''} \sqcup (X'/X)$ and therefore uniquely factors through p' giving rise to q such that $p = q \circ p'$.

c3) Furthermore

$$p' \circ \bar{i}' \quad \text{maps} \quad Y \subset Y' \text{ to the basepoint}$$
$$p' \circ \overline{f'} \quad \text{maps} \quad X \subset X' \text{ to the basepoint.}$$

Hence, these maps respectively factor uniquely through p_Y and p_X giving rise to maps r_Y and r_X such that

$$p' \circ \bar{i}' = r_Y \circ p_Y$$

and

$$p' \circ \overline{f'} = r_X \circ p_X$$

c4) $r_Y \circ f'' \circ p_A = r_Y \circ p_Y \circ f' \overset{c3)}{=} p' \circ \bar{i}' \circ f' = p' \circ \overline{f'} \circ i' \overset{c3)}{=} r_X \circ p_X \circ i' = r_X \circ i'' \circ p_A$.

But p_A is surjective, hence $r_Y \circ f'' = r_X \circ i''$ and the universal property of pushouts implies the unique existence of a map r, such that

$$r_Y = r \circ \bar{i}'' \qquad r_X = r \circ \overline{f''}$$

c5) Claim: *q and r are mutually inverse homeomorphisms.*

To prove this it suffices to show that

$$\alpha) \qquad q \circ r = 1$$

and

$$\beta) \qquad r \circ q = 1$$

α) By the universal property of pushouts it suffices to show:

$$q \circ r \circ \bar{i}'' = \bar{i}'$$

and

$$q \circ r \circ \overline{f''} = \overline{f''}$$

Since p_X and p_Y are surjections these equalities are equivalent to the following: $q \circ r \circ \bar{i}'' \circ p_Y = \bar{i}'' \circ p_Y$ and $q \circ r \circ \overline{f''} \circ p_X = \overline{f''} \circ p_X$

However

$$q \circ r \circ \bar{i}'' \circ p_Y \overset{c4)}{=} q \circ r_Y \circ p_Y \overset{c3)}{=} q \circ p' \circ \bar{i}' \overset{c2)}{=} p \circ \bar{i}' \overset{c1)}{=} \bar{i}'' \circ p_Y$$

and

$$q \circ r \circ \overline{f''} \circ p_X \overset{(4)}{=} q \circ r_X \circ p_X \overset{(3)}{=} q \circ p' \circ \overline{f'} \overset{(2)}{=} p \circ \overline{f'} \overset{(1)}{=} \overline{f''} \circ p_X$$

β) Since p' is a surjection it will suffice to show that $r \circ q \circ p' = p'$. Further, the universal property of pushouts implies that this equality is equivalent to the following two:

$$r \circ q \circ p' \circ \overline{i'} = p' \circ \overline{i'}$$

and

$$r \circ q \circ p' \circ \overline{f'} = p' \circ \overline{f'}$$

However

$$r \circ q \circ p' \circ \overline{i'} \overset{(3)}{=} r \circ q \circ r_Y \circ p_Y \overset{(4)}{=} r \circ q \circ r \circ \overline{i'} \circ p_Y \overset{(5\alpha)}{=} r \circ \overline{i'} \circ p_Y \overset{(4)}{=} r_Y \circ p_Y \overset{(3)}{=} p' \circ \overline{i'}$$

and

$$r \circ q \circ p' \circ \overline{f'} \overset{(3)}{=} r \circ q \circ r_X \circ p_X \overset{(4)}{=} r \circ q \circ r \circ \overline{f''} \circ p_X \overset{(5\alpha)}{=} r \circ \overline{f''} \circ p_X \overset{(4)}{=} r_X \circ p_X \overset{(3)}{=} p' \circ \overline{f'}$$

5) Corollary.

$$C(Y_f \sqcup X) \cong CY_{Cf} \sqcup CX$$

and

$$\Sigma(Y_f \sqcup X) \cong \Sigma Y_{\Sigma f} \sqcup \Sigma X$$

With this the compatibility between cone- and suspension-construction respectively and the adjunction operation is established.

The following theorem provides a certain amount of freedom in the representation of spaces obtained by an iteration of adjunction operations (e.g. CW-complexes).

6) Theorem.

$$\begin{array}{ccccc} Y'' & \xleftarrow{f''} & A'' & \xrightarrow{i''} & X'' \\ \uparrow{\scriptstyle g_Y} & & \uparrow{\scriptstyle g_A} & & \uparrow{\scriptstyle g_X} \\ Y & \xleftarrow{f} & A & \xrightarrow{i} & X \\ \downarrow{\scriptstyle k_Y} & & \downarrow{\scriptstyle k_A} & & \downarrow{\scriptstyle k_X} \\ Y & \xleftarrow{f'} & A' & \xrightarrow{i'} & X' \end{array}$$

If this diagram of maps commutes, and i, i', i'', k_Y, k_A, k_X are the inclusions of closed subspaces where $A = A' \cap X \neq \emptyset$ is saturated with respect to g_X and f' then there exist unique maps

$$F : A''_{g_A} \sqcup A \longrightarrow Y''_{g_Y} \sqcup Y'$$

and

$$G : Y_f \sqcup X \longrightarrow Y''_{f''} \sqcup X''$$

such that

$$(Y''_{f''} \sqcup X'')_G \sqcup (Y'_{f'} \sqcup X') \cong (Y''_{g_Y} \sqcup Y')_F \sqcup (X''_{g_X} \sqcup X)$$

Proof (cf. diagram on the next page). The solid-arrow subdiagram is commutative and the universal property of pushouts uniquely induces, one after another, the following maps:

a) $\exists!\ k$ such that $k \circ \bar{f} = \bar{f}' \circ k_X$ and $k \circ \bar{i} = \bar{i}' \circ k_Y$.

Lemma 4 implies that k is the inclusion of a closed subspace.

b) $\exists!\ I$ such that $I \circ \overline{k_A} = \overline{k_X} \circ i''$ and $I \circ g_A = \overline{g_X} \circ i'$.

Lemma 4 again implies that I is the inclusion of a closed subspace.

c) $\exists!\ G$ such that $G \circ \bar{i} = \bar{i}'' \circ g_Y$ and $G \circ \bar{f} = \bar{f}'' \circ g_X$.

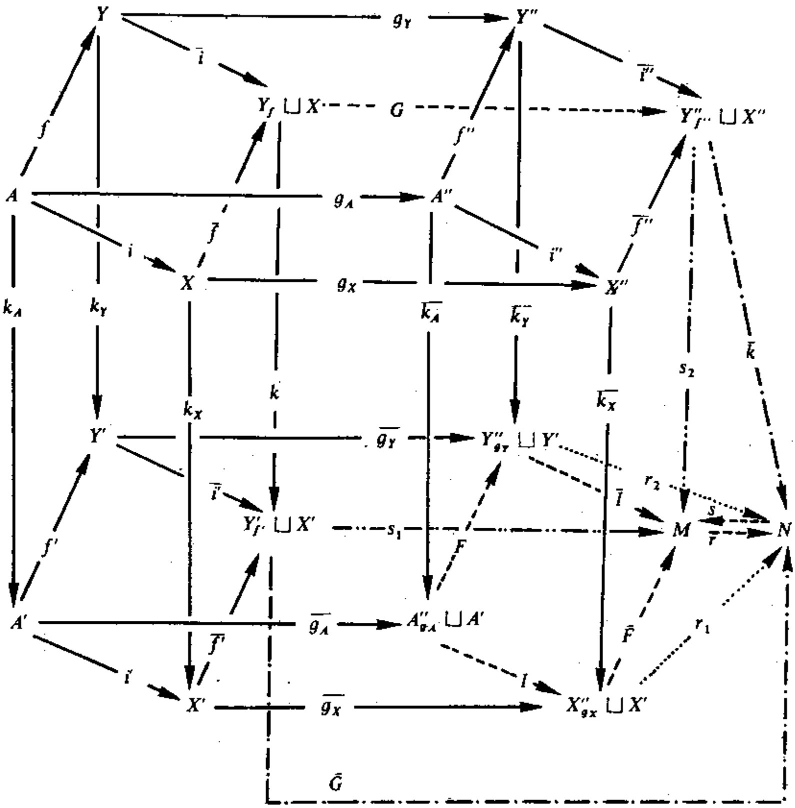
d) $\exists!\ F$ such that $F \circ \overline{g_A} = \overline{g_Y} \circ f'$ and $F \circ k_A = k_Y \circ f''$.

Using the maps of a), b), c) and d) it is possible to form in two ways new adjunction spaces, the ones which appear in the statement of the theorem and which will be shown to be homeomorphic. The existence of the mutually inverse homeomorphisms is established in the following six steps:

e) $\exists!\ r_1$ such that $r_1 \circ \overline{g_X} = \bar{G} \circ \overline{f'}$ and $r_1 \circ \overline{k_X} = \bar{k} \circ \overline{f''}$

because

$$\bar{k} \circ \bar{f}'' \circ g_X \stackrel{c)}{=} \bar{k} \circ G \circ \bar{f} = \bar{G} \circ k \circ \bar{f} = \bar{G} \circ \bar{f}' \circ k_X$$



Abbreviations: $M = (Y''_{g_Y} \sqcup Y')_f \sqcup (X''_{g_X} \sqcup X')$
$N = (Y''_{f''} \sqcup X'')_G \sqcup (Y'_{f'} \sqcup X')$

f) $\exists!\ r_2$ such that $r_2 \circ \overline{g_Y} = \bar{G} \circ \bar{i}'$ and $r_2 \circ \overline{k_Y} = \bar{k} \circ \bar{i}''$
because

$$\bar{k} \circ \bar{i}'' \circ g_Y \overset{\text{cl}}{=} \bar{k} \circ G \circ \bar{i} = \bar{G} \circ k \circ \bar{i} = \bar{G} \circ \bar{i}' \circ k_Y$$

g) Claim: $r_1 \circ I = r_2 \circ F$, or equivalently

$$r_1 \circ I \circ \overline{g_A} = r_2 \circ F \circ \overline{f_A} \quad \text{and} \quad r_1 \circ I \circ \overline{k_A} = r_2 \circ F \circ \overline{k_A}$$

But

$$r_1 \circ I \circ \overline{g_A} \overset{h)}{=} r_1 \circ \overline{g_X} \circ i' \overset{e)}{=} \bar{G} \circ \overline{f'} \circ i' = \bar{G} \circ \bar{i'} \circ f' \overset{f)}{=} r_2 \circ \overline{g_Y} \circ f' \overset{d)}{=} r_2 \circ F \circ \overline{g_A}$$

and

$$r_1 \circ I \circ \overline{k_A} \overset{h)}{=} r_1 \circ \overline{k_X} \circ i'' \overset{e)}{=} \bar{k} \circ \overline{f''} \circ i'' = \bar{k} \circ \overline{i''} \circ f'' \overset{f)}{=} r_2 \circ \overline{k_Y} \circ f'' \overset{d)}{=} r_2 \circ F \circ \overline{k_A}\,.$$

Hence $\exists!\ r$ such that $r \circ \bar{F} = r_1$ and $r \circ \bar{I} = r_2$.

h) $\exists!\ s_1$ such that $s_1 \circ \overline{f'} = \bar{F} \circ \overline{g_X}$ and $s_1 \circ \bar{i'} = \bar{I} \circ \overline{g_Y}$

because

$$\bar{F} \circ \overline{g_X} \circ i' \overset{h)}{=} \bar{F} \circ I \circ \overline{g_A} = \bar{I} \circ F \circ \overline{g_A} \overset{d)}{=} \bar{I} \circ \overline{g_Y} \circ f'.$$

i) $\exists!\ s_2$ such that $s_2 \circ \bar{i''} = \bar{I} \circ \overline{k_Y}$ and $s_2 \circ \overline{f''} = \bar{F} \circ \overline{k_X}$

because

$$\bar{F} \circ \overline{k_X} \circ i'' \overset{h)}{=} \bar{F} \circ I \circ \overline{k_A} = \bar{I} \circ F \circ \overline{k_A} \overset{d)}{=} \bar{I} \circ \overline{k_Y} \circ f''$$

j) Claim: $s_2 \circ G = s_1 \circ k$, or equivalently

$$s_1 \circ k \circ \bar{i} = s_2 \circ G \circ \bar{i} \quad \text{and} \quad s_1 \circ k \circ \bar{f} = s_2 \circ G \circ \bar{f}.$$

But

$$s_1 \circ k \circ \bar{i} = s_1 \circ \bar{i'} \circ k_Y \overset{h)}{=} \bar{I} \circ \overline{g_Y} \circ k_Y = \bar{I} \circ \overline{k_Y} \circ g_Y \overset{i)}{=} s_2 \circ \bar{i''} \circ g_Y \overset{c)}{=} s_2 \circ G \circ \bar{i}$$

and

$$s_1 \circ k \circ \bar{f} = s_1 \circ \overline{f'} \circ k_X \overset{h)}{=} \bar{F} \circ \overline{g_X} \circ k_X = \bar{F} \circ \overline{k_X} \circ g_X \overset{i)}{=} s_2 \circ \overline{f''} \circ g_X \overset{c)}{=} s_2 \circ G \circ \bar{f}$$

Hence $\exists!\ s$ such that $s \circ \bar{G} = s_1$ and $s \circ \bar{k} = s_2$.

It remains to show that r and s are mutually inverse homeomorphisms. This will be done in the final two steps.

k) Claim: $r \circ s = 1$.

Due to the universal property of pushouts it will be enough to show: $r \circ s \circ \bar{G} = \bar{G}$, or equivalently

$$r \circ s \circ \bar{G} \circ \bar{i'} = \bar{G} \circ \bar{i'} \quad \text{and} \quad r \circ s \circ \bar{G} \circ \overline{f'} = \bar{G} \circ \overline{f'}$$

as well as

$$r \circ s \circ \bar{k} = \bar{k}, \quad \text{or equivalently}$$

$$r \circ s \circ \bar{k} \circ \overline{i''} = \bar{k} \circ \overline{i''} \quad \text{and} \quad r \circ s \circ \bar{k} \circ \overline{f''} = \bar{k} \circ \overline{f''}.$$

However

$$\begin{aligned}
r \circ s \circ \bar{G} \circ \overline{i'} &\overset{i)}{=} r \circ s_1 \circ \overline{i'} \overset{h)}{=} r \circ \bar{I} \circ \overline{g_Y} \overset{g)}{=} r_2 \circ \overline{g_Y} \overset{f)}{=} \bar{G} \circ \overline{i'} \\
r \circ s \circ \bar{G} \circ \overline{f'} &\overset{i)}{=} r \circ s_1 \circ \overline{f'} \overset{h)}{=} r \circ \bar{F} \circ \overline{g_X} \overset{g)}{=} r_1 \circ \overline{g_X} \overset{e)}{=} \bar{G} \circ \overline{f'} \\
r \circ s \circ \bar{k} \circ \overline{i''} &\overset{i)}{=} r \circ s_2 \circ \overline{i''} \overset{i)}{=} r \circ \bar{I} \circ \overline{k_Y} \overset{g)}{=} r_2 \circ \overline{k_Y} \overset{f)}{=} \bar{k} \circ \overline{i''} \\
r \circ s \circ \bar{k} \circ \overline{f''} &\overset{i)}{=} r \circ s_2 \circ \overline{f''} \overset{i)}{=} r \circ \bar{F} \circ \overline{k_X} \overset{g)}{=} r_1 \circ \overline{k_X} \overset{e)}{=} \bar{k} \circ \overline{f''}
\end{aligned}$$

l) Claim: $s \circ r = 1$.

Again it is enough to verify the following:

$$s \circ r \circ \bar{F} = \bar{F}, \quad \text{or equivalently}$$

$$s \circ r \circ \bar{F} \circ \overline{g_X} = \bar{F} \circ \overline{g_X} \quad \text{and} \quad s \circ r \circ \bar{F} \circ \overline{k_X} = \bar{F} \circ \overline{k_X}$$

as well as $s \circ r \circ \bar{I} = \bar{I}$, or equivalently

$$s \circ r \circ \bar{I} \circ \overline{g_Y} = \bar{I} \circ \overline{g_Y} \quad \text{and} \quad s \circ r \circ \bar{I} \circ \overline{k_Y} = \bar{I} \circ \overline{k_Y}.$$

However

$$\begin{aligned}
s \circ r \circ \bar{F} \circ \overline{g_X} &\overset{g)}{=} s \circ r_1 \circ \overline{g_X} \overset{e)}{=} s \circ \bar{G} \circ \overline{f'} \overset{i)}{=} s_1 \circ \overline{f'} \overset{h)}{=} \bar{F} \circ \overline{g_X} \\
s \circ r \circ \bar{F} \circ \overline{k_X} &\overset{g)}{=} s \circ r_1 \circ \overline{k_X} \overset{e)}{=} s \circ \bar{k} \circ \overline{f''} \overset{i)}{=} s_2 \circ \overline{f''} \overset{i)}{=} \bar{F} \circ \overline{k_X} \\
s \circ r \circ \bar{I} \circ \overline{g_Y} &\overset{g)}{=} s \circ r_2 \circ \overline{g_Y} \overset{f)}{=} s \circ \bar{G} \circ \overline{i'} \overset{i)}{=} s_1 \circ \overline{i'} \overset{h)}{=} \bar{I} \circ \overline{g_Y} \\
s \circ r \circ \bar{I} \circ \overline{k_Y} &\overset{g)}{=} s \circ r_2 \circ \overline{k_Y} \overset{f)}{=} s \circ \bar{k} \circ \overline{i''} \overset{i)}{=} s_2 \circ \overline{i''} \overset{i)}{=} \bar{I} \circ \overline{k_Y}.
\end{aligned}$$

REMARK. The Observations 3, Lemma 4 and Theorem 6 can be obtained identically if the gluing takes place along an open instead of a closed subspace. However, only the latter case will be studied in detail, since the final aim is at analysing the stucture of finite CW-complexes which are iterated mapping cones.

For such iterated mapping cones the following is true:

7) THEOREM. *Given three spaces X, Y and Z, X compact and Hausdorff as*

well as two maps $f\colon X \longrightarrow Y$ *and* $g\colon C_f \longrightarrow Z$ *then there exists a map* $h\colon \Sigma X \longrightarrow Z_{g|_Y} \sqcup C_2 Y$ *such that*

$$Z_g \sqcup C_2(Y_f \sqcup C_1 X) \cong (Z_{g|_Y} \sqcup C_2 Y)_h \sqcup C\Sigma X$$

The proof of Theorem 7 depends on the following.

8) Lemma. *If X is a compact Hausdorff space, then there exists a homeomorphism*

$$q\colon C_2 C_1 X \longrightarrow C\Sigma X$$

extending the obvious one

$$C_2 X_{1_X} \sqcup C_1 X \longrightarrow \Sigma X$$

Proof. Let

$$K = (X \times \{-1, 1\} \times I) \cup (X \times [-1, 1] \times 1) \cup (* \times [-1, 1] \times I)$$

and

$$L = (X \times 1 \times I_2) \cup (X \times I_1 \times 1) \cup (* \times I_1 \times I_2).$$

Then

$$C\Sigma X = X \times [-1, 1] \times I / K$$

and

$$C_2 C_1 X = X \times I_1 \times I_2 / L.$$

If $f\colon X \times [-1, 1] \times I \longrightarrow X \times I_1 \times I_2$ is the map defined by

$$f(x, s, t) = \begin{cases} (x,\ s + t - st,\ t) & \text{if} \quad s \geq 0 \\ (x,\ t,\ t - s + st) & \text{if} \quad s \leq 0 \end{cases}$$

it is readily verified that $f(K) \subset L$, and therefore f induces a map $f'\colon C\Sigma X \longrightarrow C_2 C_1 X$ making the following square of maps commutative:

$$\begin{array}{ccc} X \times [-1, 1] \times I & \xrightarrow{\quad f \quad} & X \times I_1 \times I_2 \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle p_2} \\ C\Sigma X & \xrightarrow{\quad f' \quad} & C_2 C_1 X \end{array}$$

where p_1 and p_2 are the identification maps.

The construction of f implies that f' is $1-1$ and onto. Since both $C\Sigma X$ and C_2C_1X are compact Hausdorff spaces, f' actually is a homeomorphism extending the obvious one from ΣX to $C_2X_{1_X} \sqcup C_1X$.

Proof of Theorem 7 (cf. diagram on the next page). The solid-arrow subdiagram is commutative, in particular

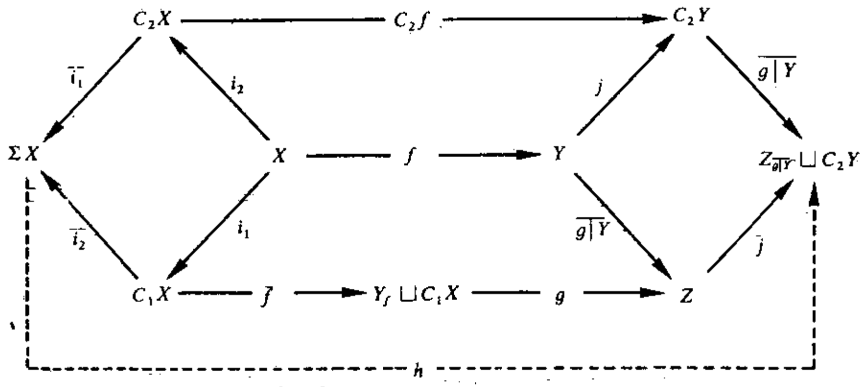
$$\bar{j}\circ 1_Z\circ g\circ f\circ i_1 = \overline{g|_Y}\circ C_2 f\circ i_2 ,$$

hence

a) $\exists!$ h such that

$$h\circ i_1 = \overline{g|_Y}\circ C_2 f \qquad \text{and} \qquad h\circ \bar{i}_2 = \bar{j}\circ 1_Z\circ g\circ f.$$

h is the map the existence of which is claimed in the theorem. The following diagram contains all information about it for future reference.



The definition of h and q imply:

$$\bar{h}\circ q\circ C_2 i = \bar{h}\circ k\circ \bar{i}_1 = \bar{k}\circ h\circ \bar{i}_1 = \bar{k}\circ \overline{g|_Y}\circ C_2 f.$$

Hence

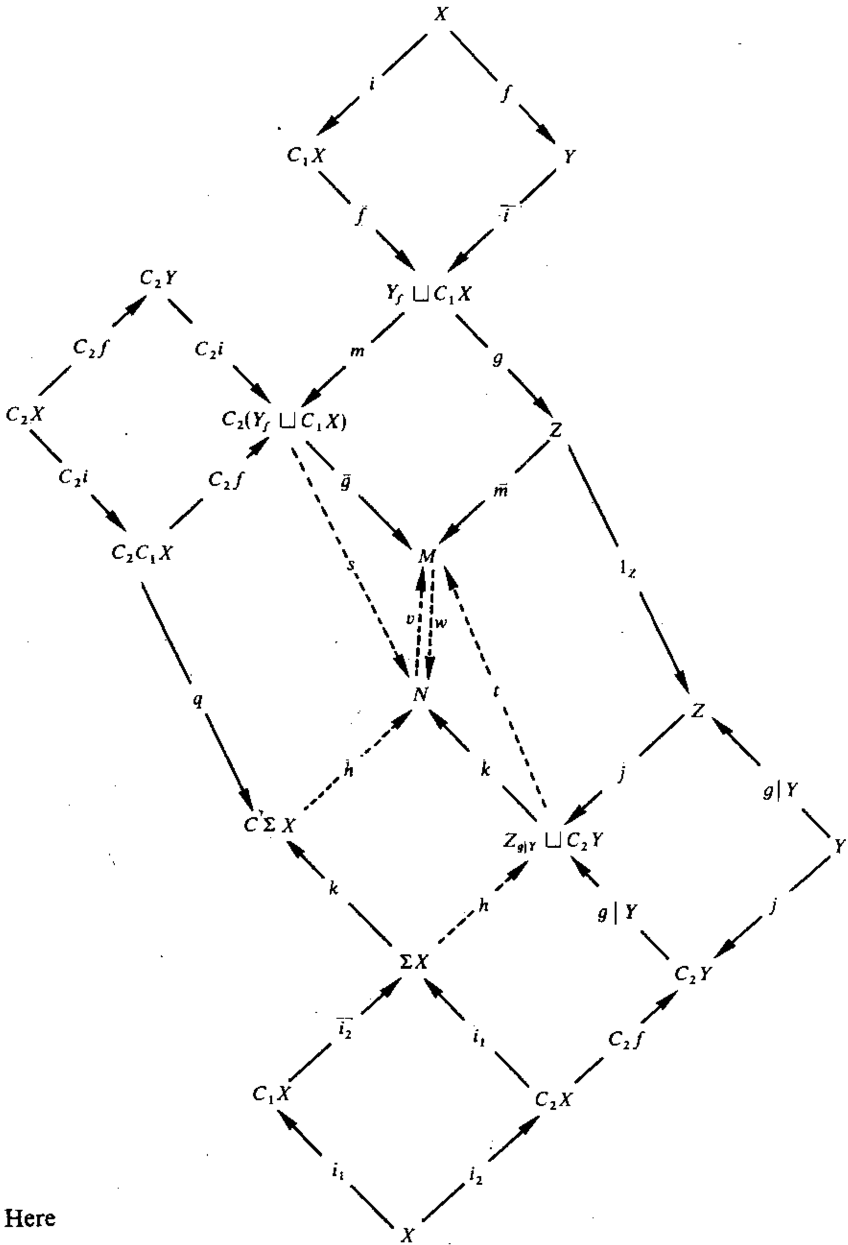
b) $\exists!$ s such that

$$s\circ \overline{C_2 i} = \bar{k}\circ \overline{g|_Y} \qquad \text{and} \qquad s\circ \overline{C_2 f} = \bar{h}\circ q.$$

c) $\exists!$ t such that

$$t\circ \bar{j} = \bar{m}\circ 1_Z{}^{-1} \qquad \text{and} \qquad t\circ \overline{g|_Y} = \bar{g}\circ \overline{C_2 i},$$

because $\bar{m}\circ 1_Z{}^{-1}\circ g|_Y = \bar{g}\circ \overline{C_2 i}\circ j$



Here

$$M = Z_g \sqcup C_2(Y_f \sqcup C_1 X) \qquad \text{and} \qquad N = (Z_{g|Y} \sqcup C_2 Y)_h \sqcup C\,\Sigma\,X$$

d) Claim: $t \circ h = \bar{g} \circ \overline{C_2 f} \circ q^{-1} \circ k$, or equivalently

$$\bar{g} \circ \overline{C_2 f} \circ q^{-1} \circ k \circ \bar{i}_1 = t \circ h \circ \bar{i}_1$$

and

$$\bar{g} \circ \overline{C_2 f} \circ q^{-1} \circ k \circ \bar{i}_2 = t \circ h \circ \bar{i}_2$$

However

$$\bar{g} \circ \overline{C_2 f} \circ q^{-1} \circ k \circ \bar{i}_1 = \bar{g} \circ \overline{C_2 f} \circ C_2 i = \bar{g} \circ \overline{C_2 i} \circ C_2 f \overset{c)}{=} t \circ \bar{g}\rceil_Y \circ C_2 f \overset{a)}{=} t \circ h \circ \bar{i}_1$$

and

$$\bar{g} \circ \overline{C_2 f} \circ q^{-1} \circ k \circ \bar{i}_2 = \bar{g} \circ m \circ \bar{f} = \bar{m} \circ g \circ \bar{f} = \bar{m} \circ 1_Z^{-1} \circ 1_Z \circ g \circ \bar{f} \overset{c)}{=} t \circ \bar{j} \circ 1_Z \circ g \circ \bar{f}$$
$$\overset{a)}{=} t \circ h \circ \bar{i}_2$$

Hence $\exists !\, v$ such that

$$v \circ \bar{k} = t \qquad \text{and} \qquad v \circ \bar{h} = \bar{g} \circ \overline{C_2 f} \circ q^{-1}$$

e) Claim: $\bar{k} \circ \bar{j} \circ 1_Z \circ g = s \circ m$, or equivalently

$$s \circ m \circ \bar{i} = \bar{k} \circ \bar{j} \circ 1_Z \circ g \circ \bar{i}$$

and

$$s \circ m \circ \bar{f} = \bar{k} \circ \bar{j} \circ 1_Z \circ g \circ \bar{f}$$

However

$$s \circ m \circ \bar{i} = s \circ \overline{C_2 i} \circ j \overset{b)}{=} \bar{k} \circ \bar{g}\rceil_Y \circ j = \bar{k} \circ \bar{j} \circ 1_Z \circ g \circ \bar{i}$$

and

$$s \circ m \circ \bar{f} = s \circ \overline{C_2 f} \circ q^{-1} \circ k \circ \bar{i}_2 \overset{b)}{=} \bar{h} \circ q \circ q^{-1} \circ k \circ \bar{i}_2 = \bar{h} \circ k \circ \bar{i}_2 = \bar{k} \circ h \circ \bar{i}_2$$
$$\overset{a)}{=} \bar{k} \circ \bar{j} \circ 1_Z \circ g \circ \bar{f}$$

Hence $\exists !\, w$ such that

$$w \circ \bar{m} = \bar{k} \circ \bar{j} \circ 1_Z \qquad \text{and} \qquad w \circ \bar{g} = s.$$

f) Claim: $v \circ w = 1$, or equivalently

$$v \circ w \circ \bar{m} = \bar{m} \qquad \text{and} \qquad v \circ w \circ \bar{g} = \bar{g}.$$

However

$$v \circ w \circ \bar{m} \overset{e)}{=} v \circ \bar{k} \circ \bar{j} \circ 1_Z \overset{d)}{=} t \circ \bar{j} \circ 1_Z \overset{b)}{=} \bar{m} \circ 1_Z{}^{-1} \circ 1_Z = \bar{m}$$

while the second equality is equivalent to the following pair:

$$v \circ w \circ \bar{g} \circ \overline{C_2 i} = \bar{g} \circ \overline{C_2 i}$$

and

$$v \circ w \circ \bar{g} \circ \overline{C_2 f} = \bar{g} \circ \overline{C_2 f}$$

But

$$v \circ w \circ g \circ \overline{C_2 i} \overset{e)}{=} v \circ s \circ \overline{C_2 i} \overset{b)}{=} v \circ \bar{k} \circ \bar{g}|_Y \overset{d)}{=} t \circ \bar{g}|_Y \overset{c)}{=} \bar{g} \circ \overline{C_2 i}$$

and

$$v \circ w \circ \bar{g} \circ \overline{C_2 f} \overset{e)}{=} v \circ s \circ \overline{C_2 f} \overset{b)}{=} v \circ \bar{h} \circ q \overset{d)}{=} \bar{g} \circ \overline{C_2 f} \circ q^{-1} \circ q = \bar{g} \circ \overline{C_2 f}$$

g) Claim: $w \circ v = 1$, or equivalently

$$w \circ v \circ \bar{h} = \bar{h} \quad \text{and} \quad w \circ v \circ \bar{k} = \bar{k}.$$

However

$$w \circ v \circ \bar{h} \overset{d)}{=} w \circ \bar{g} \circ \overline{C_2 f} \circ q^{-1} \overset{e)}{=} s \circ \overline{C_2 f} \circ q^{-1} \overset{b)}{=} \bar{h} \circ q \circ q^{-1} = \bar{h}$$

while the second equality is equivalent to the following pair:

$$w \circ v \circ \bar{k} \circ \bar{j} = \bar{k} \circ \bar{j}$$

and

$$w \circ v \circ \bar{k} \circ \bar{g}|_Y = \bar{k} \circ \bar{g}|_Y$$

But

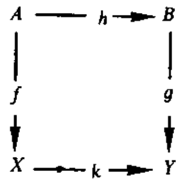
$$w \circ v \circ \bar{k} \circ \bar{j} \overset{d)}{=} w \circ t \circ \bar{j} \overset{c)}{=} w \circ \bar{m} \circ 1_Z{}^{-1} \overset{e)}{=} \bar{k} \circ \bar{j} \circ 1_Z \circ 1_Z{}^{-1} = \bar{k} \circ \bar{j}$$

and

$$w \circ v \circ \bar{k} \circ \bar{g}|_Y \overset{d)}{=} w \circ t \circ \bar{g}|_Y \overset{c)}{=} w \circ \bar{g} \circ \overline{C_2 i} \overset{e)}{=} s \circ \overline{C_2 i} \overset{b)}{=} \bar{k} \circ \bar{g}|_Y .$$

So far only the topological type of adjunction spaces has been considered. The following result will permit to deduce the central fact about the homotopy type of iterated mapping cones as a consequence of Theorem 6.

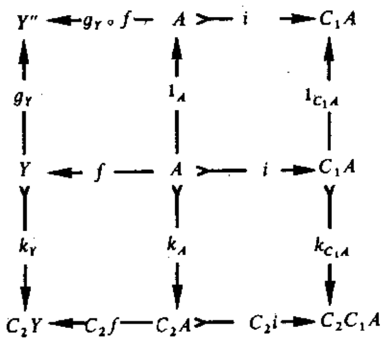
9) THEOREM (cf. (4), Theorem 6.6).



If this square of maps is commutative up to homotopy, and h, k are homotopy equivalences then

$$C_f \simeq C_g$$

i.e. the homotopy type of a mapping cone does not change when any of its ingredients are changed within their respective homotopy class.



The diagram above satisfies the hypotheses of Theorem 6. The map F which therefore exists has the special form

$$F \doteq \overline{g_Y} \circ C_2 f$$

i.e. F is homotopic to the constant map.

Hence the following topological equivalence holds:

$$(Y''_{g_Y \circ f} \sqcup C_1 A)_G \sqcup (C_2 Y_{C_2 f} \sqcup C_2 C_1 A) \cong (Y''_{g_Y} \sqcup C_2 Y)_{\overline{g_Y} \circ C_2 f} \sqcup C_2 C_1 A$$

Using Corollary 5 on the left and Theorem 9 twice on the right, this topological equivalence goes over into the following homotopy equivalence

$$(Y''_{g_Y \circ f} \sqcup C_1 A)_G \sqcup C_2 (Y_f \sqcup C_1 A) \xrightarrow{l} Y''_{g_Y} \sqcup C_2 Y$$

If

$$j : Y''_{g_Y \circ f} \sqcup C_1 A \longrightarrow (Y''_{g_Y \circ f} \sqcup C_1 A)_G \sqcup C_2(Y_f \sqcup C_1 A)$$

is the inclusion of the base space into the mapping cone of G, then two more applications of Theorem 9 infer

10) THEOREM

$$\begin{array}{ccccc} A & \xrightarrow{\;f\;} & Y & \overset{g_Y}{\dashrightarrow} & Y'' \\ \Big\downarrow i & & \Big\downarrow \bar{i} & & \\ C_1 A & \xrightarrow{\;\bar{f}\;} & Y_f \sqcup C_1 A & & \end{array}$$

$$\Sigma C_f \simeq C_{l \circ j}$$

where

$$l \circ j : C_{g_Y \circ f} \longrightarrow C_{g_Y}$$

REMARK. The value of this theorem lies in the possibility of building two new elements, namely Y'' and g_Y into the suspension of any mapping cone, without changing its homotopy type.

Chapter II: Torsion Spaces

11) DEFINITION. X be a topological space with a basepoint $*$ on which a homotopy associative coproduct structure m: $X \longrightarrow X \vee X$ is defined for which the constant map $*$: $X \longrightarrow X$ is a left and right homotopy identity. Under these conditions the set $[X, X]$ of homotopy classes of maps from X into itself forms a semigroup with respect to the operation

$$[f] \cdot [g] = [\chi \circ (f \vee g) \circ m]\cdot$$

where χ: $X \vee X \longrightarrow X$ is the folding map. The class $[*]$ is the left and right identity.

X is called a *torsion space* with respect to the coproduct structure m if $[1_X]$

has finite order in the semigroup $[X, X]$. This order will be called the torsion number of X with respect to m: τ_X.

12) REMARKS. a) Up to a homotopy equivalence, the only coproduct structure known for topological spaces is the one due to the suespension structure. In the sequal always that one is meant and m will not be mentioned any more.

b) Obviously, the space containing only its basepoint is torsion and has torsion number 1.

c) Any finite sum of torsion spaces is again torsion.

d) Simply connected pseudo-projective spaces, $S^n{}_k \sqcup CS^n$ with k, $n \geq 2$, are torsion. (cf. (8), Theorem 1.5).

e) For any finitely generated abelian group G and integer $n \geq 1$, there exists a finite CW-complex $K'(G, n)$ with abelian fundamental group and reduced integral homology as follows:

$$\tilde{H}_i(K'(G, n)) = \begin{cases} G & \text{if} \quad i = n \\ 0 & \text{if} \quad i \neq n \end{cases}$$

If $n \geq 2$, the finite CW-complex $K'(G, n)$ is determined up to homotopy type by G and n (cf. (2), I.3.e).
The spaces $K'(G, n)$ are called Moore spaces.
If G is a finite abelian group, then $G \cong Z_{k_1} \oplus \ldots \oplus Z_{k_r}$. Hence, if $n \geq 2$,

$$K'(Z_{k_1} \oplus \ldots \oplus Z_{k_r},\ n) \simeq (S^n{}_{k_1} \sqcup CS^n) \vee \ldots \vee (S^n{}_{k_r} \sqcup CS^n)$$

since for finite CW-complexes, finite sums and homology with integral coefficients are interchangable. By Remarks c) and d) it follows that Moore spaces with finite integral homology in dimension $n \geq 2$ are torsion.

The following two theorems describe methods for the construction of new torsion spaces from given ones.

13) THEOREM. *Let T be a torsion space and f: $S^{q_1} \vee \ldots \vee S^{q_r} \longrightarrow T$ some map, where $r \geq 1$ and $q_i \geq 0$, $i = 1, \ldots, r$.*

If

$$p : T_f \sqcup C(S^{q_1} \vee \ldots \vee S^{q_r}) \longrightarrow S^{q_1+1} \vee \ldots \vee S^{q_r+1}$$

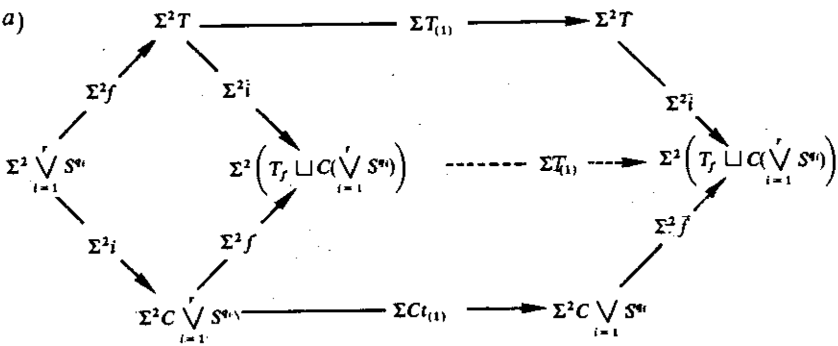
is the identification map, collapsing T to a point, and $\bigvee_i k_i = k_1 \vee \ldots \vee k_r$ is the map from

$$\bigvee_i S^{q_i+1} = S^{q_1+1} \vee \ldots \vee S^{q_r+1}$$

to itself, being the k_i-fold identitv on the component S^{q_i+1} where $k_i \geq 2$, then the two-fold suspension of the mapping cone of $(\bigvee_i k_i) \circ p$ is also torsion.

PROOF. Let $t_{(1)}$ be the τ_T-fold identity map of ΣX acting on the last suspension coordinate.

a)

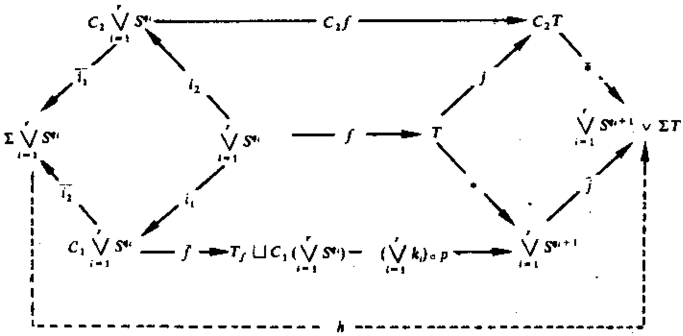


The solid-arrow subdiagram commutes and induces the unique map $\Sigma T_{(1)}$ which is the τ_T-fold identity map acting on the first coordinate of the double-suspension.

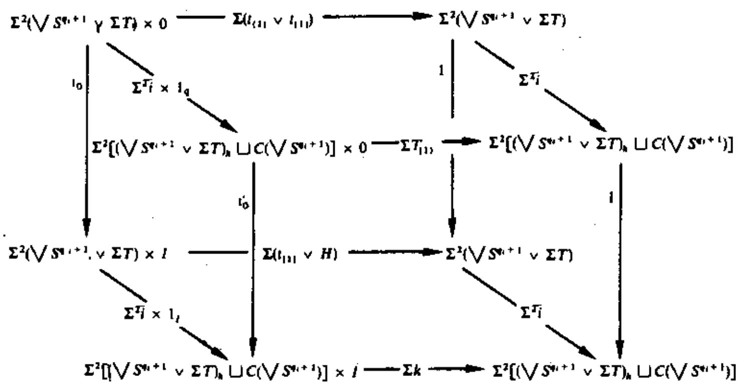
b) By Theorem 7, the two-fold suspension of the mapping cone of $(\bigvee_{i=1}^{r} k_i) \circ p$ is homeomorphic to

$$\Sigma^2[(S^{q_1+1} \vee \ldots \vee S^{q_r+1} \vee \Sigma T)_h \sqcup C(S^{q_1+1} \vee \ldots \vee S^{q_r+1})]$$

where h is uniquely determined by the next diagram (cf. p.114).

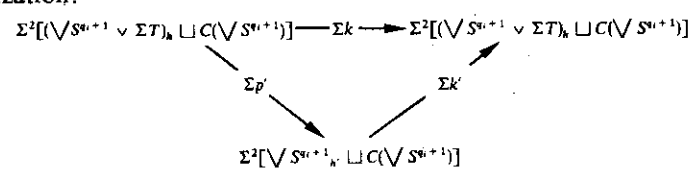


c)



As in a) $\Sigma(t_{(1)} \vee t_{(1)})$ uniquely induces the map $\Sigma T_{(1)}$ which is the τ_T-fold identity map, acting on the first coordinate of the double suspension.
By hypothesis, $\Sigma t_{(1)} : \Sigma^2 \Sigma T \longrightarrow \Sigma^2 \Sigma T$ is homotopic to the constant map. Let ΣH be this homotopy. Since $\Sigma^2 \bar{i} \times 1_I$ is a cofibration, the homotopy $\Sigma^2 \bar{i} \circ \Sigma(t_{(1)} \vee H)$ has an extension ΣK to $\Sigma^2[(\bigvee S^{q_i+1} \vee \Sigma T)_h \sqcup C(\bigvee S^{q_i+1})] \times I$, with initial map $\Sigma T_{(1)}$ and terminal map Σk, say.

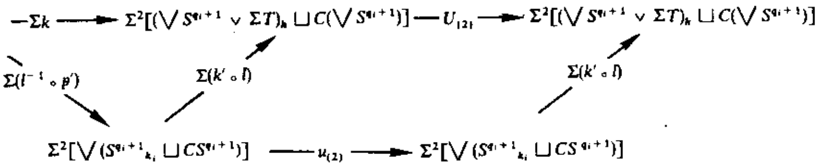
Σk maps $\Sigma^2 \Sigma T$ to the basepoint and therefore has the following unique factorization:



From b) it follows that $h' \simeq \bigvee_{i=1}^{r} k_i$. Hence

$$\Sigma^2[\bigvee S^{q_i+1}_{h'} \sqcup C(\bigvee S^{q_i+1})] \xleftarrow[\simeq]{\Sigma l} \Sigma^2[\bigvee (S^{q_i+1}_{k_i} \sqcup CS^{q_i+1})]$$

However, the last space is torsion by Remarks 12 c) and d) having torsion number u, say. The above triangle can now be extended in the following way:



where $u_{(2)}$ and $U_{(2)}$ are the u-fold identity maps on the respective spaces, acting on the second suspension coordinate of the double suspension. Hence the square is commutative. Further, $u_{(2)} \simeq *$ implies $U_{(2)} \circ \Sigma k \simeq *$.

All together it has been shown that $U_{(2)} \circ \Sigma T_{(1)}$ which is the $u \cdot t$-fold identity on $\Sigma^2[(\bigvee S^{q_i+1} \vee \Sigma T)_h \sqcup C(\bigvee S^{q_i+1})]$ is homotopic to the constant map, i.e. the latter space is torsion.

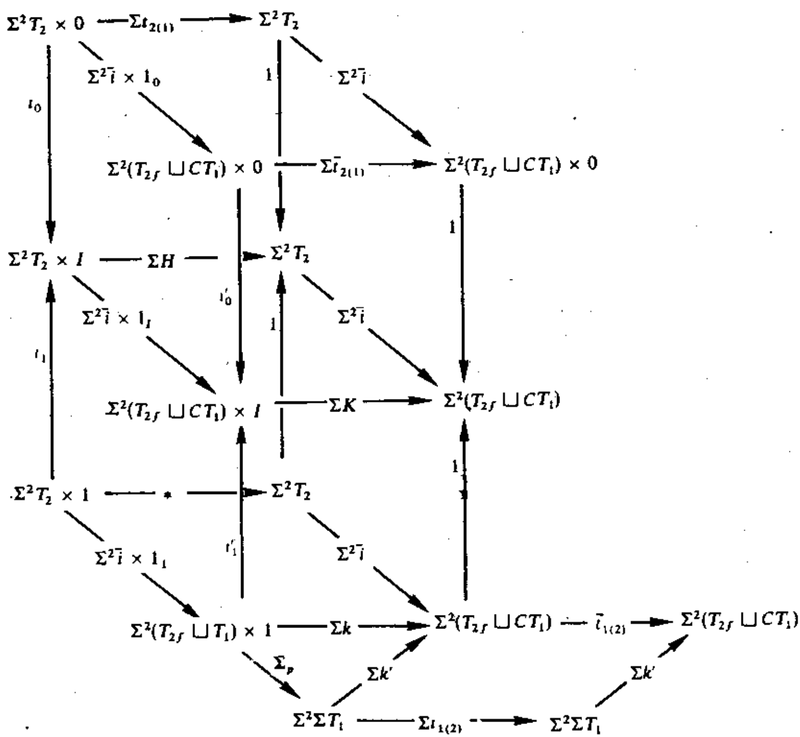
14) THEOREM *Lt $f\colon T_1 \longrightarrow T_2$ be anv map between torsion spaces, then $\Sigma^2 C_f$ is also torsion.*

Cf. diagram on the next page.

PROOF. If $\Sigma t_{2(1)}$ is the τ_{T_2}-fold identity of $\Sigma^2 T_2$ acting on the first coordinate of the double suspension it uniquely induces $\Sigma \tilde{t}_{2(1)}$, the τ_{T_2}-fold identity map on $\Sigma^2(T_{2f} \sqcup CT_1)$ acting on the first coordinate.

By hypothesis, $\Sigma t_{2(1)} \simeq *$, by means of a homotopy ΣH. Also $\Sigma^2 \bar{i} \times 1_I$ is a cofibration and $\Sigma^2 \bar{i} \circ \Sigma H$ therefore has an extension ΣK with initial map $\Sigma \tilde{t}_{2(1)}$ and terminal map Σk, say. Σk uniquely factors through Σp giving rise to $\Sigma k'$.

If $t_{1(2)}$ and $\tilde{t}_{1(2)}$ are the τ_{T_1}-fold identity maps of $\Sigma^2 \Sigma T_1$ and $\Sigma^2(T_{2f} \sqcup CT_1)$ respectively, acting on the second coordinate of the respective double sus-



pension, then the last square of the diagram is commutative. By hypothesis $t_{1(2)} \simeq *$. Hence $\tilde{t}_{1(2)} \circ \Sigma \tilde{t}_{2(1)} \simeq *$. This means that the $\tau_{T_1} \cdot \tau_{T_2}$-fold identity map of the twice suspended mapping cone of f is homotopic to the constant map.

15) COROLLARY. *If X is a finite simplv connected CW-complex, then $\Sigma^2 X$ is torsion iff the reduced integral homologv of X is finite.*

PROOF. X has a homology decomposition (cf. (4), Theorem 8.2), i.e. it has the same homotopy type as some space X' obtained inductively by adjoining cones $CK'(\tilde{H}_n(X), n-1)$ along their base space to what already has been constructed, adjoining the first one to the basepoint. Suspending X' twice and applying Theorem 14) to each adjunction operation gives one implication.

For the other observe that for CW-complexes integral homology and finite sums are compatible; hence the assumptions on the coproduct structure on $\Sigma^2 X$ imply the following equalities, where A stands for $\tilde{H}_n(\Sigma^2 X)$:

a) $$A \xrightarrow{m*} \begin{array}{c} A \xrightarrow{1*} A \\ \oplus \qquad \oplus \\ A \xrightarrow{0} A \end{array} \xrightarrow{\chi*} A = A \xrightarrow{1*} A = A \xrightarrow{m*} \begin{array}{c} A \xrightarrow{0} A \\ \oplus \qquad \oplus \\ A \xrightarrow{1*} A \end{array} \xrightarrow{\chi*} A$$

b) $$\begin{array}{ccccccc} & & & A & A & \xleftarrow{1*} & A \\ & & A \xrightarrow{m*} & \oplus & \oplus & & \\ & & & & & & \oplus \xleftarrow{m*} A \\ A \xrightarrow{m*} & \oplus & & A & = A & & \\ & & & \oplus & \oplus & \xleftarrow{m*} & A \\ & & A \xrightarrow{1*} & A & A & & \end{array}$$

Combinations of these equalities yield:

$$\begin{array}{c} A \\ A \xrightarrow{m*} \oplus \\ A \end{array} = A \xrightarrow{m*} \begin{array}{c} A \xrightarrow{1*} A \xrightarrow{1*} A \\ \qquad\quad \oplus \qquad \oplus \xrightarrow{\chi*} A \\ \oplus \qquad A \xrightarrow{0} A \qquad\quad \\ \qquad\qquad\qquad\qquad \oplus \\ A \xrightarrow{m*} \oplus \qquad \oplus \qquad\quad \\ \qquad A \xrightarrow{1*} A \xrightarrow{1*} A \end{array}$$

$$= A \xrightarrow{m*} \begin{array}{c} \qquad A \xrightarrow{1*} A \xrightarrow{1*} A \\ A \xrightarrow{m*} \oplus \qquad \oplus \qquad\quad \\ \oplus \qquad A \xrightarrow{0} A \qquad \oplus \\ \qquad\quad \oplus \qquad \oplus \xrightarrow{\chi*} A \\ A \xrightarrow{1*} A \xrightarrow{1*} A \end{array}$$

Hence m_* is the diagonal. It further follows from a) that χ_* is the ordinary addition. The fact that $\Sigma^2 X$ is torsion then implies that 1_* has finite order, or equivalently, that any element of $\tilde{H}_n(\Sigma^2 X)$ has finite order. Since the integral homology of a finite CW-complex is finitely generated, the second implication also follows.

REMARK. Modulo suspension, all finite CW-complexes which are torsion can be obtained from the trivial torsion space, $*$, using the two methods described in Theorems 13 and 14.

16) DEFINITION. $\{S^{q_k}\}_{k=1}^{n}$ be a sequence of spheres, $-1 \leq q_1 \leq \ldots \leq q_n$ where $S^{-1} = *$.

$\{X_k\}_{k=0}^{n}$ be a sequence of spaces.

$\{f_k\}_{k=1}^{n}$ be a sequence of maps $f_k : S^{q_k} \longrightarrow X_{k-1}$ such that $X_0 = *$, and inductively $X_k = X_{k-1\,f_k} \sqcup CS^{q_k}$, then $X = X_n$ is the *finite* CW-*complex* determined by these three sequences.

By Theorem 9, this definition of finite CW-complexes corresponds to the usual one, where the spheres are not ordered by dimension, up to homotopy type.

In the next and final chapter it will be shown that any finite CW-complex X, after it has been sufficiently often suspended, has a torsion decomposition, i.e. has the same homotopy type as the mapping cone of some map

$$f : S^{q_1} \vee \ldots \vee S^{q_k} \longrightarrow T$$

where T is a torsion space.

In case $X = *$ the claim is evident. The general case will be established by induction where the induction step can be described as follows:

17) *Induction step.* Given two maps $f : S^{p_1} \vee \ldots \vee S^{p_r} \longrightarrow T$ and $g : S^p \longrightarrow C_f$, where T is torsion and $-1 \leq p_1 \leq \ldots \leq p_n$, then there exists another torsion space T' and a map $f' : S^{q_1} \vee \ldots \vee S^{q_s} \longrightarrow T'$ such that

$$\Sigma^m C_g \simeq C_{f'} \qquad \text{for some} \qquad m \geq 0$$

18) Theorem 10 suggests the following:

$$\begin{array}{ccccc} S^p & \xrightarrow{\;g\;} & T_f \sqcup C(S^{p_1} \vee \ldots \vee S^{p_r}) & \xrightarrow{\;g_Y\;} & Y \\ \big\downarrow i & & \big\downarrow i & & \\ C_1 S^p & \xrightarrow{\;\bar{g}\;} & [T_f \sqcup C(S^{p_1} \vee \ldots \vee S^{p_r})]_g \sqcup C_1 S^p & & \end{array}$$

With the prolongation g_Y of g, $\Sigma C_g \simeq C_{l \circ j}$ with $l \circ j : C_{g_Y \circ g} \longrightarrow C_{g_Y}$.

19) *Choice of the space Y in* 18. Theorem 13 suggests to choose for

$$g_Y\colon T_f \sqcup C(S^{p_1} \vee \ldots \vee S^{p_r}) \longrightarrow Y$$

some map of the form

$$T_f \sqcup C(S^{p_1} \vee \ldots \vee S^{p_r}) \xrightarrow{\pi} S^{p_1+1} \vee \ldots \vee S^{p_r+1} \xrightarrow{k_1 \vee \ldots \vee k_r} S^{p_1+1} \vee \ldots \vee S^{p_r+1}$$

where π is the identification of T to a point, and $k_i \geq 2$ for $i = 1, \ldots, r$, because then the twice suspended base space of the mapping cone $C_{l \circ j}$ will be torsion, while the top space will have the form:

$$(S^{p_1+1} \vee \ldots \vee S^{p_r+1})_{(k_1 \vee \cdots \vee k_r) \circ \pi \circ g} \sqcup CS^p$$

With this the space Y in 18 has been fixed. The rest of this chapter is devoted to the choice of $k_1 \vee \ldots \vee k_r$ which determines g_Y, and to a convenient reinterpretation of the top space of the mapping cone $C_{l \circ j}$.

20) Theorem (cf. (6), Theorem 3.1), *If (X, x_0) and (Y, y_0) are two spaces with basepoints then for $p \geq 2$*

$$\pi_p(X \vee Y, *) \cong \pi_p(X, x_0) \oplus \pi_p(Y, y_0) \oplus \pi_{p+1}(X \times Y, X \vee Y)$$

where the left hand side is the direct sum of the images of the monomorphisms induced by the natural injections

$$(X, x_0) \longrightarrow (X \vee Y, *) \longleftarrow (Y, y_0)$$

as well as the boundary homomorphism

$$\partial\colon \pi_{p+1}(X \times Y, X \vee Y) \longrightarrow \pi_p(X \vee Y, *)$$

of the exact homotopy sequence for the triple $(X \times Y, X \vee Y, *)$ which also proves to be a monomorphism.

21) Theorem. *If $2 \leq q_1 \leq \ldots \leq q_r$ and $q_1 + q_2 \geq p + 2 \geq 4$ then*

$$\pi_p(\bigvee_{i=1}^{r} S^{q_i}) \cong \bigoplus_{i=1}^{r} \pi_p(S^{q_i})$$

and the left hand side is the direct sum of the images of the r monomorphisms induced by the natural injections of the spheres into their sum.

Proof. Induction with respect to the number r of summands. The claim is trivial for $r = 1$. Let B_{r-1} be the sum of the first $r-1$ spheres, and assume the theorem for $r-1$, $r > 1$. Then by Theorem 20 it suffices to show that

$$\pi_{p+1}(B_{r-1} \times S^{q_r},\ B_{r-1} \vee S^{q_r}) = 0$$

for the induction step.

By the Künneth Formula for integral homology

$$[H(B_{r-1},\ *) \oplus H(S^{q_r},\ *)]_n \cong H_n(B_{r-1} \times S^{q_r},\ B_{r-1} \vee S^{q_r})$$

The first non-trivial group on the left hand side may appear in dimensions $n \geq q_1 + q_r \geq q_1 + q_2 \geq p + 2 \geq 4$. Hence by the relative Hurewicz Isomorphism Theorem also the first non-trivial homotopy groups of $(B_{r-1} \times S^{q_r}, B_{r-1} \vee S^{q_r})$ may appear in dimensions $n \geq p + 2$; in particular the above homotopy group vanishes.

Remark. This is a corollary of a much more general result due to Hilton (cf. (5)).

22) Theorem. *If* $2 \leq q_1 \leq \ldots \leq q_r$ *and* $2q_1 \geq p + 2 \geq 4$ *and exactlv* $t \geq 0$ *of the q's are equal to p, then if*

$$f: S^p \longrightarrow S^{q_1} \vee \ldots \vee S^{q_r}$$

is a suspension and

a) $t = 0$, *then there exists an integer* $n \geq 1$ *such that* $(\bigvee_{i=1}^{r} n) \circ f \simeq *$.

b) $t > 0$ *and*

$$\mathrm{i}: S_1^p \vee \ldots \vee S_t^p \longrightarrow S^{q_1} \vee \ldots \vee S^{q_r}$$

is the natural inclusion, then there exists an integer $n \geq 1$ *such that*

$$(\bigvee_{i=1}^{r} n) \circ f \simeq i \circ f' \quad \text{for some map} \quad f': S^p \longrightarrow \bigvee_{i=1}^{t} S^p.$$

Proof. By Theorem 21

$$\pi_p(\bigvee_{i=1}^{r} S^{q_i}) \cong \bigoplus_{i=1}^{r} \pi_p(S^{q_i})$$

The homotopy groups of spheres are finite except for the following two cases: (cf. (7), Theorem 9, p. 516)

$$\pi_p(S^p) \quad \text{for all} \quad p \geq 1, \quad \text{and} \quad \pi_p(S^{(p+1)/2}) \quad \text{for odd } p.$$

By hypothesis $2q_1 \geq p + 2$ so that the second case cannot appear in the above direct sum, while the first case will furnish precisely t infinite cyclic direct summands. Assume that the finite part is of order n, say.

The element in $\bigoplus_{i=1}^{r} \pi_p(S^{q_i})$ corresponding to $[f] \in \pi_p(\bigvee_{i=1}^{r} S^{q_i})$ is of the form: $(h_1, \ldots, h_{k+1}, \ldots, h_{k+t}, \ldots, h_r)$ with $h_i \in \pi_p(S^{q_i})$ and such that precisely $q_{k+1} = \ldots = q_{k+t} = p$. Since f is a suspension, $[(\bigvee_{i=1}^{r} n) \circ f] = n \cdot [f]$.

However, $n \cdot (h_1, \ldots, h_r) = (0, \ldots, n \cdot h_{k+1}, \ldots, n \cdot h_{k+t}, \ldots, 0)$, which corresponds to $i_*([f'])$ for some $[f'] \in \pi_p(\bigvee_1^t S^p)$, where $i : \bigvee_1^t S^p \longrightarrow \bigvee_{i=1}^{r} S^{q_i}$ is the natural injection.

23) *Choice of the map* $\bigvee_{i=1}^{r} k_i$ *in* 19).

To satisfy the conditions of Theorem 22 it may be necessary to suspend

$$S^{p_1+1} \vee \ldots \vee S^{p_r+1}{}_{(k_1 \vee \cdots \vee k_r) \circ \pi \circ g} \sqcup CS^p$$

s times, say. Then if n is the order of the torsion part of

$$\pi_{p+s}(\bigvee_{i=1}^{r} S^{p_i+1+s}), \quad \text{the choice will be} \quad \bigvee_{i=1}^{r} k_i = \bigvee_{i}^{r} n.$$

Under these circumstances the top-space of $\Sigma^s C_{l \circ j}$ mentioned in 19) will be of the form:

$$S^{p_1+1+s} \vee \ldots \vee S^{p_k+1+s} \vee [(\bigvee_1^t S^{p+s})_\phi \sqcup CS^{p+s}] \vee S^{p_{k+t+1}+1+s} \vee \\ \vee \ldots \vee S^{p_r+1+s}$$

up to homotopy type.

It remains to give $(\bigvee_1^t S^{p+s})_\phi \sqcup CS^{p+s}$ a more convenient form (cf. Remark 27).

24) LEMA. *Let* $2 \leq q_1 \leq \ldots \leq q_r$ *be such that* $q_1 + q_2 \geq p + 2 \geq 4$. $\beta_j : \pi_p(S^{q_j}) \longrightarrow \bigoplus_{i=1}^r \pi_p(S^{q_i})$ *be the natural injection, and* $k_j : S^{q_1} \vee \ldots \vee S^{q_r} \longrightarrow S^{q_j}$ *be the map which sends all the spheres to the basepoint with the exception of* S^{q_j} *on which* k_j *is the identity map. Then*

$$(\beta_1 \circ (k_1)_*, \ldots, \beta_r \circ (k_r)_*) : \pi_p(\bigvee_{i=1}^r S^{q_i}) \longrightarrow \bigoplus_{i=1}^r \pi_p(S^{q_i})$$

is the inverse of the isomorphism $(i_1)_* + \ldots + (i_r)_*$.

PROOF: Enough to show: $(\beta_1 \circ (k_1)_*, \ldots, \beta_r \circ (k_r)_*) \circ ((i_1)_* + \ldots + (i_r)_*)$ is the identity.

Let $([a_1], \ldots, [a_r]) \in \pi_p(S^{q_1}) \oplus \ldots \oplus \pi_p(S^{q_r})$, then

$$(\beta_1 \circ (k_1)_*, \ldots, \beta_r \circ (k_r)_*) \circ ((i_1)_* + \ldots + (i_r)_*)([a_1], \ldots, [a_r]) =$$
$$(\beta_1 \circ (k_1)_*, \ldots, \beta_r \circ (k_r)_*)([i_1 \circ a_1] + \ldots + [i_r \circ a_r]) =$$
$$([k_1 \circ i_1 \circ a_1] + \ldots + [k_1 \circ i_r \circ a_r], \ldots, [k_r \circ i_1 \circ a_1] + \ldots + [k_r \circ i_r \circ a_r]) =$$
$$([a_1], \ldots, [a_r])$$

For $p \geq 1$, let $G_r^p = \{[x, t] \in \Sigma S^{p-1} \mid t = 0, 1/r, \ldots, k/r, \ldots, 1\}$ and define $F_r^p = \Sigma S^{p-1} / G_r^p$.

25) COROLLARY. *If* $2 \leq q_1 \ldots \leq q_r$ *such that* $q_1 + q_2 \geq p + 2 \geq 4$, *then any map*

$$f : S^p \longrightarrow \bigvee_{i=1}^r S^{q_i}$$

is homotopic to one that factors through the identification $\kappa : S^p \longrightarrow F_r^p$.

PROOF. The element corresponding to $([a_1], \ldots, [a_r])$ under the isomorphism $((i_1)_* + \ldots + (i_r)_*)$ may be represented by $[(a_1 \vee \ldots \vee a_r) \circ \kappa]$, because by Lemma 24

$$(\beta_1 \circ (k_1)_*, \ldots, \beta_r \circ (k_r)_*)([(a_1 \vee \ldots \vee a_r)] \circ \kappa) =$$
$$([k_1 \circ (a_1 \vee \ldots \vee a_r) \circ], \ldots, [k_r \circ (a_1 \vee \ldots \vee a_r) \circ \kappa]) = ([a_1], \ldots, [a_r]).$$

In particular $[f] \in \pi_p(S^{q_1} \vee \ldots \vee S^{q_r})$ may be represented by

$$[(k_1 \circ f \vee \ldots \vee k_r \circ f) \circ \kappa]$$

i.e.

$$f \simeq (k_1 \circ f \vee \ldots \vee k_r \circ f) \circ \kappa$$

26) THEOREM. *Let $p \geq 2$ and $t \geq 1$. If $f : S^p \longrightarrow \bigvee_1^t S^p$ is not homotopic to the constante map, then C_f has the same homotopy type as the following Moore space*

$$\bigvee_1^{t-1} S^p \vee (S^p{}_k \sqcup CS^p)$$

where

$$k = g.c.d.([k_1 \circ f], \ldots, [k_r \circ f]).$$

PROOF. Consider the following part of the exact homotopy sentence of the triplet $(C_f, \bigvee_1^t S^p, *)$, using the abbreviation $V = \bigvee_1^t S^p$

$$\longrightarrow \pi_{p+1}(C_f, V, *) \xrightarrow{\partial} \pi_p(V, *) \longrightarrow \pi_p(C_f, *) \longrightarrow \pi_p(C_f, V, *). \longrightarrow$$

$\pi_{p+1}(C_f, V, *)$ is generated by the characteristic map $\phi : (I^{p+1}, I^{p+1}, *) \longrightarrow (C_f, V, *)$ (cf. (9), Theorem 2.6.17). Also $\partial([\phi]) = [f]$ and $\pi_q(C_f, V, *) = 0$ for $q < p + 1$. Hence

$$\pi_p(C_f, *) \cong \pi_p(V, *)/\mathrm{Im}(\partial).$$

By Theorem 21,

$$\pi_p(V, *) \cong \bigoplus_1^t \pi_p(S^p, *) \cong \bigoplus_1^t Z.$$

Further by Corollary 25 $[f]$ may be represented by

$$[(k_1 \circ f \vee \ldots \vee k_t \circ f) \circ \kappa]$$

and the element corresponding to it in $\bigoplus_1^t Z$ is

$$([k_1 \circ f], \ldots, [k_t \circ f]).$$

Hence

$$\pi_p(C_f, *) \cong \bigoplus_1^t Z / [[k_1 \circ f], \ldots, [k_t \circ f]] \cong \bigoplus_1^{t-1} Z \oplus Z_k$$

where

$$k = g.c.d. \{[k_1 \circ f], \ldots, [k_t \circ f]\}.$$

The relative Hurewicz Isomorphism Theorem now implies

$$\tilde{H}_q(C_f) \cong \begin{cases} \bigoplus_1^{t-1} Z \oplus Z_k & \text{if} \quad q = p \\ 0 & \text{if} \quad q \neq p \end{cases}$$

Hence by Remark 12 e) $C_f \simeq \bigvee_1^{t-1} S^p \vee (S^p{}_k \sqcup CS^p)$

27) Remark. Hence the top-space mentioned in 17) under the choice made in 23) has the same homotopy type as the sum of some spheres together with possibly one pseudo-projective space, which is torsion by Remark 12 d).

Chapter IV: Torsion Decomposition

28) Theorem. *There exists a function $\sigma : |\mathscr{C}_f| \longrightarrow \mathbb{N}^+$ from the set of spaces which have the same homotopy type as a finite CW-complex to the set of non-negative integers such that, whenever $X \in |\mathscr{C}_f|$ there exists a map*

$$f : S^{p_1} \vee \ldots \vee S^{p_k} \longrightarrow T \qquad T \ \text{torsion}$$

such that

$$\Sigma^{\sigma(X)} X \simeq T_f \sqcup C(S^{p_1} \vee \ldots \vee S^{p_k}).$$

The sum of spheres is uniquely determined by $\sigma(X)$ and the infinite cyclic part of the reduced integral homology of X. On the other hand, T is not even determined up to homotopy type.

Proof. It suffices to carry out the induction step annaunced in 17), observing the special choices made in 19) and 23) as well as their consequences for the top-space mentioned in 27).

a) In case the top-space is a sum of spheres only, the desired form is already present, since the base space is torsion by 19).

b) If the top-space contains a pseudo-projective space in its sum the result follows from Theorem 7 and Theorem 14, combining the base space, which is torsion, with the torsion part of the top-space to a new torsion space.

This argument only applies after the initial situation has been sufficiently often suspended.

From the homology-properties of the relative CW-complex (C_f, T) it follows immediately that the infinite cyclic part of the reduced integral homology together with the number $\sigma(X)$ determines the sum of spheres uniquely.

It remains to give an example which will show that the torsion space is not determined up to homotopy type:

$$S^{n+1} \simeq (S^n{}_k \sqcup CS^n)_{\bar{i}} \sqcup CS^n \simeq *_* \sqcup CS^n \quad \text{for any} \quad k,\ n \geq 2$$

where $\bar{i}$ is, as usually, the inclusion of the base space into a mapping cone.

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