A Note on First Integrals of Vector Fields and Endomorphisms*

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Introduction

The purpose of this paper is to provide a simple proof that C^r-generically $(1 \leq r \leq \infty)$ vector fields on a compact, smooth, boundaryless n-dimensional connected manifold have the following property:

($\mathscr{P}$) *C^n-first integrals are constant functions.*

We recall that a C^k-first integral of a vector field X on a manifold M is a C^k-function $f: M \to \mathbb{R}$ (the real numbers) such that is constant on the orbits of the flow generated by X. We shall also prove that a similar property is C^r-generically true in $\mathscr{C}^r(M, M)$, the space of C^r-endomorphisms of M.

Recently, T. Bewley [5] proved the genericity of $\mathscr{P}$ in the case $1 \leq r < \infty$. In the two dimensional case this property follows from Peixoto's theorem [2]. R. Thom, in an unpublished paper [4] observed that the C^r-closing lemma implies the C^r-genericity of $\mathscr{P}$. For ε-structurally stable vector fields J. Arraut [1] proved that property $\mathscr{P}$ is true.

In [3], Peixoto observed that if a vector field has a Morse first integral then it can be approximated by a gradient like vector field.

Our proof of the genericity of $\mathscr{P}$ uses a lemma due to F. Takens [6] (lemma 1.1 in this paper). In section 2 this lemma is generalized and applied to the case of endomorphisms of M.

REMARK 1. Peixoto's observation is not true, even taking C^0-approximations, if the first integral isn't a Morse function. To see this, consider an Anosov

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Let V be a C^n-first integral of X. Then $(\nabla_p V, X_p) = 0 \ \forall\, p \in M$. If V is non constant, by Sard's theorem there is a regular value $\alpha \in \mathbb{R}$, and by well known facts about differentiable functions, there is neighborhood U of $V^{-1}(\alpha)$, an $\varepsilon > 0$. and a diffeomorphism:

$$F\colon (\alpha - \varepsilon,\ \alpha + \varepsilon) \times V^{-1}(\alpha) \to U$$

such that the following diagram is commutative

$$\begin{array}{ccc} (\alpha - \varepsilon,\ \alpha + \varepsilon) \times V^{-1}(\alpha) & \xrightarrow{F} & U \\ & \overset{\pi}{\searrow} & \downarrow V \\ & & \mathbb{R} \end{array}$$

where π is the projection onto the first factor. Let

$$W = F\left(\left(\alpha - \frac{\varepsilon}{2},\ \alpha + \frac{\varepsilon}{2}\right) \times V^{-1}(\alpha)\right).$$

Obviously $\Omega(X) \cap W \neq \varnothing$. Then there is a neighborhood $\mathscr{U}$ of zero in $\chi_r(M)$ such that:

$$y \in (X + \mathscr{U}) \cap \mathscr{B}_r \Rightarrow \Omega(y) \cap W \neq \varnothing. \tag{1}$$

Let $\mathscr{W} = \{Z \in \chi_r(M) : (\nabla_p V, Z_p) > 0 \ \ \forall\, p \in W\}$. $\mathscr{W}$ is non vacuous because a C^r-vector field C^0-near to $\nabla_p V$ is in $\mathscr{W}$. Then $\mathscr{W}$ is an open non vacuous cone in $\mathscr{X}_r(M)$ and it follows that:

$$(X + \mathscr{U} \cap \mathscr{W}) \cap \mathscr{B}_r \neq \varnothing.$$

But if $Z \in (X + \mathscr{U} \cap \mathscr{W}) \cap \mathscr{B}_r$ then it is easy to see that $\Omega(Z) \cap W \neq \varnothing$ and this is in contradiction with (1).

2. First Integrals of Endomorphisms

Let $f \in \mathscr{C}^r(M, M)$, $1 \leq r \leq \infty$. We say that $p \in M$ is a non wandering point of f if for every neighborhood U of p, there is an integer $N \geq 1$ such that $f^N(U) \cap U \neq \varnothing$. We define $\Omega\colon \mathscr{C}^r(M, M) \to S(M)$ as the function that to each endomorphism assigns the set of his non wandering points. We shall consider $\mathscr{C}^r(M, M)$ as a metric space with the usual C^r-metric.

diffeomorphism $f: T^2 \to T^2$ such that $f_*: H_1(M) \to H_1(M)$ is hyperbolic [7]. The diffeomorphism f: $T^2 \times S^1 \to T^2 \times S^1$ defined by $f(p, q) = (f(p), q)$ has C^∞-first integrals non constant on open sets. If g is C^0-near to f then $g_* = f_*$ because they are homotopic. But f_* is not unipotent on homology [8]. Then by [8], g is not Morse-Smale. The suspension of f is then a vector field, with a first integral non constant on open sets, and cannot be C^0-approximated by a Morse-Smale vector field.

REMARK 2. The method used in the proof of Lemma 2.1 can be used to prove that given a connected Lie group G there is a residual subset S of $\mathscr{A}_r(G, M)$, the space of C^r-actions of G on M with the C^r-topology such that if $S(M)$ is the space of subsets of M with the Hausdorff pseudometric then $\Omega: S \to S(M)$ is continuous and is upper semicontinuous at points of S.

1. Genericity of P in $\chi_r(M)$

Let M be a compact connected smooth n-dimensional boundaryless manifold, with a smooth Riemannian structure. Let $\chi_r(M)$, $1 \leq r \leq \infty$, be the Frechet space (Banach if $r < \infty$) of C^r-vector fields with the usual C^r-metric. Let $S(M)$ be the pseudometric space of subsets of M with the Hausdorff pseudometric.

If $X \in \chi_r(M)$, and $\phi.$ is the flow generated by X, we recall that $p \in M$ is a non wandering point of X if for every neighborhood U of P there exists $T \in \mathbb{R}$, $T \geq 1$, such that $\phi_t(U) \cap U \neq \varnothing$. Let Ω: $\chi_r(M) \to S(M)$ be the function that to each vector field assigns the set of his non wandering points.

LEMMA 1.1. *There is a residual subset $\mathscr{B}_r \subset \chi_r(M)$ such that Ω: $\mathscr{B} \to S(M)$ is continuous.*

PROOF. See F. Takens [6], or apply a method similar to the one used in the proof of Lemma 2.1.

THEOREM 1.2. *If $X \in \mathscr{B}_r$, then X has property $\mathscr{P}$.*

PROOF. Let $(\cdot, \cdot)_p$ and ∇_p be the scalar product and the gradient operator of the Riemannian structure of M.

LEMMA 2.1. *There is a residual subset* $\mathscr{B}_r \subset \mathscr{C}^r(M, M)$ *such that:*

$$\Omega: \mathscr{B}_r \to S(M)$$

is continuous.

PROOF. Let $\mathscr{N}$ be a countable basis of compact neighborhoods of M. Given $U \in \mathscr{U}$ we define:

$$\mathscr{O}_U^{N,m}: \mathscr{C}^r(M, M) \to S(M)$$

as

$$\mathscr{O}_U^{N,m}(f) = \bigcup_{N \le k \le m} f^k(U).$$

We also define

$$\bar{\mathscr{O}}_U^N: \mathscr{C}^r(M, M) \to S(M)$$

as:

$$\bar{\mathscr{O}}_U^N(f) = \overline{\bigcup_{N \le k} f^k(U)}.$$

It is easy to see that $\mathscr{O}_U^{N,m}$ is continuous and

$$\bar{\mathscr{O}}_U^N(f) = \lim \mathscr{O}_U^{N,m}(f).$$

By a well known lemma due to Baire, there is a residual subset $\mathscr{B}_r^{N,U} \subset \mathscr{C}^r(M, M)$ such that $f \in \mathscr{B}_r^{N_0 U} \Rightarrow f$ is a point of continuity of $\bar{\mathscr{O}}_U^N$. Then if we define:

$$\hat{\mathscr{B}}_r = \bigcap_{\substack{U \in \mathscr{N} \\ 1 \le N}} \mathscr{B}_r^{N_0 U}$$

it is easy to see that Ω: $\hat{\mathscr{B}}_r \to S(M)$ is upper semicontinous. By well known properties of set valued functions, we deduce that there is a residual subset $\mathscr{B}_r$ of $\hat{\mathscr{B}}_r$ such that Ω: $\mathscr{B}_r \to S(M)$ is continuous.

THEOREM 2.1. *If* $f \in \mathscr{B}_r$ *then it has the following property: P') If* $V \in \mathscr{C}^n(M, \mathbb{R})$ *is such that* $V \cdot f = V$ *then V is a constant function.*

PROOF. If $f \in \mathscr{B}_r$, and $V \in \mathscr{C}^n(M, \mathbb{R})$ is such that $V \cdot f = V$ and if V is non constant, then there is a regular value $\alpha \in \mathbb{R}$, and $\varepsilon > 0$, an open neighborhood U of $V^{-1}(\alpha)$ and a diffeomorphism F such that the following diagram is commutative:

$$\begin{array}{ccc} (\alpha - \varepsilon,\ \alpha + \varepsilon) \times V^{-1}(\alpha) & \xrightarrow{F} & U \\ & \searrow^{\pi} & \downarrow \\ & & \mathbb{R} \end{array}$$

where π is the projection onto the first factor. Let

$$W_1 = F\left(\left(a - \frac{\varepsilon}{2},\ \alpha + \frac{\varepsilon}{2}\right) \times V^{-1}(\alpha)\right)$$

and

$$W_2 = F\left(\left(\alpha - \frac{\varepsilon}{4},\ \alpha + \frac{\varepsilon}{4}\right) \times V^{-1}(\alpha)\right).$$

There is an open neighborhood $\mathscr{U}$ of f such that:

$$g \in \mathscr{U} \cap \mathscr{B}_r \Rightarrow \Omega(g) \cap W_2 \neq \varnothing. \tag{2}$$

Let $X \in \chi_\infty(M)$ such that $X_p = 0$ if $p \notin W_1$, and $(X_p, \nabla_p V) > 0$ if $p \in W_1$. Let $\lambda \in \mathbb{R}$ be such that the flow ϕ_t generated by λX has $\phi_1 \circ g \in \mathscr{U}$ $(\lambda \neq 0)$. Let:

$$c = \inf_{p \in \overline{W}_2} ((V \circ \phi_1 \circ f)(p) - V(p)).$$

By the density of $\mathscr{B}_r \cap \mathscr{U}$ in $\mathscr{U}$, and because

$$(V \circ \phi_1 \circ f)(p) - V(p) \geq 0 \ \forall p \in M$$

we can take $g \in \mathscr{B}_r \cap \mathscr{U}$ such that:

$$\left|(V \circ \phi_1 \circ f)(p) - (V \circ g)(p)\right| \leq \frac{c}{4} \quad \forall p \in M.$$

Then $\Omega(g) \cap W_2 = \varnothing$ in contradiction with (2).

BIBLIOGRAPHY

[1] J. ARRAUT, *Note on Structural Stability*, Bull. Amer. Math. Soc. 72(1966), 543-544.

[2] M. PEIXOTO, *Structural Stability on Two-Dimensional Manifolds*, Topology 1 (1962), 101-120.

[3] M. PEIXOTO, Qualitative Theory of Differential Equations, Hale/La Salle – Differencial Equations and Dynamical Systems, Academic Press, New York (1967).

[4] R. THOM, *Les integrales Premières d'un Systeme Différentiel sur une Variété Compacte* (Unpublished).

[5] T. BEWLEY, Non Genericity of First Integrals, Thesis, University of California, Bekerley, 1971.

[6] F. TAKENS, *On Zeeman's Tolerance Stability Conjecture*, Proceedings of the Nuffic School on Manifolds, 1971, Springer Verlag 197.

[7] J. FRANKS, *Anosov Diffeomorphisms on Tori*, Trans. Amer. Math. Soc. 145 (1969).

[8] M. SHUB, *Morse-Smale Diffeomorphism are unipotent on Homology*, Proceedings of the 1971 Dynamical Systems Symposium of Salvador, Brasil.

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