An Extension of Sard's Theorem*

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A classical form of the theorem of Sard affirms that if $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is a function $x = g(y)$ in $\mathscr{C}^1(\mathbb{R}^n)$, then the set of critical values, i.e. the set of points $x_0 \in \mathbb{R}^n$ such that for some y_0, $g(y_0) = x_0$ and $\left[\det\left(\frac{\partial g}{\partial y}\right)\right]_{y=y_0} = 0$, is of zero measure.

This theorem can be considerably extended in the following way. One first gives a generalization of the notion of the Jacobian determinant, then one defines correspondingly the critical values and proves that the set of critical values has measure zero. It turns out that one can present the theorem in a very general setting as follows.

Let $V \subset Y = \mathbb{R}^n$ be an open set. Let X be another set and ρ an exterior measure defined on the subsets of X. Let $g\colon V \to X$ be an arbitrary function. A point $y \in V$ will be called *critical point* of g if there is a sequence $\{Q_k(y)\}$ of open cubic intervals centered at y and contracting to y such that

$$\frac{\rho(g(Q_k(y))}{|Q_k(y)|} \to 0.$$

Here $|Q|$ is the Lebesgue measure of Q.

For the proof of the theorem we want to present we shall make use of the following covering lemma whose proof can be found in [1].

LEMMA. *Let S be a bounded set of $\mathbb{R}^n$. For each $y \in S$ an open cubic interval $Q(y)$ centered at y is given. Then one can choose from among $(Q(y))_{y\in S}$ a sequence $\{Q_k\}$ of such cubes so that*

a) $S \subset \bigcup_k Q_k$;

*Recebido pela SBM em 29 de agosto de 1972.

b) $\Sigma \chi_{Q_k}(x) \leq \theta(n)$ *for every* $x \in \mathbb{R}^n$.

Here χ_P *is the characteristic function of* P *and* $\theta(n)$ *is a constant that only depends on the dimension.*

THEOREM. *Let* $g: V \to X$ *be as in the preceding definition. If* C *is the set of critical points, then* $\rho(g(C)) = 0$, *i.e. the set of critical values has* ρ *measure zero.*

PROOF. We first take a bounded subset S of C and prove $\rho(g(S)) = 0$. Since $C = \bigcup_k S_k$, S_k bounded, we then have

$$\rho(g(C)) = \rho(g(\bigcup_k S_k)) \leq \rho(\bigcup_k g(S_k)) \leq \sum_k \rho(g(S_k)) = 0.$$

Let then S be an arbitrary bounded subset of C. Let $S \subset G$, G open, $|G| < \infty$, $\varepsilon > 0$. If $y \in S$ we can choose an open cubic interval centered at y so that $Q(y) \subset G$, $\dfrac{\rho(g(Q(y)))}{|Q(y)|} < \varepsilon$.

By the lemma we can choose a sequence $\{Q_k\}$ from among such sets $(Q(y))_{y \in S}$ satisfying

$$S \subset \bigcup_k Q_k \subset G, \ \Sigma \chi_{Q_k} \leq \theta(n), \ \frac{\rho(g(Q_k))}{|Q_k|} < \varepsilon$$

Therefore

$$\rho(g(S) \leq \rho\,(g(\bigcup_k Q_k)) \leq \rho(\bigcup_k g(Q_k))$$

$$\leq \sum_k \rho(g(Q_k)) \leq \varepsilon \sum_k |Q_k| = \varepsilon \sum_k \int \chi_{Q_k}(x)\, dx$$

$$\leq \varepsilon\theta(n) \,|\bigcup_k Q_k| \leq \varepsilon\theta(n)\,|G|.$$

Since ε is arbitrary, $\rho(g(S)) = 0$ and the theorem is proved.

REMARK 1. It is rather easy to show, by means of the mean value inequality that if in the theorem $X = \mathbb{R}^n$, ρ is the exterior measure associated to Lebesgue measure and $g \in \mathscr{C}^1$ then

$$\det\left[\left(\frac{\partial g}{\partial y}\right)\right]_{y=y_0} = 0$$

implies that y_0 is a critical point in the sense of the above definition. So the theorem of Sard is an easy consequence of the theorem.

Remark 2. It is easy to show that if $X = \mathbb{R}^n$, ρ is the exterior measure and $g\colon V \to X$ is one-to-one and transforms Lebesgue measurable sets into Lebesgue measurable sets, then the limit

$$\lim_{Q(y)\to y} \frac{|g(Q(y))|}{|Q(y)|}$$

exists for almost every $y \in V$. Here $Q(y) \to y$ means that one considers all open cubic intervals centered at y and makes their diameters shrink to 0.

In fact, for $\gamma > s > 0$ and B a bounded subset of $\mathbb{R}^n$ we consider

$$A_{rs} = \left\{ y \in V \cap B : \exists Q_k(y) \to y, \exists Q_k^*(y) \to y, \frac{|g(Q_k(y))|}{|Q_k(y)|} > r > s > \frac{|g(Q_k^*(y))|}{|Q_k^*(y)|} \right\}$$

If $|P|_e$ means the exterior measure of the set P, we have $|A_{rs}|_e = 0$. Let $|A_{rs}|_e = \alpha$. We take an open set G, $G \subset A_{rs}$, $|G| < \alpha + \eta$, for $\eta > 0$ given. For $y \in A_{rs}$ we have a sequence $\{Q_k^*(y)\}$ with

$$Q_k^*(y) \to y,\ Q_k^*(y) \subset G,\ \frac{|g(Q_k^*(y))|}{|Q_k^*(y)|} < s.$$

We apply the theorem of Vitali and so we obtain a disjoint sequence $(Q_{k'}^*(y))$ from among $(Q_k^*(y))_{k, y\in A_{rs}}$ such that

$$s(\alpha + \eta) > s|G| \geq s\sum_{k'} |Q_{k'}^*| \geq \sum_{k'} |g(Q_{k'}^*)|$$

and

$$|A_{rs} - \bigcup_{k'} Q_{k'}^*| = 0.$$

Let now $y^* \in (\bigcup_{k'} Q_{k'}^*) \cap A_{rs} = C$. Observe that almost every point in A_{rs} is also in C. For y^* we have a sequence $\{Q_k(y^*)\}$ with $Q_k(y^*) \to y^*$, $Q_k(y^*) \subset Q_h^* \in (Q_{k'}^*(y))$ for some h such that

$$\frac{|g(Q_k(y^*))|}{|Q_k(y^*)|} > r.$$

We apply again the theorem of Vitali to obtain $\{Q_k\}$ another disjoint sequence from among $(Q_k(y^*))_{y^* \in C, k}$ such that we can write

$$\alpha = |A_{rs}|_e = |C|_e \leq \left| C - \bigcup_k Q_k \right|_e + \left| C \cap (\bigcup_k Q_k) \right|_e$$

$$= \left| C \cap (\bigcup_k Q_k) \right|_e \leq \sum_k |Q_k| \leq \frac{1}{r} \sum_k |g(Q_k)|$$

$$\leq \frac{1}{r} \sum_h |g(Q_h^*)| \leq \frac{s}{r} (\alpha + \eta).$$

Hence $\alpha \leq \frac{\eta s}{r - s}$. Since η is arbitrary, $\alpha = 0$ and so $|A_{rs}|_e = 0$.

REFERENCE

[1] M. DE GUZMAN, *A covering lemma with applications to the differentiability of measures and singular integral operators*, Studia Math. **34** (1970), 299-317.

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