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Warped products with special Riemannian curvature*

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Abstract. We study the geometry of particular classes of Riemannian manifolds obtained as warped products. We focus on the case of constant curvature which is completely classified and on the Einstein case. This study provides nontrivial instances of Einstein manifolds which are warped product of Einstein factors.

Keywords: warped product, curvature, Einstein manifold.

Mathematical subject classification: 53B20, 53C25.

1 Introduction

Given two Riemannian manifolds (B, g_B) and (F, g_F), then the space $M := B \times F$ endowed with the *warped metric* $g_B + \omega^2 g_F$ is said to be a *warped product* and it is also denoted as $B \times_\omega F$.

Historically such spaces have been used in order to prove that some classes of Riemannian manifolds are not empty and to produce large families of examples.

In [2], Bishop and O'Neill constructed this way a class of Riemannian manifolds of negative curvature.

In the seventies, Tanno gave locally symmetric warped products whose factors have constant curvature: $K(B) = 0$ and $K(F) \leq 0$; however such manifolds do not have constant curvature. He worked with the further hypotheses that $R(X, Y) \circ R = 0$ is satisfied and the scalar curvature is constant, [10]. Later, Takagi, [9], generalized this result substituting the two hypothesis with "M is homogeneous or Ricci-symmetric".

In [8], Sekigawa enlarged the class introduced by Bishop and O'Neill. Moreover he gave an example of a warped product which is curvature homogeneous

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but non-homogeneous. Finally, Tricerri generalized this result, giving an infinite class (depending on a non-countable set of parameters) of Riemannian manifolds which are curvature homogeneous, non-locally homogeneous, non-isometric to each other, [11].

Since the class of warped products is so rich of interesting examples, there is a natural geometrical interest in the study of the Riemannian properties of a generic warped product.

Our paper classifies certain warped products by the condition of constant sectional curvature or the Einstein condition.

In particular the basis B of a warped product of constant sectional curvature admits a function (namely the warping function) ω which satisfies

$$H^{\omega} = -\varkappa\omega\, g\ . \tag{1}$$

Such an equation has been studied by Obata [3] in the case $\varkappa = 1$. Our Theorem 3 provides another proof of his Main Theorem, generalized to any $\varkappa$. The proof brings us to the classification of the warped products of constant sectional curvature.

Furthermore, the motivation of the study of Einstein warped products with Einstein basis is given in [1], Section 9J. In fact Besse describes the Einstein warped products with basis either one or two dimensional. A natural generalization is given by the cases studied here. The results are those expected. Moreover it is well known that the equation $\nabla_X\, grad\, \omega = -\varkappa\omega X$ (equivalent to Eq. 1) plays an important role. Thus, by Theorem 3.4, solutions allow us to reduce the manifold to a warped product with one–dimensional basis. This parameter is given just by the level surfaces of the function ω itself. All of these warping functions can be integrated in terms of elementary functions [4, 6].

Equation (1) is intimately related to the nature of a warped product; this is clearly shown by Theorem 3.4. Notice that pairs $((B, g_B), (F, g_F))$ not satisfying Eq. (1) provide interesting examples of warped products. In the case that both the factors are of constant curvature the warped products constructed by Tanno and Takagi are of this type. Moreover if (B, g_B) and (F, g_F) are Einstein manifolds and ω is constant, (M, g) has constant scalar curvature and hence is Ricci-symmetric but, in the generic case, not Einstein. In particular, the Ricci tensor has just two distinct eigenvalues λ^B and λ^F/ω^2.

For example, an interesting class of non-Einstein manifolds which are Ricci-symmetric is given by the following mixed case. Let (B, g_B) have constant curvature $\varkappa$ and (F, g_F) be an Einstein manifold: $\rho_F = \lambda^F g_F$. Then consider a function ω satisfying the Eq. (1) with first integral $||\Omega||^2 + \varkappa\omega^2 = \mathfrak{h}$ and

$\mathfrak{h} \neq \frac{\lambda^F}{f-1}$. Since (B, g_B) is Einstein with $\lambda^B = \varkappa(b-1)$, the warped product (M, g) is not an Einstein manifold, while a direct computation shows that $D\rho$ vanishes.

2 Notations and preliminaries

In this section we recall some definitions and notations.

Given a Riemannian manifold (M, g), its Levi-Civita connection is denoted by ∇. We use R for the Riemannian curvature tensor and associated $(4, 0)$ tensor, ρ for the Ricci tensor and K for the sectional curvature.

For a smooth function ω, Ω will denote its gradient, H^ω its Hessian and $\Delta\omega$ will be the Laplacian namely *plus* the trace of the Hessian (it is sometimes defined with a minus).

Let now (B, g_B) and (F, g_F) be two Riemannian manifolds of dimension $b > 1$ and f respectively. Consider the smooth manifold $M := B \times F$ with the canonical projections denoted by $\pi : M \to B$ and $\sigma : M \to F$. Given an arbitrary smooth map $\omega : B \to \mathbb{R}^+$ we correspondingly define a Riemannian metric $g = g_\omega$ on M (called *warped metric*)

$$g_\omega := \pi^* g_B + (\omega \circ \pi)^2 \sigma^* g_F.$$

Notice that $\omega : B \to \mathbb{R}^+$ is just for convenience: what is necessary is that ω never vanishes on B (and hence it has constant sign -say positive- on B which we assume connected). The pair (M, g) is also denoted as $M = B \times_\omega F$ and it is said to be a *warped product*. We shall often denote the scalar product $g_\omega(X, Y)$ as $< X, Y >$, while g_B and g_F will be explicitly written.

The *fibers* $\pi^{-1}(p) = \{p\} \times F$ and the *leaves* $\sigma^{-1}(q) = B \times \{q\}$ are Riemannian submanifolds of M. Moreover, the projections have the following properties, [7]:

a) the map $\pi\lceil_{(B\times\{q\})}$ is an isometry onto B;

b) the map $\sigma\lceil_{(\{p\}\times F)}$ is a homothety onto F with factor $\frac{1}{\omega^2(p)}$;

c) the fibers and the leaves are orthogonal.

The Riemannian manifold (M, g_ω) is complete if and only if both (B, g_B) and (F, g_F) are complete and ω never vanishes [2].

For the reader's commodity we report the expressions of its Levi-Civita connection ∇, of the Riemannian tensor R, of the sectional curvature K and of the Ricci tensor ρ (to be found in [2]).

We shall abuse slightly the notation and confuse the connection ∇^B of B and the lift $\pi^*\nabla$ of the connection ∇. Hereafter X, Y, Z will be sections of $\Gamma(\pi^*TB)$ and U, V, W of $\Gamma(\sigma^*TF)$, while Ω will denote the gradient of ω. The Levi-Civita connection is given by

$$\nabla_X Y = \nabla^B_X Y; \quad \nabla_X V = \nabla_V X = \frac{X(\omega)}{\omega} V; \quad \nabla_V W = \nabla^F_V W - \frac{\langle V, W\rangle}{\omega}\Omega.$$

Via a direct computation, we have the Riemannian curvature tensor R,

$$R_{XY}Z = R^B_{XY}Z; \quad R_{VX}Y = \frac{H^\omega(X,Y)}{\omega}V; \quad R_{XY}V = R_{VW}X = 0$$
$$R_{XV}W = \frac{\langle V, W\rangle}{\omega}\nabla_X(\Omega); \quad R_{VW}U = R^F_{VW}U - \frac{\|\Omega\|^2}{\omega^2}\left\{\langle V, U\rangle W - \langle W, U\rangle V\right\},$$

and the Ricci tensor

$$\begin{aligned} \rho(X,Y) &= \rho^B(X,Y) - \frac{f}{\omega}H^\omega(X,Y) \\ \rho(X,V) &= 0 \\ \rho(V,W) &= \rho^F(V,W) - < V, W > \omega^\#, \end{aligned} \tag{2}$$

where $\omega^\# := \frac{\Delta\omega}{\omega} + (f-1)\frac{\|\Omega\|^2}{\omega^2}$. Finally, we give the sectional curvature

$$K_{XY} = K^B_{XY}; \quad K_{XV} = -\frac{H^\omega(X,X)}{\omega||X||^2}; \quad K_{UV} = \frac{K^F_{UV} - ||\Omega||^2}{\omega^2}. \tag{3}$$

3 Warped products with constant curvature

We now classify the warped products with constant sectional curvature; we will also address the issue as to whether such an M can be taken complete preserving the structure of warped product.

Consider the warped product $M = B \times_\omega F$ with the further hypothesis $K = \varkappa$, $\varkappa \in \mathbb{R}$.

Proposition 3.1. *If $M = B \times_\omega F$ has constant sectional curvature equal to $\varkappa$ then both B and F have constant sectional curvature, $K^B = \varkappa$ and $K^F = \mathfrak{h}$ for some $\mathfrak{h} \in \mathbb{R}$. Without loss of generality we can assume that F is complete.*

Proof. The formulae (3) show that

$$
\begin{aligned}
&K^B = \varkappa \\
&H^\omega(X, Y) = -\varkappa\omega\, g_B(X, Y) && (4)\\
&K^F_{UV} = \varkappa\omega^2 + ||\Omega||^2. && (5)
\end{aligned}
$$

The first equation tells that B has constant sectional curvature, while the last implies that K^F is some constant $\mathfrak{h} \in \mathbb{R}$: in fact, the second member is a first integral of (4).

Finally, since for any warped product M the projection of a geodesic γ in M onto the second factor F is a pre-geodesic in F [7], then if F itself is not complete we can isometrically embed it into its completion preserving the structure of warped product. □

We see that the only possible obstruction to take M complete preserving the warped structure is that the differential equations for ω (4) may imply that ω vanishes somewhere; in this occurrence the metric on M would be degenerate at such points and hence we should remove them from B thus destroying completeness.

Equation (4) is equivalent to

$$\nabla_X \Omega = -\varkappa\omega X.$$

In particular $\nabla_\Omega \Omega = -\varkappa\omega\Omega$, thus we have the

Proposition 3.2. *The integral curves of Ω are pre-geodesics in B, i.e. curves which admit a reparametrization as a geodesic.*

Remark 3.3. The equation (4) is highly overdetermined.

The compatibility of the tensor equation rigidifies the structure of B: indeed, in explicit coordinate notation $H^\omega_{\mu\nu} = \nabla_\mu \nabla_\nu \omega = \nabla_\nu \nabla_\mu \omega$ (by vanishing of the torsion), hence the equations are $\nabla_\mu \nabla_\nu \omega = -\varkappa\omega g_{\mu\nu}$ (we have suppressed the index B from the metric for the sake of simplicity). Taking one further covariant derivative ∇_ρ and subtracting the same equation with the exchange $\rho \leftrightarrow \mu$ we obtain

$$[\nabla_\mu, \nabla_\rho]\nabla_\nu \omega := R^\lambda_{\mu\rho\nu} \nabla_\lambda \omega = -\varkappa \nabla_\mu \omega g_{\rho\nu} + \varkappa \nabla_\rho \omega g_{\mu\rho} = \frac{\varkappa}{2}(g \owedge g(\Omega))_{\mu\rho\nu}.$$

Thus we see that compatibility of the equations boil down to

$$\mathfrak{Q}_{XY}\Omega := (R_{XY} - \varkappa R^0_{XY})\Omega = 0,$$

where we have put $R^0 = \frac{1}{2} g \owedge g$ the Riemann curvature tensor of a manifold with constant sectional curvature 1 (we have used the Nomizu–Kulkarni symbol $\owedge$[1]).

This remark shows that a sufficient condition for the existence of such a function ω is that B is of constant curvature, as it is our case now: it turns out that this is also a necessary condition in some circumstances, depending on the completeness of B and the relative signs of $\varkappa$ and $\mathfrak{h}$.

In the following theorem we are going to study under which conditions the system $H^\omega = -\varkappa\omega g$ admits a nontrivial solution on a given manifold (N, g) not *a priori* of constant curvature: we will focus on the constraints that the compatibility imposes on the geometry of the manifold N while we will lift the requirements of positivity of ω itself. Clearly we will re-impose the requirement $\omega > 0$ when (N, g) will play the role of base of a warped product (at which point we will use the notation (B, g_B). In particular it will be shown that (N, g) (or some suitable open maximal subset) must be itself a warped product for some warping function α.

Theorem 3.4. *Let (N, g) be any complete Riemannian manifold of dimension greater than one (to avoid trivialities) such that there exists a function ω satisfying $H^\omega = -\varkappa\omega\, g$ with first integral $\varkappa\omega^2 + \|\Omega\|^2 = \mathfrak{h}$ for suitable (real) constants $\mathfrak{h}$, $\varkappa$: denoting $\Delta := \{x \in N\,;\, \|\Omega_x\| = 0\}$ the critical locus of ω, then*

i) *$(N \setminus \Delta, g)$ is isometric to a warped product $I \times_\alpha \Sigma_\mathfrak{q}$ where $I \subseteq \mathbb{R}$, $\Sigma_\mathfrak{q} := \omega^{-1}(\mathfrak{q})$ for a regular value $\mathfrak{q}$, and $\alpha(t)$ is a suitable function to be specified in the proof.*

ii) *if $\Delta \neq \emptyset$ then (N, g) is of constant curvature $K^{(N)} = \varkappa$;*

iii) *If $\varkappa \leq 0 \leq \mathfrak{h}$ then the above holds globally (and $I = \mathbb{R}$).*

iv) *The surface $\Sigma_0 := \omega^{-1}(0)$ is always regular (if not empty) and totally geodesic.*

Proof. Let $\mathfrak{q}$ be a regular value for ω. The level surfaces $\Sigma_{\tilde{\mathfrak{q}}} := \omega^{-1}(\tilde{\mathfrak{q}})$ are regular hypersurfaces for $\tilde{\mathfrak{q}}$ in a suitable neighborhood of $\mathfrak{q}$. Moreover, all the $\Sigma_{\tilde{\mathfrak{q}}}$ are diffeomorphic to $\Sigma_\mathfrak{q}$ via the flux generated by the gradient.

Let $i : \Sigma_\mathfrak{q} \hookrightarrow N$ be the natural injection and, for an interval $I \subset \mathbb{R}$ containing 0, define the map $\psi : I \times \Sigma_\mathfrak{q} \to N : (t, x) \mapsto \psi(t, x)$ as follows: the point $\psi(t, x)$ is the unique point of $\Sigma_{\chi(t)}$ lying on the integral curve of Ω (a geodesic) through the point $i(x) \in \Sigma_\mathfrak{q}$ at a distance t from $\Sigma_\mathfrak{q}$ (t is the oriented distance).

This definition implies clearly that $\psi_* \partial_t = \frac{\Omega}{\|\Omega\|}$ and that $\psi(\{t\} \times \Sigma_q)$ are the level sets of ω.

The function $\chi := \omega \circ \psi$ is a function only of $t \in U$ and satisfies

$$\begin{cases} (\chi')^2 = \mathfrak{h} - \varkappa \chi^2 \\ \chi(0) = \mathfrak{q} > 0 \end{cases}$$

According to the signs of $\varkappa$ and $\mathfrak{h}$ and after a suitable shift of the affine parameter t we have

$$\chi(t) = \begin{cases} \sqrt{\mathfrak{h}/\varkappa} \cos\left(\sqrt{\varkappa} t\right) & \varkappa > 0,\ \mathfrak{h} > 0 \quad (1) \\ \sqrt{\mathfrak{h}/|\varkappa|} \sinh\left(\sqrt{|\varkappa|} t\right) & \varkappa < 0,\ \mathfrak{h} > 0 \quad (2) \\ \sqrt{|\mathfrak{h}|/|\varkappa|} \cosh\left(\sqrt{|\varkappa|} t\right) & \varkappa < 0,\ \mathfrak{h} < 0 \quad (3) \\ \sqrt{1/|\varkappa|} \exp\left(\sqrt{|\varkappa|} t\right) & \varkappa < 0,\ \mathfrak{h} = 0 \quad (4) \\ \sqrt{\mathfrak{h}}\, t & \varkappa = 0,\ \mathfrak{h} > 0 \quad (5) \end{cases}$$

where now $\mathfrak{q} = \chi(t_0)$ for some $t_0 \in I$. Notice that ψ is an isomorphism of manifolds outside the stationary points of ω and that now ω takes on negative values in some cases (but -as we said- here ω is not a warping function, just a solution of $H^\omega = -\varkappa \omega g$).

We now prove that the metric $\tilde{g} := \psi^* g$ gives $I \times \Sigma_\mathfrak{q}$ the structure of warped product. Let p_1 and p_2 denote the projections onto the two factors of $I \times \Sigma_\mathfrak{q}$ and note that for all X, Y in the tangent bundle of $\Sigma_\mathfrak{q}$ in $I \times \Sigma_\mathfrak{q}$ (i.e. in $\Gamma(p_2^* T \Sigma_\mathfrak{q})$)

$$\tilde{g}(\partial_t, \partial_t) = 1; \quad \tilde{g}(\partial_t, X) = 0; \quad \tilde{g}(X, Y) = g(\psi_* X, \psi_* Y)$$
$$\psi_* \partial_t = \frac{\Omega}{\|\Omega\|} = \frac{1}{\sqrt{\mathfrak{h} - \varkappa \omega^2}} \Omega$$
$$\omega \circ \psi(t, x) = \chi(t),$$

Let now $X, Y \in \Gamma(p_2^* T \Sigma_\mathfrak{q})$ such that $[\partial_t, X] = [\partial_t, Y] = 0$ and thus $[\Omega, \psi_* X] = [\Omega, \psi_* Y] = 0$: if we now compute $\mathcal{L}_{\partial_t} \tilde{g}$ we get

$$\begin{aligned} \partial_t(\tilde{g}(X, Y)) &= (\mathcal{L}_{\partial_t} \tilde{g})(X, Y) = \frac{\Omega}{\|\Omega\|} g(\psi_* X, \psi_* Y) = \\ &= \frac{1}{\|\Omega\|} \{g(\nabla_\Omega \psi_* X, \psi_* Y) + g(\psi_* X, \nabla_\Omega \psi_* Y)\} = \\ &= \frac{1}{\|\Omega\|} \{g(\nabla_{\psi_* X} \Omega, \psi_* Y) + g(\psi_* X, \nabla_{\psi_* Y} \Omega)\} = \frac{2H^\omega(X, Y)}{\|\Omega\|} = \end{aligned}$$

$$= \frac{-2\varkappa\omega}{\|\Omega\|} g(\psi_* X, \psi_* Y) = -2\frac{\varkappa\chi(t)}{\chi'(t)} \tilde{g}(X, Y) =$$
$$= \frac{-2\varkappa\chi\chi'}{\mathfrak{h} - \varkappa\chi^2} \tilde{g}(X, Y) = \frac{d}{dt} \log(\chi'(t)^2) \tilde{g}(X, Y).$$

Hence

$$\tilde{g} = dt^2 + \frac{(\chi'(t))^2}{(\chi'(t_0))^2} i^* g = dt^2 + \frac{\mathfrak{h} - \varkappa\chi^2}{\mathfrak{h} - \varkappa\mathfrak{q}^2} i^* g$$

This proves the first part with warping function

$$\alpha(t) = \frac{(\chi'(t))}{\sqrt{\mathfrak{h} - \varkappa\mathfrak{q}^2}} = \sqrt{\frac{\mathfrak{h} - \varkappa\chi^2(t)}{\mathfrak{h} - \varkappa\mathfrak{q}^2}}.$$

In the five cases above we have that $\alpha(t) = \frac{1}{\sqrt{\mathfrak{h} - \varkappa\mathfrak{q}^2}} f(t)$ and the corresponding maximal intervals are given by

$$f(t) = \begin{cases} \sqrt{\mathfrak{h}} \sin\left(\sqrt{\varkappa} t\right) & \varkappa > 0, \ (\mathfrak{h} > 0), \quad I = (0, \pi/\sqrt{\varkappa}) & (a) \\ \sqrt{\mathfrak{h}} \cosh\left(\sqrt{|\varkappa|} t\right) & \varkappa < 0, \ (\mathfrak{h} > 0), \quad I = \mathbb{R} & (b) \\ \sqrt{|\mathfrak{h}|} \sinh\left(\sqrt{|\varkappa|} t\right) & \varkappa < 0, \ (\mathfrak{h} < 0), \quad I = (0, \infty) & (c) \\ \exp\left(\sqrt{|\varkappa|} t\right) & \varkappa < 0, \ (\mathfrak{h} = 0), \quad I = \mathbb{R} & (d) \\ \sqrt{\mathfrak{h}} & \varkappa = 0, \ (\mathfrak{h} > 0), \quad I = \mathbb{R}. & (e) \end{cases}$$

We remark that the critical set Δ corresponds to the zero locus of α (since α is proportional to $\|\Omega\|$), or more precisely that $\Delta \neq \emptyset$ iff α is not everywhere positive, i.e. in cases (a) and (c).

To prove assertion ii) we now compute the sectional curvature of N on a plane spanned by U, V vectors in $\Gamma(T\Sigma_{\mathfrak{q}})$. The calculation follows from the expression of the sectional curvature of a warped product

$$K_{UV}^{(N)} = \frac{K_{UV}^{\Sigma_{\mathfrak{q}}} - (\alpha')^2}{\alpha^2}. \tag{6}$$

The cases when the critical locus Δ is not empty are (a) and (c) and Δ is constituted by isolated points;

if $\varkappa > 0$ (and thus $\mathfrak{h} > 0$, case (a)) Δ is constituted by isolated points since the Hessian is nondegenerate (i.e. there are isolated maxima or minima). It is easy to realize that there exist exactly one maximum and one minimum $x_\pm$ (where the critical values are $\omega_{cr} = \pm\sqrt{\frac{\mathfrak{h}}{\varkappa}}$), and that ψ is an isometry on $N/\{x_+, x_-\}$;

if $\varkappa < 0 > \mathfrak{h}$ (case (c)) then Δ is just an isolated point of minimum x_0 for ω ($\omega_{cr} = \sqrt{|\mathfrak{h}|/|\varkappa|}$), and the isometry is defined onto $N/\{x_0\}$.

Before proceeding let us point out that, since all critical points of ω are nondegenerate and the Hessian is of definite signature (being proportional to the metric), the level sets $\Sigma_{\mathfrak{q}} = \omega^{-1}(\mathfrak{q})$ are all topological spheres (from Morse theory).

We shrink this topological sphere ($\alpha \to 0$ i.e. $\omega \to \omega_{cr}$), by parallel translating the two vertical vectors U, V up to the critical point x_{cr} along the flow generated by the gradient Ω (remember that Ω and so the gradient of α generate pregeodesics). Notice that for each such flow line γ the projection on the fiber $\Sigma_{\mathfrak{q}}$ is constant, and the 2-plane spanned by U, V does not change (each vector is just rescaled). At the end of this shrinking process we obtain two vectors in the tangent space $T_{x_{cr}}N$. Since we must obtain a well defined value of the sectional curvature of N then we must have $K_{UV}^{\Sigma_{\mathfrak{q}}} = \left(\alpha'(0)\right)^2$ independently of the "direction" of the geodesic, namely of the point on $\Sigma_{\mathfrak{q}}$, and of the two-plane. This proves that $\Sigma_{\mathfrak{q}}$ is a sphere.

Then, from Eq. (6) and from the explicit form of α, it follows that also $K^{(N)} = \varkappa$ and hence (N, g) is globally (by continuity) of constant sectional curvature, which proves part ii).

If the critical locus Δ is empty (which corresponds to the remaining cases (b), (d), (e)) we have no constraint on the curvature of the leaf $\Sigma_{\mathfrak{q}}$, which can be any Riemannian manifold.
Indeed if $\varkappa < 0 \leq \mathfrak{h}$, from $\|\Omega\|^2 = \mathfrak{h} - \varkappa\omega^2$ we see that ω has no stationary points and hence ψ is defined globally. Now χ is an hyperbolic sine or an exponential if $\mathfrak{h} = 0$ or a linear function if $\varkappa = 0$, and correspondingly α is an hyperbolic cosine, exponential, or constant. Therefore the maximal interval I is exactly $\mathbb{R}$ and the manifold N is complete and globally isometric to a warped product, which proves assertion iii).

Finally, $\Sigma_0 = \omega^{-1}(0)$ is not empty only in the cases (a), (b), (e) and is clearly regular because $\|\Omega\|^2 = \mathfrak{h}$.
To prove that it is a totally geodesic hypersurface, take a geodesic γ of N such that $\gamma(0)$ is in Σ_0 and $\dot{\gamma}(0)$ is orthogonal to $\Omega(\gamma(0))$, namely $\dot{\gamma}(0) \in T_{\gamma(0)}\Sigma_0$. Consider now the map $\mu(t) := \omega(\gamma(t))$. Its first and second derivatives are

$$\frac{d}{dt}\mu = \langle \Omega, \dot{\gamma} \rangle; \qquad \frac{d^2}{dt^2}\mu = \langle D_{\dot{\gamma}}\Omega, \dot{\gamma} \rangle = -\varkappa\mu.$$

The initial data for this Cauchy problem are $\mu(0) = 0$ and $\frac{d}{dt}\mu(0) = 0$, hence μ vanishes identically, i.e. γ stays always in Σ_0. Thus Σ_0 is totally geodesic and therefore has the same curvature as N. □

First of all we have to integrate equation (4): from $\varkappa\omega^2 + \|\Omega\|^2 = \mathfrak{h} = K^F$ and $H^\omega = -\varkappa\omega g$ it follows that when ω is constant then both $\mathfrak{h}$ and $\varkappa$ vanish, that is M, B and F are flat and vice versa. Therefore we will consider only the case ω nonconstant in the following. Summarizing the contents of our investigation we can state the following corollary

Corollary 3.5. *Suppose that (N, g) has constant curvature $\varkappa$, is geodesically complete and ω satisfies $H^\omega = -\varkappa\omega g$, then*

i) *if $\varkappa = 0 < \mathfrak{h}$ then $\omega(x) = a \cdot x + c$ (an affine function), where $\cdot$ here denotes the euclidean scalar product and $\|a\|^2 = \mathfrak{h}$;*

ii) *if $\varkappa > 0$ then $\mathfrak{h} > 0$ and $\omega(x) = \sqrt{\frac{\mathfrak{h}}{\varkappa}} \cos(\sqrt{\varkappa} d(x, x_0))$, for some $x_0 \in B$;*

iii) *if $\varkappa < 0 = \mathfrak{h}$ then $\omega(x) = \omega_0 e^{\sqrt{|\varkappa|} d(x, \Sigma_0)}$ where Σ_0 is a suitable hypersurface of zero intrinsic curvature;*

iv) *if $\varkappa < 0 < \mathfrak{h}$ then $\omega(x) = \sqrt{\frac{\mathfrak{h}}{\varkappa}} \sinh(\sqrt{|\varkappa|} d(x, \Sigma_0))$ where Σ_0 is a suitable totally geodesic hypersurface;*

v) *if $\varkappa < 0 > \mathfrak{h}$ then $\omega(x) = \sqrt{\frac{\mathfrak{h}}{\varkappa}} \cosh(\sqrt{|\varkappa|} d(x, x_0))$, $x_0 \in B$.*

Proof. This is a specification of Thm. (3.4) with the aid of the formula (3) for the sectional curvature of a warped product. □

After studying the geometry implied by the system (4) in Theorem 3.4 we now turn back to the case of warped products: thus we are going to specify the setting of Theorem 3.4 to the base (B, g_B) of a warped product. The only difference is that now we must impose $\omega \neq 0$.

We can apply Corollary 3.5 (with (B, g_B) playing the role of (N, g) in the statement) to the classification of warped product where M has sectional curvature $\varkappa$ (and hence B as well): we will have to restrict ω to the maximal connected set where it never vanishes.

Corollary 3.6 *The possible simply–connected $M = B \times_\omega F$ with constant sectional curvature $\varkappa$ are in Table 1: they are complete manifold iff both factors in the table are. In the cases where M is not complete, no completion is possible preserving the structure of warped product.*

$\varkappa$	$\mathfrak{h}$	M	ω
$\varkappa = 0$	$\mathfrak{h} = 0$	$M_0 = \mathbb{R}^b \times_\omega \mathbb{R}^f$	$\omega(x) = cost$
$\varkappa = 0$	$\mathfrak{h} > 0$	$M_1^\pm = \left(\mathbb{R}^\pm \times \mathbb{R}^{b-1}\right) \times_\omega S^f\left(1/\sqrt{\mathfrak{h}}\right)$	$\omega(x) =< a, x > +x_0$ and $\|a\|^2 = \mathfrak{h}$
$\varkappa > 0$	$\mathfrak{h} > 0$	$M_2^\pm = \left(S^b\left(1/\sqrt{\varkappa}\right)\right)^\pm \times_\omega S^f(1/\sqrt{\mathfrak{h}})$	$\omega(x) = \sqrt{\mathfrak{h}/\varkappa}\cos\left(\sqrt{\varkappa}d(x, x_0)\right)$
$\varkappa < 0$	$\mathfrak{h} > 0$	$M_3^\pm = \left(H^b\left(1/\sqrt{\lvert\varkappa\rvert}\right)\right)^\pm \times_\omega S^f\left(1/\sqrt{\mathfrak{h}}\right)$	$\omega(x) = \sqrt{\mathfrak{h}/\lvert\varkappa\rvert}\sinh\left(\sqrt{\lvert\varkappa\rvert}d(x, \Sigma_0)\right)$
$\varkappa < 0$	$\mathfrak{h} < 0$	$M_4 = H^b\left(1/\sqrt{\lvert\varkappa\rvert}\right) \times_\omega H^f\left(1/\sqrt{\lvert\mathfrak{h}\rvert}\right)$	$\omega(x) = \sqrt{\mathfrak{h}/\varkappa}\cosh\left(\sqrt{\lvert\varkappa\rvert}d(x, x_0)\right)$
$\varkappa < 0$	$\mathfrak{h} = 0$	$M_5 = \mathbb{R} \times_{\tilde{\omega}} \mathbb{R}^{b+f-1}$	$\tilde{\omega}(t) = \omega_0 e^{\sqrt{\lvert\varkappa\rvert}t}$

Table 1: The possible warped products with curvature $\varkappa$ and fibers with curvature $\mathfrak{h}$. The superscripts $\pm$ means the maximal connected regions of the manifold where ω has signum $\pm$ (e.g. $\left(S^b\left(1/\sqrt{\varkappa}\right)\right)^\pm$ are hemispheres or radius $1/\sqrt{\varkappa}$), and $H^b\left(1/\sqrt{\lvert\varkappa\rvert}\right)$ means the simply–connected hyperbolic space with sectional curvature $-\lvert\varkappa\rvert$.

4 Einstein warped products

An Einstein warped product is a warped product $M = B \times_\omega F$ whose metric g is Einstein: $\rho = \lambda g$. In order to avoid trivialities we will assume that the dimension of B is greater than one.

By (2) the equation $\rho = \lambda g$ now reads

$$\begin{aligned} \rho^B(X, Y) &= \lambda g_B(X, Y) + \frac{f}{\omega} H^\omega(X, Y) \\ \rho^F(V, W) &= (\lambda + \omega^\#)\omega^2 g_F(V, W). \end{aligned} \tag{7}$$

Proposition 4.1. *If (M, g) is an Einstein manifold with $\rho = \lambda g$ then (F, g_F) is Einstein and the following equation is satisfied*

$$(\lambda + \omega^\#)\omega^2 = \lambda_F,$$

where λ_F is a suitable constant.

Proof. This follows from the above formulae and the fact that the Ricci tensor ρ_F and g_F depend only on the point in F while the expression in brackets on the point in B. □

The Hessian H^ω of ω and the Ricci tensor ρ_B have the same eigenspaces, while their eigenvalues are related by

$$\frac{\lambda_i - \lambda}{f}\omega \in \sigma(H^\omega) \Leftrightarrow \lambda_i \in \sigma(\rho_B).$$

For fixed g_B and g_F, it is possible that there are no $\omega's$ which are solutions of the above system. In fact, (7) gives a constraint on the Riemannian curvature of B. A computation similar to that of Remark 3.3 shows that the compatibility of the tensor equation (7) is equivalent to

$$\begin{aligned} R^B_{XYZ\Omega} =& \frac{\lambda}{2f} g_B \owedge g_B(X, Y, Z, \Omega) + \frac{1}{f}\{g_B(X, \Omega)\rho_B(Y, Z) - \\ &- g_B(Y, \Omega)\rho_B(X, Z) + -\omega\nabla_X\rho_B(Y, Z) + \omega\nabla_Y\rho_B(X, Z)\}. \end{aligned}$$

This situation is too general for the purpose of a first classification so we will consider the case where B is Einstein as well. Then we have some relations between the Einstein constants of M, B and F: for instance, if we want that M has Einstein's nonvanishing λ then B is not flat, otherwise we have the equation $H^\omega = -\frac{\lambda}{f}\omega g_B$ which has only the null solution (apart from the trivial case when the dimension of B is one). Furthermore, it is possible to give nontrivial Einstein warped products, whose curvature is not constant.

Example 1. Let B be a manifold with constant curvature $\varkappa$, choosen in Table 1, and let ω be the corresponding function satisfying the equation (4).
Take F Einstein with $\lambda_F = \mathfrak{h}(f-1)$. Then $M = B \times_\omega F$ is an Einstein manifold with $\lambda = \varkappa(n-1)$. In fact, a direct computation shows that

$$\begin{aligned} \rho(X, Y) &= \rho^B(X, Y) - \frac{f}{\omega}H^\omega(X, Y) = \varkappa(b + f - 1)g_B(X, Y) = \\ &= \varkappa(n-1)g(X, Y); \\ \rho(V, W) &= \rho^F(V, W) - g(V, W)\omega^\# = (\mathfrak{h}(f-1)\omega^{-2} - \omega^\#)g(V, W) = \\ &= (\mathfrak{h}(f-1)\omega^{-2} - \frac{\Delta\omega}{\omega} - (f-1)\frac{||\Omega||^2}{\omega^2})g(V, W) = \\ &= (\mathfrak{h}(f-1)\omega^{-2} + \varkappa b - \mathfrak{h}(f-1)\omega^{-2} - \varkappa(f-1))g(V, W) = \\ &= \varkappa(n-1)g(V, W). \end{aligned}$$

The case in this example is quite paradigmatic of the situation in view of Theorem 3.4 which classified the manifolds with nontrivial solutions of $H^\omega = -\varkappa\omega\, g$. More generally we have

Proposition 4.2. *If $M = B \times_\omega F$ is Einstein with both factors Einstein then either*

i) *ω is constant and then $\lambda_B = \lambda$ and $\omega = \sqrt{\frac{\lambda_F}{\lambda}}$ or*

ii) *ω is nonconstant and $\lambda_B = \frac{b-1}{n-1}\lambda$. In this case $B = I \times_\alpha \Sigma_\mathfrak{q}$ with $I \subset \mathbb{R}$ and $\Sigma_\mathfrak{q} = \omega^{-1}(\mathfrak{q})$ ($\mathfrak{q}$ regular value for ω) is Einstein with constant*

$$\lambda_\Sigma = \frac{(b-2)\mathfrak{h}\varkappa}{\mathfrak{h} - \varkappa\mathfrak{q}^2}.$$

Proof. We only have to prove the relation between the Einstein's constants; the rest of the proof is an application of Theorem 3.4.

Letting $\varkappa = (\lambda - \lambda_B)/f$, it follows from equations (7) that the function ω must satisfy

$$H^\omega = -\varkappa\omega\, g_B \Rightarrow \begin{cases} \Delta\omega = -\varkappa\, b\, \omega \\ \varkappa\omega^2 + \|\Omega\|^2 = const \end{cases} \tag{8}$$
$$(\lambda + \omega^\#)\omega^2 = \omega\,\Delta\omega + (f-1)\|\Omega\|^2 + \lambda\,\omega^2 = \lambda_F = const.$$

Substituting the expression for the laplacian from the first into the second equation we get the two scalar equations

$$\frac{\lambda - b\varkappa}{f-1}\omega^2 + \|\Omega\|^2 = \frac{\lambda_F}{f-1}$$
$$\varkappa\omega^2 + \|\Omega\|^2 =: \mathfrak{j} = const.$$

Subtracting them we get

$$\left[\frac{\lambda - b\varkappa}{f-1} - \varkappa\right]\omega^2 = \frac{\lambda_F}{f-1} - \mathfrak{j} = const.$$

Hence

i) either ω is constant and then from (8) we have $\omega = \sqrt{\frac{\lambda_F}{\lambda}}$ and from the tensor equation $\lambda = \lambda_B$ or

ii) $\dfrac{\lambda - b\varkappa}{f-1} = \varkappa$, which is the same as saying $\lambda_B = \dfrac{b-1}{n-1}\lambda = (b-1)\varkappa$.

In the latter case, as we saw in Theorem 3.4 at least locally $B = I \times_\alpha \Sigma_\mathfrak{q}$ with α satisfying

$$(\alpha')^2 = \frac{\mathfrak{h}\varkappa}{\mathfrak{h} - \varkappa\mathfrak{q}^2} - \varkappa\alpha^2; \qquad \alpha(0) = 1.$$

Using formulae (2) with the substitution $M = B$, $\omega = \alpha$ and $f = b - 1$ we compute (observe that, from the above, $\alpha'' = -\varkappa\alpha$)

$$\begin{aligned}\lambda_\Sigma &= \alpha^2(\alpha^\# + \lambda_B) = \alpha\alpha'' + (b-2)(\alpha')^2 + \varkappa(b-1)\alpha^2 = \\ &= (b-2)\left[\varkappa\alpha^2 + (\alpha')^2\right] = \frac{(b-2)\mathfrak{h}\varkappa}{\mathfrak{h} - \varkappa\mathfrak{q}^2}.\end{aligned}$$

This ends the proof. □

As we saw in the proof, compatibility between the tensor equation and the scalar one gives constraint on the values of Einstein constants.

Proposition 4.3. *Let $M = B \times_\omega F$ be a warped product (with ω not constant) Einstein manifold with constant λ with both factors Einstein with constants λ_B and λ_F, respectively. Then, letting $g_B^1 := \omega^{-2} g_B$, (B, g_B^1) is Einstein as well with constant $\mu_B = -\frac{b-1}{f-1}\lambda_F$.*

Proof: The proof is based on the formula of the Ricci tensor of the metric $g_B^1 = e^{2\alpha} g_B$ given in [1][1]

$$\rho_B^1 = \rho_B - (b-2)(H^\alpha - d\alpha \circ d\alpha) - (\Delta\alpha + (b-2)||A||^2) g_B,$$

where A denotes the gradient of α. In order to prove the Proposition, we substitute $\omega = e^{-\alpha}$. We have, for the gradient and the hessian,

$$\begin{aligned}\Omega &= -\omega A; & d\omega &= -\omega d\alpha; \\ H^\omega &= \omega(d\alpha \circ d\alpha - H^\alpha); & \Delta\omega &= \omega(||A||^2 - \Delta\alpha).\end{aligned}$$

These relations imply that

$$\rho_B^1 = \rho + (b-2)\frac{H^\omega}{\omega} + (\frac{\Delta\omega}{\omega} - (b-1)\frac{||\Omega||^2}{\omega^2}) g_B.$$

From the equation $H^\omega = -\varkappa\,\omega\, g_B$ and recalling from Proposition 4.2 that

$$\varkappa = \frac{\lambda}{n-1}, \quad \lambda_B = \frac{b-1}{n-1}\lambda = (b-1)\varkappa,$$

[1] This formula is not strictly the same as in [1], since the definitions of the Laplacian have different sign.

we get

$$\begin{aligned}\rho_B^1 &= \left(\lambda_B - \varkappa(b-2) + \frac{\Delta\omega}{\omega} - (b-1)\frac{||\Omega||^2}{\omega^2}\right) g_B = \\ &= \left((\lambda_B - 2\varkappa(b-1))\omega^2 - (b-1)||\Omega||^2\right) g_B^1 = \\ &= \left(-\varkappa(b-1)\,\omega^2 - (b-1)\|\Omega\|^2\right) g_B.\end{aligned}$$

Thus the metric g_B^1 is Einstein and the new constant μ_B is given by

$$\mu_B = \left(-\varkappa(b-1)\,\omega^2 - (b-1)\|\Omega\|^2\right) = -\frac{b-1}{f-1}\lambda_F$$

It follows immediately that

Corollary 4.4. *Under the assumptions of Proposition (4.3) (M, g) is conformal to (M, g^1), where g^1 is the product $g_B^1 + g_F$; both g_B^1 and g_F are Einstein and their constants satisfy the relation*

$$\mu_B = -\frac{b-1}{f-1}\lambda_F.$$

Such a result suggests a remark about the warped products with constant curvature.

Remark 4.5. Let (M, g) have constant curvature equal to $\varkappa$. Then the conformal metric $g^1 = \omega^{-2}g = g_B^1 + g_F$ is the product of two metrics with the opposite constant sectional curvature:

$$\begin{aligned}R^1 &= \omega^{-2}\left(\frac{\varkappa}{2} g_B \owedge g_B - g_B \owedge\left(H^\alpha - d\alpha \circ d\alpha + \frac{1}{2}||A||^2 g_B\right)\right) = \\ &= \omega^{-2}\left(\frac{\varkappa}{2} g_B \owedge g_B - g_B \owedge\left(\varkappa\, g_B + \frac{1}{2}\frac{||\Omega||^2}{\omega^2} g_B\right)\right) = \\ &= -\frac{1}{2}\omega^{-2}\left(\frac{||\Omega||^2}{\omega^2} + \varkappa\right) g_B \owedge g_B = -\frac{1}{2}\frac{\mathfrak{h}}{\omega^4} g_B \owedge g_B = -\frac{\mathfrak{h}}{2} g_B^1 \owedge g_B^1.\end{aligned}$$

□

A natural question is whether these kind of Einstein manifolds can be both warped–products and geodesically complete. More generally we will consider what are the necessary conditions for the existence of a complete Einstein manifold (M, g) such that it possesses an open maximal subset $M_0 \hookrightarrow M$ isometric to a warped product $M_0 \simeq B \times_\omega F$ (both factors Einstein). We will see that in some cases the condition imposes constraints on the curvature of B, F or both.

In the trivial case when $\omega = \sqrt{\lambda/\lambda_F}$ (and $\varkappa = 0$) the manifold is just a direct product of Einstein manifolds so that completeness is equivalent to completeness of both factors. We consider this case as uninteresting and therefore we are going to exclude it from the following discussion.

The necessary (and sufficient) condition to have $M = M_0 = B \times_\omega F$ geodesically complete and globally warped product, is that both (B, g_B) and (F, g_F) are complete and ω never vanishes on B. Since the warping factor ω must satisfy the system (4) then according to the relative signs of $\varkappa = \frac{\lambda}{n-1}$ and $\mathfrak{h} = \frac{\lambda_F}{f-1}$ its form is dictated by the expressions in Table 1.

Thus we see that there are only three classes of cases in which ω never vanishes on a complete manifold (B, g_B) (and thus $M_0 = M$):

1. $\lambda < 0 > \lambda_F$. In this case the critical locus of ω is not empty since $\omega = \sqrt{\mathfrak{h}/\varkappa}\cosh(\sqrt{\varkappa} d_B(x, x_0))$ for some $x_0 \in B$. From Thm. 3.4 we know that g_B is of constant sectional curvature while (F, g_F) can be any complete Einstein manifold with the appropriate constant.

2. $\lambda < 0 = \lambda_F$. Then (again from Thm. 3.4), (B, g_B) must be a complete warped product itself of the form $\mathbb{R} \times_{\exp(\sqrt{|\varkappa|}t)} \Sigma$ with Σ complete and Einstein, while (F, g_F) can be any complete Einstein fiber (with constant λ_F).

3. $\lambda = 0 = \lambda_F$. Then ω is a constant and (M, g_M) is a direct product of complete Einstein manifolds (of zero constants).

In the remaining cases (B, g_B) could not possibly be complete and have a never vanishing solution ω. In these circumstances we consider $M_0 = B \times_\omega F$ where (B, g_B) is a maximal open subset of a complete simply–connected manifold $(\overline{B}, g_{\overline{B}})$ restricted to which ω is not zero and satisfies the system $H^\omega = -\varkappa\omega g$. In this case ω (defined on B) can be extended to a smooth function (denoted by the same symbol) on M so that the boundary $\partial M_0 = \omega^{-1}(0)$. Under these circumstances we have:

4. If $\lambda = 0 < \lambda_F$ then B is itself a direct product $\mathbb{R}_+^\times \times \Sigma$ for some complete Einstein (in this case) manifold Σ (see Thm. 3.4). In this case ω is proportional to the geodesic coordinate $t \in \mathbb{R}_+^\times$. A computation similar to that in Thm 3.4 for the sectional curvature of F (which plays the role of Σ_q in said Thm.) shows that F must be of constant positive sectional

curvature. Indeed, for any two vertical vectors $U, V \in T_x M_0$, the sectional curvature is

$$K_{UV} = \frac{K^F_{\sigma^*U,\sigma^*V} - ||\omega'(t)||^2}{\omega(t)^2}$$

(where $t = t(\pi(x))$, π denoting the projection on B and σ the projection on F). If we parallel transport the two vectors U, V along the gradient of ω (which generates a geodesic γ starting at x) they keep spanning the same two-plane in $T_{\sigma(x)}F$ because they are simply rescaled, while $\sigma(x) = \sigma(\gamma)$ is constant. By taking the limit $t \to 0$ and from the fact that this limit must exist finite (since (M, g) is smooth), we obtain the constancy of the sectional curvature of F.

Thus, assuming simply–connectedness of F, we have

$$M \hookleftarrow M_0 = (\mathbb{R}_+^\times \times \Sigma) \times_\omega S^f(1/\sqrt{\mathfrak{h}}) .$$

5. If $\lambda > 0$, then ω must be the cosine of the distance from a fixed point $x_0 \in B$, i.e. $\omega(x) = \sqrt{\mathfrak{h}/\varkappa}\cos(\sqrt{\varkappa}d(x, x_0))$: then B is a hemisphere (Thm. 3.4). As above, considering the sectional curvature of (F, g_F) in a neighborhood of $\omega^{-1}(0) \hookrightarrow M$, we find that F too must be of positive constant sectional curvature ($K^F = \mathfrak{h}$). Therefore M_0 (and by continuity M too) is of positive curvature (a sphere if we assume simply–connectedness).
6. If $\lambda < 0 < \lambda_F$ then $\omega = \sqrt{\mathfrak{h}/|\varkappa|}\sinh(\sqrt{|\varkappa|}d(x, \Sigma_0))$ for some totally geodesic hypersurface $\Sigma_0 \hookrightarrow \overline{B}$. Then $B = \mathbb{R}_+^\times \times_\alpha \Sigma_0$ (Thm. 3.4) where Σ_0 must be complete and Einstein. Here $\alpha = \frac{1}{\sqrt{\mathfrak{h}-\varkappa}}\|\Omega\|_B$ (i.e. it is a hyperbolic cosine). Again, smoothness of M at the boundary of M_0 implies that F is of constant positive sectional curvature.

Concluding we see that -as anticipated- the requirement of completeness for (M, g) joint with the smoothness at the points of $\partial M_0 \hookrightarrow M$ (in the setting above) "rigidifies" the manifold M completely to be of constant sectional curvature in case (5) or rigidifies the fiber F in cases (4) and (6).

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