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Congruence of hypersurfaces in S^6 and in CP^n

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Abstract. The invariants needed to decide when a pair of hypersurfaces of S^6 or CP^n are respectively G_2-congruent or holomorpic congruent are determined and this result is used to characterize the hypersurfaces of these spaces whose Hopf vector fields are also Killing fields.

Keywords: rigidity, hypersurfaces, nearly Kähler 6-sphere, complex projective space.

1. Introduction

We investigate a special type of rigidity for hypersurfaces of the 6-sphere. Our inspiration to tackle this problem came from a series of three papers [10],[9] and [5] where holomorphic congruence of hypersurfaces of complex projective spaces is studied.

Let $g : M \longrightarrow \overline{M}$ be an isometric immersion of a Riemannian manifold M^{2n-1} into a nearly Kahler manifold $(\overline{M}^n, J)$ and let $\tilde{\xi}$ be a normal vector field on the hypersurface $\widetilde{M} = g(M)$ of $\overline{M}$. Then we can define on M a **Hopf vector field** $\hat{U}$ and tensors $\hat{\phi}$ and $\hat{A}$ of type $(1, 1)$, as follows

$$g_*\hat{U}_q = J\tilde{\xi} \tag{1}$$
$$g_*\hat{\phi}(X) = J(g_*X) - < Jg_*X, \tilde{\xi} > \tilde{\xi} \tag{2}$$
$$g_*\hat{A}(X) = -\overline{\nabla}_X\tilde{\xi}. \tag{3}$$

When M is a submanifold of $\overline{M}$ and g is taken as the inclusion map then we denote these induced structures on the hypersurface M by ξ, U, ϕ, A. In this case they are more simply expressed by

$$U = J\xi, \tag{4}$$
$$\phi(X) = JX + < X, U > \xi. \tag{5}$$

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We will make extensive use later on of some basic properties, listed below, of these induced structures. They are immediate consequences of the definitions above and the properties of the almost complex structure J. We should also point out that they are also valid for the induced structures $(\hat{U}, \hat{\phi})$.

$$\begin{aligned}
\phi^2 X &= -X + \langle X, U\rangle U, && (6)\\
\langle \phi X, \phi Y\rangle &= \langle X, Y\rangle - \langle X, U\rangle\langle Y, U\rangle, && (7)\\
\phi &\quad \text{is skew symmetric}, && (8)\\
Ker(\phi) &= span\{U\}, && (9)\\
\phi : U^{\perp} \to U^{\perp} &\quad \text{is a linear isometry}, && (10)\\
(\nabla_X \phi)Y &= \langle AX, Y\rangle U - \langle Y, U\rangle AX. && (11)
\end{aligned}$$

We say that M is a **Hopf hypersurface** of $\overline{M}$ if the integral curves of U are geodesics of M, that is

$$\nabla_U U = 0. \tag{12}$$

Berndt, Bolton and Woodward [1] have given a complete characterization of the Hopf hypersurfaces of the 6-sphere as tubular hypersurfaces around almost complex curves, these curves being fully classified in [3]. For the case of complex projective spaces the works of Cecil and Ryan [4] and Martins [7] give the caracterization of these hypersurfaces as tubes around complex submanifolds.

We recall the vector cross product $\times$ obtained by identifying $\mathbb{R}^7$ with the imaginary part of the Cayley numbers $\mathbb{O}$ and defining $u \times v = \Im\mathrm{m}(uv)$ where $u, v \in \mathbb{R}^7 = \Im\mathrm{m}\mathbb{O}$. If $\overline{M}$ is an oriented hypersurface of $\mathbb{R}^7$ with orientation determined by a choice of a unit normal vector field N, then $\overline{M}$ admits an orthogonal almost complex structure J defined by $JX = N(p) \times X$, $p \in \overline{M}$ and $X \in T_p\overline{M}$. In the particular case where $\overline{M}$ is the unit sphere S^6 with its standard orientation, we identify the unit normal $N(p)$ at $p \in S^6$ with the vector $p \in \mathbb{R}^7$ so that $JX = p \times X$, for $p \in S^6$ and $X \in T_pS^6$. Moreover, the tensor J is a nearly Kähler structure on S^6.

The subgroup of the group $SO(7)$ of isometries which preserve J is the exceptional Lie group G_2. Indeed, the group G_2 is the group of isometries of $\mathbb{R}^7$ which preserve the vector cross-product. A G_2-basis $\{e_1, \dots, e_7\}$ for $\mathbb{R}^7$ is an orthonormal basis satisfying the following multiplication properties:

$$e_1 \times e_2 = e_3, \quad e_1 \times e_4 = e_5, \quad e_2 \times e_4 = e_6, \quad e_3 \times e_4 = e_7.$$

It is easy to verify that a linear transformation of $\mathbb{R}^7$ lies in G_2 if and only if it maps a G_2-basis to another G_2-basis.

We use the transitive action of the excepcional Lie group G_2 on the 6-sphere to obtain a special type of rigidity for hypersurfaces of this sphere. Namely, given an isometric immersion $f : M \to S^6$ of a non totally umbilic hypersurface M of the 6-sphere whose second fundamental form has rank greater or equal to 3, we prove that this immersion is extendable to an element of G_2 if and only if its derivative maps the Hopf vector field of M to the Hopf vector field of $f(M)$.

When we consider the case $\overline{M} = S^6$ and g as the restriction of a linear map (lying on SO(7)) to a hypersurface M of S^6, we can write the definition of J in terms of the vector cross product of $\mathbb{R}^7$ and so those structures can be rewritten as

$$g\hat{U}_q = gq \times g\xi, \tag{13}$$
$$g\hat{\phi}(X) = gq \times gX + \langle X, \hat{U}\rangle g\xi. \tag{14}$$

In this case the rate of change of the Hopf vector field will play an important role when dealing with the induced structures $(\phi, \hat{\phi})$ because it gives a direct relation between these structures and the corresponding second fundamental forms A and $\hat{A}$. Namely

$$\nabla_X U = -\phi AX + X \times \xi + \langle X, U\rangle q, \tag{15}$$
$$g\nabla_X \hat{U} = -g\hat{\phi}\hat{A}X + gX \times g\xi + \langle X, \hat{U}\rangle gq. \tag{16}$$

These equations can be easily obtained by using the Riemannian connections $\widetilde{\nabla}$ and $\overline{\nabla}$ of $\mathbb{R}^7$ and S^6 respectively. For example we obtain (15) as follows.

$$\nabla_X U = \overline{\nabla}_X(J\xi) - \langle AX, U\rangle\xi,$$

[illegible]

$$= X \times \xi + q \times \overline{\nabla}_X\xi - \langle X \times \xi, q\rangle q - \langle AX, U\rangle\xi,$$
$$= -\phi AX + X \times \xi + \langle X, U\rangle q.$$

Proposition 1. *Let M be a hypersurface of S^6 and $g \in SO(7)$. Consider the*

On the other hand $A = \hat{A}$ because g is an isometry of S^6. This together with the assumptions gives

$$g(q \times X) \quad = gq \times gX \text{ for every } X \in T_q M \tag{17}$$
$$g(\xi \times X) \quad = g\xi \times gX \text{ for every } X \in T_q M. \tag{18}$$

Indeed, the first equation comes from equations (5) and (14), and the second one comes from equations (15) and (16). Now, since $U \times X = (q \times \xi) \times X = \xi \times JX = \xi \times \phi X$, we obtain

$$\begin{aligned} g(U \times X) &= g(\xi \times \phi X) = g\xi \times g\phi X && \text{[from (18)]} \\ &= g\xi \times g(q \times X) = g\xi \times (gq \times gX) && \text{[from (17)]} \\ &= gU \times gX. && (19) \end{aligned}$$

If $q \in M$ and $X = X_q \in T_q M$ is a unit tangent vector orthogonal to U_q then elementary calculations using the basic properties of the cross-product of $\mathbb{R}^7$ show that the ordered set

$$\{q, \xi, U, X, q \times X, \xi \times X, U \times X\}$$

is a G_2-basis for $\mathbb{R}^7$. Observe that equations (17).(18) and (19) say that

$$\{gq, g\xi, gU, gX, g(q \times X), g(\xi \times X), g(U \times X)\}$$

is also a G_2-basis and hence $g \in G_2$. □

Lemma 1. *Let $g \in SO(7)$ and let M be a hypersurface of S^6 endowed with the induced structures $(\phi, \hat{\phi}, A = \hat{A})$ as described above, then $\phi A - \hat{\phi} A = A\phi - A\hat{\phi}$.*

Proof: If M is totally umbilic then the lemma holds trivially. Thus assume that M is not totally umbilic. Let a and b be distinct eigenvalues of A and let X and Y be corresponding principal vector fields of M, then from (15) and (16) we have

$$\begin{cases} -a\langle \phi X, Y\rangle + \langle X \times \xi, Y\rangle = \langle \nabla_X U, Y\rangle = -a\langle \hat{\phi} X, Y\rangle + \langle gX \times g\xi, gY\rangle \\ -b\langle \phi Y, X\rangle + \langle Y \times \xi, X\rangle = \langle \nabla_Y U, X\rangle = -b\langle \hat{\phi} Y, X\rangle + \langle gY \times g\xi, gX\rangle. \end{cases}$$

As ϕ is skew symmetric, we get

$$\begin{cases} a(\langle \phi X, Y\rangle - \langle \hat{\phi} X, Y\rangle) &= \langle X \times \xi, Y\rangle - \langle gX \times g\xi, gY\rangle, \\ b(\langle \phi X, Y\rangle - \langle \hat{\phi} X, Y\rangle) &= \langle gY \times g\xi, gX\rangle - \langle Y \times \xi, X\rangle. \end{cases}$$

And thus we have $(a-b)(\langle\phi X, Y\rangle - \langle\hat{\phi}X, Y\rangle) = 0$. Which implies $\langle(\phi - \hat{\phi})X, Y\rangle = 0$. Therefore, since A is symmetric, we conclude that $(\phi - \hat{\phi})$ leaves all the eigenspaces invariant and consequently the equality $\phi A - \hat{\phi}A = A\phi - A\hat{\phi}$ holds on every eigenspace, and hence everywhere. □

When M is a Hopf hypersurface of S^6 then M is an open subset of the tube $\Phi_r(\perp {}^1S)$ around an almost complex curve of S^6 which is given in turn as the focal set of the focal map Φ_r of M (see [1] for details). In this case, we can give an explicit description of the integral curves of the Hopf vector field U of M. Indeed, given a point q of M, say the end point of the geodesic

$$q = \gamma_{(p,\eta)}(r) = \cos(r)p + \sin(r)\eta. \tag{20}$$

Consider the curve $\sigma(t)$ of M, passing through q, given by

$$\sigma(t) = \gamma_{(p,\delta(t))}(r) = \cos(r)p + \sin(r)\delta(t), \tag{21}$$

where $\delta(t) = \cos(\bar{t})\eta + \sin(\bar{t})p \times \eta$ with $\bar{t} = \frac{t}{\sin(r)}$.

In the following equations we use **dot** and **prime** to denote derivatives with respect to the variables s and t respectively. Now, by elementary calculations we obtain

$$\sigma' = p \times \delta = U(\sigma). \tag{22}$$

Thus σ is the integral curve of U through q and this proves that the The flow $\mathbb{C}F_t$ of the Hopf vector field of a Hopf hypersurface $M \subset \Phi_r(\perp {}^1S)$ is given by

$$\mathbb{C}F_t(\gamma_{(p,\eta)}(r)) = \gamma_{(p,\delta(t))}(r) = \cos(r)p + \sin(r)\delta(t). \tag{23}$$

In particular, we note in passing that the integral curve of the Hopf vector field starting at the point $\gamma_{(p,\eta)(r)}$ is geometrically originated from the rotation of the complex 2-plane at p spanned by the vectors $\{\eta, J\eta\}$.

We name as generic Hopf hypersurfaces of S^6 those ones which are neither the geodesic hyperspheres nor subsets of a tube around totally geodesic almost complex curves.

Proposition 2. *Let g be an isometry of S^6 ($g \in SO(7)$) and let M be a generic Hopf hypersurface of S^6, then the following conditions are equivalent*

- **(i)** $\widetilde{M} = g(M)$ *is a Hopf hypersurface,*
- **(ii)** $g(p \times X) = gp \times gX$ *for every* $p \in S$ *and* $X \in T_pS^6$,
- **(iii)** g *maps the Hopf vector field U of M to the Hopf vector field $\widetilde{U}$ of $\widetilde{M}$ ($U = \hat{U}$), that is* $g(q \times \xi) = gq \times g\xi$.

Proof: $(iii) \Longrightarrow (i)$ The isometry $f = g\,|_M\colon M \to \widetilde{M}$ maps the geodesics which are integral curves of U to geodesics which are integral curves of $\widetilde{U}$ and hence by definition $\widetilde{M}$ is a Hopf hypersurface.

$(ii) \Longleftrightarrow (iii)$ As an isometry of S^6, g maps the geodesic $\gamma_{(p,\delta)}(r)$ to the geodesic $\widetilde{\gamma}_{(gp,g\delta)}(r)$ thus from (22) we obtain respectively

$$\begin{aligned} g\sigma' &= \sin(\bar{t})g\eta + \cos(\bar{t})g(p \times \eta). && (24)\\ \widetilde{U}_{g\sigma} &= g\sigma \times \dot{\widetilde{\gamma}}_{(gp,g\delta)}(r) = g\sigma \times g\dot{\gamma}_{(p,\delta)}(r) = gp \times g\delta \\ &= \cos(\bar{t})(gp \times g\eta) - \sin(\bar{t})[gp \times g(p \times \eta)]. && (25)\end{aligned}$$

Thus if we use that $g \in SO(7)$ and the vectors $\{g\eta, gp \times g(p \times \eta)\}$ are orthogonal to the vectors $\{g(p \times \eta), gp \times g\eta\}$, then the equivalence $(ii) \Longleftrightarrow (iii)$ follows from equations (24) and (25). Note that under the assumption of either condition (i) or (iii), condition (ii) is trivially satisfied for every $X \in T_pS$, because in both cases the isometry g will map the almost complex curve S into the almost complex curve $g(S)$.

$(i) \Longrightarrow (ii)$ Since g is an isometry, it takes the focal set S of M into the focal set $\widetilde{S}$ of $\widetilde{M}$, moreover the hypersurfaces M and $\widetilde{M}$ lie on tubes around the almost complex curves given by these focal sets.

In order to prove that g and J commute along S we first note that the images of the second fundamental forms h and $\widetilde{h}$ of S and $\widetilde{S}$ span 2-dimensional J-invariant subspaces V_1 and $\widetilde{V}_1$ of the normal spaces $\perp_p S$ and $\perp_{gp}\widetilde{S}$.

As S and $\widetilde{S}$ are almost complex curves we also have that these normal spaces are J-invariant. Thus they can be decomposed as direct sum of J-invariant subspaces $\perp_p S = V_1 \oplus V_2$ and $\perp_{gp}\widetilde{S} = \widetilde{V}_1 \oplus \widetilde{V}_2$.

However, since g is an isometry of S^6, mapping S to $\widetilde{S}$, we have $g(T_pS) = T_{gp}\widetilde{S}$, $g(\perp_p S) = \perp_{gp}\widetilde{S}$, $g(V_1) = \widetilde{V}_1$, and $g(V_2) = \widetilde{V}_2$. Thus using that $g \in SO(7)$ plus the orthogonal properties of J we see that these maps commute on the subspace T_pS and on each subspace V_j and hence they commute on T_pS^6 for every $p \in S$. □

It is worthwhile observing that the condition (ii) in the proposition above gives a way to improve this result by proving that actually an element $g \in SO(7)$ satisfying those conditions lies in fact in G_2. Although the following examples show that this would not be true for every Hopf hypersurface of S^6, we will prove after the examples that it is true for any generic Hopf hypersurfaces.

Example 1. *Let M be a geodesic hypersphere of S^6 centred at the point e_4. Consider the element F of $SO(7)$ defined by $F(e_j) = e_j$ for $j \neq 3, 7$, $F(e_3) = e_7$, $F(e_7) = -e_3$, then F is the unique extension of the isometry $f = F\,|_M$:*

$M \to M$ and obviously F is not an element of G_2. Moreover, F maps the Hopf vector field to itself.

Example 2. *Let M be a Hopf hypersurface contained in a tube around the almost complex curve $S = V^3 \cap S^6$ where $V^3 = span\{e_3, e_4, e_7\}$. Consider the map $F \in SO(7)$ given by $F(e_j) = e_j$ for $j = 3, 4, 7$, $F(e_1) = e_2$, $F(e_2) = e_1$, $F(e_5) = e_6$, $F(e_6) = e_5$, then $F \notin G_2$ and F is the unique extension of the isometry $f = F|_M\colon M \to M$. Furthermore, F does not map the Hopf vector field U of M to the Hopf vector field of $F(M)$, that is $U \neq \hat{U}$.*

In order to see the later part of each example above we just remark that as F is a linear map then from (22) we have that at each point $q = \gamma_{(p,\eta)} \in M$ and $F(q)$ the Hopf vectors are given respectively by

$$U_q = p \times \eta \qquad \text{and} \qquad F(\hat{U}_q) = Fp \times F\eta. \tag{26}$$

Therefore, $U_q = \hat{U}_q$ if and only if

$$F(p \times \eta) = Fp \times F\eta,$$

from which the properties stated in the examples follow.

Proposition 3. *Let M be a generic Hopf hypersurface of the 6-sphere. Let $g \in SO(7)$, then $\widetilde{M} = g(M)$ is a Hopf hypersurface if and only if $g \in G_2$.*

Proof: ($\Longleftarrow$) If $g \in G_2$ then g maps the Hopf vector field U of M to the Hopf vector field $\widetilde{U}$ of $\widetilde{M}$ so that from Proposition 2, $\widetilde{M}$ is a Hopf hypersurface.
($\Longrightarrow$) If $\widetilde{M}$ is a Hopf hypersurface, then from Proposition 2 we know that $U = \hat{U}$, that is

$$g(q \times \xi) = gq \times g\xi \text{ for every } q \in M \tag{27}$$

As we have just noted in the proof of Proposition 1, in order to prove that $g \in G_2$ it suffices to find a unit vector $X = X_q \in T_qM$ orthogonal to U_q and satisfying the following equations

$$g(q \times X) = g(q) \times g(X) \tag{28}$$

$$g(\xi \times X) = g(\xi) \times g(X) \tag{29}$$

$$g(U \times X) = g(U) \times g(X) \tag{30}$$

Let us consider M as an open subset of the tube $\Phi_r(\perp^1 S)$ of radius $r \in (0, \frac{\pi}{2})$ around an almost complex curve S. Let $p \in S$ and $\eta \in \perp^1_p S$. Let γ denote the geodesic $\gamma_{(p,\eta)}(s) = \cos(s)p + \sin(s)\eta$ of S^6. The unit vector $\xi := \dot{\gamma}(r)$ is normal to the hypersurface M at the point $q := \gamma(r)$.

Given $X \in T_pS^6 \cap \{\mathbb{R}\eta\}^\perp$, we can define a Jacobi field W_X along γ complying with the conditions

$$W_X(0) = X^T \qquad \text{(orthogonal projection of } T_pS^6 \text{ onto } T_pS)$$
$$\dot{W}_X(0) = X^\perp - A_\eta X^T \qquad \text{(orthogonal projection of } T_pS^6 \text{ onto } \perp_p S),$$

where A_η denotes the shape operator of S with respect to η.

Let us denote by $B_v(s)$ the parallel transport of a vector $v \in T_pS^6$ along γ. Then the Jacobi field W_X can be written as

$$W_X(s) = \cos(s)B_{X^T}(s) + \sin(s)B_{X^\perp - A_\eta X^T}(s). \tag{31}$$

Thus, we can distinguish two particular cases. The first being when X is an eigenvector of A_η, say $A_\eta X = \lambda X$. This implies

$$W_X(s) = (\cos s - \lambda \sin s)B_X(s).$$

The second case is when X lies in $(\perp_p S) \cap (\mathbb{R}\eta)^\perp$, for which we have

$$W_X(s) = (\sin s)B_X(s).$$

Noting that W_X is a M-Jacobi field and writting the principal curvature λ as $\lambda = \tan(\theta)$, we have that $B_X(r)$ is a principal vector of A_ξ with eigenvalues $\tan(r \pm \theta)$ and $-\cot(r)$ corresponding to the first and second cases respectively.

Therefore the orthonormal eigenvectors $\{B_i\}$ $(i = 3, ..., 7)$ of the shape operator A of M at a point $q = \gamma_{p,\eta}(r) \in M$ are just the parallel transport $B_i(t)$ along γ of orthonormal vectors $\{X_1 = p, X_2 = \eta, X_3 = p \times \eta, X_4, X_5 = p \times X_4, X_6, X_7 = p \times X_6\}$ satisfying the following conditions

- $\{X_1, ..., X_7\}$ is a G_2-basis for $\mathbb{R}^7$;
- $\{X_6, X_7 = p \times X_6\}$ is a basis for T_pS;
- $\{X_2 = \eta, X_3 = p \times \eta, X_4, X_5 = p \times X_4\}$ is a basis for $\perp_p^1 S$.

We will show now that $X = B_6 = B_6(r)$ satisfies the equations (28) (29) and (30) and therefore g is an element of G_2. Consider the vector field

$$L(s) = g(\gamma_\eta \times B_6) - g\gamma_\eta \times gB_6,$$

then L is a Jacobi field along γ_η. Indeed, $\ddot{\gamma}_\eta = -\gamma_\eta$ and so $\ddot{L} = -L$. Moreover L also satisfies

$$L(0) = g(p \times X_6) - gp \times gX_6,$$
$$L(r) = g(q \times B_6) - gq \times gB_6,$$
$$\dot{L}(0) = g(\eta \times X_6) - g\eta \times gX_6.$$

It follows from Proposition (2-ii) that for every curve σ in S and every vector field $Z \in \mathfrak{X}(S^6)$ along σ, we have

$$g(\sigma \times Z) \quad = \quad g\sigma \times gZ. \tag{32}$$

Considering Z as parallel vector field along σ and differentiating this last equation, we see that for each vector $X \in T_pS$ and $Z \in T_pS^6$,

$$g(X \times Z) \quad = \quad gX \times gZ. \tag{33}$$

Thus it follows from (32) and (33) that $L(0) = 0$ and $\dot{L}(0) = 0$ respectively. Therefore, the Jacobi field L vanishes identically. In particular $L(r) = 0$ which proves that B_6 satisfies the equation 28.

We can similarly prove that B_6 satisfies the equations (29) and (30) by using respectively the following Jacobi vector fields
$L(s) = g(\dot{\gamma}_\eta \times B_6) - g\dot{\gamma}_\eta \times gB_6,$
$L(s) = g(B_6 \times (\gamma_\eta \times \dot{\gamma}_\eta)) - gB_6 \times g(\gamma_\eta \times \dot{\gamma}_\eta).$ □

Corollary 1. *Given a non-totally umbilic Hopf hypersurface M of S^6, that is M is not a geodesic hypersphere, and $g \in SO(7)$ then $U = \hat{U}$ if and only if $g \in G_2$.*

Proof: If M is a generic Hopf hypersurface this is just a consequence of Proposition 2 and Proposition 3. Therefore, we just need to prove the Corollary for the case when M is an open subset of a tube around a totally geodesic almost complex curve S. We can assume without loss of generality that S is the intersection of S^6 with the subspace of $\mathbb{R}^7$ spanned by the cannonical vectors $\{e_1, e_2, e_3\}$.

Now, we know from (22) that $U = \hat{U}$ if and only if for every $p \in S^6 \cap span\{e_1, e_2, e_3\}$ and $\eta \in span\{e_4, e_5, e_6, e_7\}$ we have $g(p \times \eta) = gp \times g\eta$, which implies that g maps the cannonical G_2-basis of $\mathbb{R}^7$ to another G_2-basis and hence $g \in G_2$. □

Theorem 1. *Let M be a non totally umbilic hypersurface of S^6 and $g \in SO(7)$, then g maps the Hopf vector field of M to the Hopf vector field of $g(M)$, that is $U = \hat{U}$, if and only if g is an element of G_2.*

Proof: The converse of the theorem is trivial.

If M is a Hopf hypersurface then the Theorem has already been proved by Corollary (1), thus we may assume that U and AU are linearly independent vector fields.

Now, let us assume that $U = \hat{U}$. Looking at Proposition (1), we see that it is only necessary to prove that $\phi = \hat{\phi}$. Moreover, if $U = \hat{U}$ then (15) and (16) yield

$$g\phi AX - g\hat{\phi}AX = g(X \times \xi) - gX \times g\xi. \tag{34}$$

In particular, for $X = U$ we have

$$\phi AU = \hat{\phi}AU. \tag{35}$$

Using this and the fact that $\phi^2 = \hat{\phi}^2$ when $U = \hat{U}$, then we get

$$\phi(\phi AU) = \phi^2(AU) = \hat{\phi}^2(AU) = \hat{\phi}(\phi AU). \tag{36}$$

Hence $\phi = \hat{\phi}$ on the space $V = span\{U, AU, \phi AU\}$. Note that this space has always dimension three because (8) and (9) imply that $\phi AU \neq 0$ and ϕAU is orthogonal to U and AU. Thus we have $T_qM = V_q \oplus W_q$, where W_q is the 2-dimensional orthogonal complement $V^{\perp}$.

By using those properties, (6) and (8), of ϕ and $\hat{\phi}$, we can also see that W is invariant under these maps and it follows particularly from (7) that $\phi, \hat{\phi} : W \longrightarrow W$ are isometries.

Now, since by their definitions ϕ and $\hat{\phi}$ realize $\frac{\pi}{2}$-rotations and $dim(W) = 2$ then $\phi = \hat{\phi}$ or $\phi = -\hat{\phi}$ on W.

Therefore we just need now to prove that $\phi = -\hat{\phi}$ on W leads us to a contradiction. Henceforth let us assume $\phi = -\hat{\phi}$ on W. First we observe that W is invariant under the tensor A. Indeed, given any vector $X \in W$, we have $\phi X \in W$ and so $\langle A\phi X, U\rangle = \langle \phi X, AU\rangle = 0$. Using Lemma 1 together with (8) and (35) we get $\langle A\phi X, AU\rangle = 0$, and from Lemma 1 together with (35) and (7) we obtain $\langle A\phi X, \phi AU\rangle = 0$. Thus $A\phi X \in W$ for every $X \in W$ and consequently $A(W) \subset W$.

The invariance of W under A together with Lemma 1 imply that A and ϕ commute on W and hence for each $X \in W$ we have $\langle A\phi X, \phi X\rangle = \langle \phi AX, \phi X\rangle = \langle AX, X\rangle$, and so

$$\langle AX, \phi X\rangle = -\langle \phi AX, X\rangle = -\langle A\phi X, X\rangle = -\langle AX, \phi X\rangle,$$

which implies $\langle AX, \phi X\rangle = 0$. However, $\{X, \phi X\}$ is an orthonormal basis for W, so $AX = kX$ for every $X \in W$. Considering this in (34) we have $2kg\phi X = g(X \times \xi) - gX \times g\xi$, and since $g\phi X = -g\hat{\phi}X = -gq \times gX$, we deduce that $k = 0$. Therefore the second fundamental form A vanishes on W and consequently (15) and (16) are reduced to

$$\nabla_X U = X \times \xi \tag{37}$$

$$g\nabla_X U = gX \times g\xi. \tag{38}$$

Substituting X by ϕX in these equations and using (5), (13) and (14), we have

$$g(X \times U) = gU \times gX, \tag{39}$$

and hence we obtain respectively from (37), (39) and (38):

$$g(q \times \nabla_X U) = g(U \times X) = gX \times gU = g\nabla_X U \times gq. \tag{40}$$

Therefore, we have proved that

$$\phi(\nabla_X U) = -\hat{\phi}(\nabla_X U).$$

However, $\nabla_X U \in V$ since in accordance with (37) this vector is orthogonal to W. This contradicts the fact that ϕ and $\hat{\phi}$ coincide on V. □

Corollary 2. *Let M be a non totally umbilic hypersurface of the 6-sphere whose second fundamental form has rank greater or equal to 3. Let $f : M \to S^6$ be an isometric immersion of M. Then f maps the Hopf vector field of M to the Hopf vector field of $f(M)$ if and only if there exists an element $g \in G_2$ such that f is the restriction of g to the hypersurface M.*

Proof: Indeed, from the classical rigidity for hypersurfaces of real space forms mentioned in the introduction of this chapter we have that the map f can be extended to an isometry of S^6 and hence the corollary follows from the previous theorem. □

A vector field X in a Riemannian manifold $(M, <>, \nabla)$ is a Killing field when its flow is locally an isometry, or equivalently if X satisfies the so called **Killing equation**:

$$\langle \nabla_Y X, Z\rangle = -\langle \nabla_Z X, Y\rangle. \tag{41}$$

Lemma 2. *If M is a Riemannian manifold which admits a Killing field X of constant length, then the integral curves of X are geodesics.*

Proof: From the Killing equation (41) we have for every $Y \in \mathfrak{X}(M)$ that $\langle \nabla_X X, Y \rangle = -\langle \nabla_Y X, X \rangle = 0$. Thus, $\nabla_X X = 0$. □

Theorem 2. *The geodesic hyperspheres are the only connected hypersurfaces of S^6 whose Hopf vector fields are Killing fields.*

Proof: $(\Longrightarrow)$
Let us first consider a great hypersphere. There is no loss of generality if we choose $M = V \cap S^6$ where $V = e_4{}^{\perp}$ because this hypersphere can be mapped to any other one via an element $g \in G_2$ and this transformation will certainly map the Killing Hopf vector field of M to the Killing Hopf vector field of $g(M)$. In this case, the unit normal vector field $\xi = e_4$ to M is constant and the Hopf vector field at a point $q \in M$ is just $U_q = e_4 \times q$. Thus

$$\langle \nabla_X U, Y \rangle = \langle \nabla_X (e_4 \times q), Y \rangle = \langle e_4 \times X, Y \rangle. \tag{42}$$

Therefore, using equation (41) and the fact that the product $\langle X \times Y, Z \rangle$ is skew-symmetric we conclude that U is a Killing field.

Now, let M be the small hypersphere of S^6 centred at the point $p = e_4$. This hypersurface is just a degenerate tube of radius r around the degenerate curve $S = \{p\}$. However, we note that all of our calculations to determine the flow $\mathbb{C}F_t$ of the Hopf vector field remain valid in this situation.

In order to prove that U is a Killing field, we start by assuming this to be true and out of that assumption we deduce the natural candidate for the local isometry $\mathbb{C}F_t$ which describes the flow of U.

As the rank of the second fundamental form of M is 5, we have by the rigidity of the hypersurfaces of spheres that $\mathbb{C}F_t$ can be extended yielding a 1-parameter subgroup of $SO(7)$ which we still denote by $\mathbb{C}F_t$. From linearity of $\mathbb{C}F_t$ and (23) we obtain

$$\cos(r)\mathbb{C}F_t e_4 + \sin(r)\mathbb{C}F_t \eta = \cos(r)e_4 + \sin(r)\delta(t). \tag{43}$$

Each $\mathbb{C}F_t$ must map the focal set of M to itself and since the focal set of M is just $\{e_4\}$, we have $\mathbb{C}F_t e_4 = e_4$. Thus (43) can be simplified to

$$\mathbb{C}F_t \eta = \cos(\bar{t})\eta + \sin(\bar{t})(e_4 \times \eta) \text{ for every } \eta \in e_4^{\perp}. \tag{44}$$

It is immediate to verify that the map $\mathbb{C}F_t$ defined as above is indeed an element of $SO(7)$. Moreover, it is worth mentioning that $\mathbb{C}F_t$ lies in G_2 only for the values $t = 0$ and $t = \pi \sin(r)$.

In the following, we determine the action of $\mathbb{C}F_t$ on an integral curve σ of U in order to check that $\mathbb{C}F_t$ corresponds, indeed, to the flow of U.

$$\mathbb{C}F_t\sigma(0) = \cos(r)e_4 + \sin(r)\mathbb{C}F_t\eta = \cos(r)e_4 + \sin(r)\delta(t) = \sigma(t).$$

$(\Longleftarrow)$

Let M be a connected hypersurface of S^6 with unit normal field ξ and assume that its Hopf vector field U is a Killing field. It follows from Lemma 2 that M is a Hopf hypersurface, say that M is a subset of a tube $\Phi_r(\perp^1 S)$ where S is an almost complex curve of S^6. We assume that the Hopf principal curvature $\alpha = -\cot(r)$ is not zero, that is, $r \neq \frac{\pi}{2}$. Then we know from [3] that the second fundamental form of M has rank at least 3 everywhere because the α-eigenspace of M has dimension at least 3.

From the well known rigidity of hypersurfaces of a real space form [8] we have that under the assumption that the second fundamental form having rank at least 3 everywhere, any isometry between hypersurfaces of a sphere is extendable to an ambient isometry. Therefore, the flow $\mathbb{C}F_t$ of the vector field U can be realised as the restriction to the hypersurface of an isometry of S^6, which we still name as $\mathbb{C}F_t$.

Now, we prove that the almost complex curve is a connected component of the fixed point set of each isometry $\mathbb{C}F_t$. Geometrically, this is almost evident for since $\mathbb{C}F_t$ is the flow of the Hopf vector field, we can expect that the action of the isometry $\mathbb{C}F_t$ on M, and similarly on each tubular hypersurface around S, is just to turn it around the curve S. We call attention to the rather subtle fact that the proof we give in the sequel does rely only upon the formula obtained in (23) for the flow of the Hopf vector field and this formula in turn depends only on the fact that the focal map of a Hopf hypersurface is constant along the integral curves of the Hopf vector field.

S is connected because it is the image of the connected tubular hypersurface M under the focal map, which is a continuous map.

Since $\mathbb{C}F_t$ is an isometry of S^6 and maps open subsets of M to open subsets of M then $\mathbb{C}F_t$ also maps open subsets of the focal set S to open subsets of S.

On the other hand, by construction, the map $\mathbb{C}F_t$ maps an integral curve σ of the Hopf vector field of M to itself. Thus it follows from (23) that $\mathbb{C}F_t$ must fix the point $p \in S$ which corresponds to the integral curve σ. Therefore, $\mathbb{C}F_t$ fixes every point of S.

Moreover, it follows from the fact that S is the set fixed by $\mathbb{C}F_t$, the linearity of $\mathbb{C}F_t$ and (23) that for $s \in (0, \frac{\pi}{2})$ we also have $\mathbb{C}F_t(\gamma_{(p,\eta)}(s)) = \gamma_{(p,\delta)}(s)$, that is the isometries $\mathbb{C}F_t$ perform a non-trivial rotation of each tube of constant radius s around the curve S.

Using that $\mathbb{C}F_t$ is linear and fixes S, from (23) we obtain

$$\mathbb{C}F_t\eta = \cos(\bar{t})\eta - \sin(\bar{t})(p \times \eta) \text{ for every } (p, \eta) \in \perp^1 S. \tag{45}$$

We want to prove that the almost complex curve S is totally geodesic and we already know that S is a connected component of the fixed point set of the isometry $\mathbb{C}F = \mathbb{C}F_{\bar{t}}$. In order to do that we consider a conjugation $\widetilde{\mathbb{C}F} = g^{-1}\mathbb{C}Fg$ by an element $g \in SO(7)$ such that $\widetilde{\mathbb{C}F}$ be an element of the standard maximal torus of $SO(7)$. Then g maps the fixed point set of $\mathbb{C}F$ exactly to the fixed point set of $\widetilde{\mathbb{C}F}$ and hence $\widetilde{S} = g(S)$ is also a connected component of $\widetilde{\mathbb{C}F}$.

Therefore, we can describe $\widetilde{\mathbb{C}F}$ by $\widetilde{\mathbb{C}F}X = AX$ where A is the matrix

$$A = \begin{pmatrix} R_0 & 0 & 0 & 0 \\ 0 & R_1 & 0 & 0 \\ 0 & 0 & R_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad R_j = \begin{pmatrix} \cos\theta_j(t) & \sin\theta_j(t) \\ -\sin\theta_j(t) & \cos\theta_j(t) \end{pmatrix}$$

Since $\widetilde{\mathbb{C}F}$ fixes any point $p = (p_0, ..., p_6) \in \widetilde{S}$ then our matricial representation for $\widetilde{\mathbb{C}F}$ yields for each $j \in \{0, 1, 2\}$ a homogeneous system as follows

$$\begin{cases} (\cos(\theta_j) - 1)p_{2j} + \sin(\theta_j)p_{2j+1} = 0 \\ -\sin(\theta_j)p_{2j} + (\cos(\theta_j) - 1)p_{2j+1} = 0 \end{cases}$$

This homogeneous system must hold for every real value t and every point $p \in \widetilde{S}$, thus its discriminant $\Delta_j = 2(1 - \cos\theta_j)$ vanishes if and only if the function $\cos\theta_j(t)$ is identically equal to 1 so that for at least one value $j \in \{0, 1, 2\}$, say $j = 0$, we must have $\Delta_0 \neq 0$ otherwise $\widetilde{\mathbb{C}F}$ would be the Identity map. Therefore, the two first coordinates of any point of $\widetilde{S}$ vanish.

By using these systems, we can also conclude that $R_j = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if and only if for some point $p \in \widetilde{S}$ we have $p_{2j} \neq 0$ or $p_{2j+1} \neq 0$. Consequently, there are only three possibilities for the isometry $\widetilde{\mathbb{C}F}$ and the corresponding fixed point sets in each case are given by $V_1 = S^6 \cap span\{e_3, .., e_7\}$, $V_2 = S^6 \cap span\{e_5, e_6, e_7\}$ and $V_3 = S^6 \cap span\{e_3, e_4, e_7\}$.

Since in all these possibilities the set V would be connected, we should have $\widetilde{S} = V$. However, in the first case $dim V = 4$. Since the other two possibilities give $\widetilde{S}$ totally geodesic and g is an isometry then S is also totally geodesic.

The proof of the theorem comes from the fact that a totally geodesic almost complex curve of S^6 is given by $S = V^3 \cap S^6$ where V^3 is spanned by vectors $\{v_1, v_2, v_3\}$ of a G_2-basis $\{v_1, ..., v_7\}$. Indeed, this gives us an obvious contradiction in the equation (45). since for $\eta = v_4$ and $p \in \{v_1, v_2\}$ we have

$$v_5 = v_1 \times v_4 = v_2 \times v_4 = v_6.$$

□

In 1973, Takagi ([10]) gave a rigidity theorem for hypersurfaces of the complex projective spaces which was equivalent to that well known rigidity theorem for hypersurfaces of a real space form, namely, he proved:

Theorem 3. *Let M be a hypersurface of $\mathbb{C}P^n$ whose second fundamental form A has rank at least 3 everywhere. Let f denote an isometric immersion of M into $\mathbb{C}P^n$ ($n \geq 3$). Then,*

(i) $\phi = \hat{\phi}$ *if and only if* $A = \hat{A}$,

(ii) *If $A = \hat{A}$ then there exists a holomorphic isometry F of $\mathbb{C}P^n$ such that $F|_M = f$.*

We call attention here to the fact that the second part of the theorem above, as observed by Takagi, can be proved by following the same method used to deal with rigidity of hypersurfaces in real space forms.

Recently, Takagi et al ([5]) have improved this result by showing that the rigidity of hypersurfaces in $\mathbb{C}P^n$ depends in general only on the invariance of the Hopf vector field, that is $U = \hat{U}$. More precisely they have shown:

Theorem 4. *Let M be a hypersurface of $\mathbb{C}P^n$ whose second fundamental form A has rank at least 3 everywhere. Let f denote an isometric immersion of M into $\mathbb{C}P^n$ ($n \geq 3$). If $U = \hat{U}$ then f is a restriction of a holomorphic isometry of $\mathbb{C}P^n$.*

In this section, we shall give a new proof for this result, using the same method as in the case of hypersurfaces of S^6. It turns out that the approach we give here makes the proof clearer, simpler and more geometrical.

Consider the complex projective space $(\mathbb{C}P^n, J, <>, \overline{\nabla}, \overline{R})$ endowed with the Fubini-Study metric of constant holomorphic sectional curvature 4. Then its curvature tensor $\overline{R}$ is given by

$$\begin{aligned}\overline{R}(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y - \langle Y,JZ\rangle JX \\ + \langle X,JZ\rangle JY + 2\langle X,JY\rangle JZ. \end{aligned} \tag{46}$$

Let M be a hypersurface of $\mathbb{C}P^n$ with second fundamental form h and induced structures $<>, \nabla, R, etc.$ Let ξ be a unit normal vector field on M. The Gauss and Codazzi equations for M are respectively:

$$\begin{aligned}\langle \overline{R}(X,Y)Z, W\rangle &= \langle R(X,Y)Z,W\rangle + \langle AX,Z\rangle\langle AY,W\rangle \\ &\quad -\langle AX,W\rangle\langle AY,Z\rangle \end{aligned} \tag{47}$$

$$\langle \overline{R}(X,Y)Z,\xi\rangle = \langle(\tilde{\nabla}_X h)(Y,Z),\xi\rangle - \langle(\tilde{\nabla}_Y h)(X,Z),\xi\rangle, \tag{48}$$

where the covariant derivative of the tensor h is given by

$$(\tilde{\nabla}_X h)(Y,Z) := \overline{\nabla}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

In terms of the shape operator A of M, we can also write (48) as

$$\begin{aligned}\langle \overline{R}(X,Y)Z,\xi\rangle =&\langle AX,\nabla_Y Z\rangle - \langle AY,\nabla_X Z\rangle + \langle AZ,\nabla_Y X\rangle - \\ &- \langle AZ,\nabla_X Y\rangle + X\langle AY,Z\rangle - Y\langle AX,Z\rangle.\end{aligned} \tag{49}$$

Thus, using (46), we have that for every hypersurface M of $\mathbb{C}P^n$, the Gauss and Codazzi equations are simplified to

$$\begin{aligned} R(X,Y)Z &= \langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle \phi Y,Z\rangle \phi X - \langle \phi X,Z\rangle \phi Y \\ &\quad -2\langle \phi X,Y\rangle \phi Z + \langle AY,Z\rangle AX - \langle AX,Z\rangle AY, \quad (50)\\ (\nabla_X A)Y - (\nabla_Y A)X &= 2\langle \phi X,Y\rangle U + \langle Y,U\rangle \phi X - \langle X,U\rangle \phi Y. \quad (51)\end{aligned}$$

The rate of change of the induced vector fields U and $\hat{U}$ (if we are considering an isometric immersion $g : M \longrightarrow \mathbb{C}P^n$) is given by

$$\nabla_X U = -\phi AX \tag{52}$$

$$\nabla_X \hat{U} = -\hat{\phi}\hat{A}X \tag{53}$$

This follows immediately from the Kähler condition $\overline{\nabla}_X(JY) = J(\overline{\nabla}_X Y)$ on $\mathbb{C}P^n$.

Theorem 5. *Let M be a hypersurface of $\mathbb{C}P^n$ whose second fundamental form has rank at least 3 everywhere and let g be an isometric immersion of M into $\mathbb{C}P^n$. If g maps the Hopf vector field of M to the Hopf vector field of $g(M)$, that is $U = \hat{U}$, then g is the restriction of a holomorphic isometry of $\mathbb{C}P^n$.*

Proof: Since $U = \hat{U}$, it follows from (52) and (53) that:

$$\phi AX = \hat{\phi}\hat{A}X \text{ for every } X \in \mathfrak{X}(M). \tag{54}$$

As g preserves the curvature, that is $R = \hat{R}$, we obtain from (46) and (49) that

$$\begin{aligned}&\langle X,\phi Z\rangle \phi Y + 2\langle X,\phi Y\rangle \phi Z - \langle Y,\phi Z\rangle \phi X - \langle AX,Z\rangle AY + \langle AY,Z\rangle AX = \\ &\langle X,\hat{\phi} Z\rangle \hat{\phi} Y + 2\langle X,\hat{\phi} Y\rangle \hat{\phi} Z - \langle Y,\hat{\phi} Z\rangle \hat{\phi} X - \langle \hat{A}X,Z\rangle \hat{A}Y + \langle \hat{A}Y,Z\rangle \hat{A}X.\end{aligned} \tag{55}$$

Specializing this equation for $Z = U$ we get:

$$\langle X,AU\rangle AY - \langle Y,AU\rangle AX = \langle X,\hat{A}U\rangle \hat{A}Y - \langle Y,\hat{A}U\rangle \hat{A}X. \tag{56}$$

Specializing again this equation for $Y = U$ we have for every $X \in \mathfrak{X}(M)$:

$$\langle X, AU\rangle AU - \langle U, AU\rangle AX = \langle X, \hat{A}U\rangle \hat{A}U - \langle U, \hat{A}U\rangle \hat{A}X. \tag{57}$$

If W denotes the orthogonal complement of the vector space $span\{AU, \hat{A}U\}$ then for any $Y \in \mathfrak{X}(M)$ and $X \in W$, the equations (56) and (57) give respectively:

$$\langle Y, AU\rangle AX = \langle Y, \hat{A}U\rangle \hat{A}X, \tag{58}$$

$$\langle U, AU\rangle AX = \langle U, \hat{A}U\rangle \hat{A}X. \tag{59}$$

Taking $Y = AU$ and $Y = \hat{A}U$ in (58) we have for every $X \in W$ respectively:

$$|AU|^2 AX = \langle \hat{A}U, AU\rangle \hat{A}X, \tag{60}$$

$$\langle \hat{A}U, AU\rangle AX = |\hat{A}U|^2 \hat{A}X. \tag{61}$$

Now we shall split our proof into two cases.

Case 1: $AU \neq 0$.

Since rank$A$ is at least 3, there exists a vector $X \in W$ such that $AX \neq 0$, so from (60) and (61) we have $\hat{A}X \neq 0$, $\hat{A}U \neq 0$ and $\langle \hat{A}U, AU\rangle \neq 0$, moreover by taking the quotient between those equations we get $|\langle \hat{A}U, AU\rangle| = |\hat{A}U||AU|$ and hence

$$\hat{A}U = \delta AU,$$

where $\delta = \pm\frac{|\hat{A}U|}{|AU|}$. Using this in (60), it follows that $AX = \delta\hat{A}X$ for every $X \in W$. However, from (54) and (7) we also have $|AX| = |\hat{A}X|$ for every $X \in W$ and so $\delta = \pm 1$. Choosing if necessary, the oposite normal vector field on $g(M)$, we can assume $\delta = 1$. Thus,

$$AX = \hat{A}X, \tag{62}$$

$$AU = \hat{A}U. \tag{63}$$

If $\langle AU, U\rangle \neq 0$ then substituting (63) in (57) we obtain $A = \hat{A}$.

If $\langle AU, U\rangle = 0$ then from (10) we can choose a vector $X \in U^{\perp}$ such that $\phi X = AU$ and so

$$X = -\phi AU = -\hat{\phi} AU,$$

$$A(AU) = A(\phi X) = \hat{A}\hat{\phi}X = -\hat{A}\hat{\phi}^2(AU) = \hat{A}(AU).$$

This together with (62) implies $A = \hat{A}$, which reduces (55) to

$$\langle X, \phi Y\rangle \phi Y = \langle X, \hat{\phi}Y\rangle \hat{\phi}Y.$$

therefore, for every $X \in U^{\perp}$we have $\phi X = \pm\hat{\phi}X$.

Because $Ker(\phi) = Ker(\hat{\phi}) = span\{U\}$, we must have $\phi = \pm\hat{\phi}$. But we know that $\phi AX = \hat{\phi}AX$ and hence $\phi = \hat{\phi}$.

Case 2: $AU = 0$.

In this case the Codazzi equation (49) for the hypersurfaces M and $g(M)$ are written respectively as:

$$\langle \overline{R}(X, Y)U, \xi\rangle = 2\langle \phi AX, AY\rangle \tag{64}$$

$$\langle \overline{R}(g_*X, g_*Y)g_*U, \tilde{\xi}\rangle = 2\langle \hat{\phi}\hat{A}X, \hat{A}Y\rangle \tag{65}$$

On the other hand, using the curvature tensor of $\mathbb{C}P^n$ as given in (46), we have for every $X, Y \in U^\perp$:

$$\overline{R}(X, Y)U = 2\langle \phi X, Y\rangle \xi$$
$$\overline{R}(g_*X, g_*Y)g_*U = 2\langle \hat{\phi}X, Y\rangle \tilde{\xi}.$$

thus

$$\langle \phi X, Y\rangle = \langle \phi AX, AY\rangle$$
$$\langle \hat{\phi}X, Y\rangle = \langle \hat{\phi}\hat{A}X, \hat{A}Y\rangle. \tag{66}$$

In other words, recalling that $\phi A = \hat{\phi}\hat{A}$, we have

$$A\phi A = \phi \tag{67}$$

$$A\phi\hat{A} = \hat{\phi} \tag{68}$$

Now, taking $Z = Y$ in (55) we have

$$\begin{aligned} &3\langle X, \phi Y\rangle\phi Y - \langle AX, Y\rangle AY + \langle AY, Y\rangle AX = \\ &3\langle X, \hat{\phi}Y\rangle\hat{\phi}Y - \langle \hat{A}X, Y\rangle\hat{A}Y + \langle \hat{A}Y, Y\rangle\hat{A}X \end{aligned} \tag{69}$$

Putting $Y = -\phi AX$ in this equation and using that

$$\phi Y = AX \qquad \text{and} \qquad AY = -A\phi AX = -\phi X,$$

we obtain for every $X \in U^\perp$:

$$< X, AX > AX =< X, \hat{A}X > \hat{A}X.$$

However,

$$|AX| = |\phi AX| = |\hat{\phi}\hat{A}X| = |\hat{A}X|$$

and hence $AX = \pm\hat{A}X$. From (67) we see that the restriction $A : U^\perp \longrightarrow U^\perp$is non-singular so that $Ker(A) = span\{U\}$ and hence $A = \pm\hat{A}$ on $U^\perp$. Choosing an apropriate normal vector field, if necessary, we can assume $A = \hat{A}$. Therefore, from (67) and (68) we have $\phi = \hat{\phi}$.

Therefore, the proof of the theorem follows from Theorem (3). □

We shall make use of Theorem (5) on holomorphic congruence for hypersurfaces in $\mathbb{C}P^n$, to prove that the hypersurfaces of $\mathbb{C}P^n$ which have a Killing Hopf vector field are exactly the open subsets of tubes around totally geodesic complex submanifolds. This result has already been proved by Berndt [2] but we give here a simpler and more geometrical proof.

We will think of S^{2n+1} as naturally included in $\mathbb{C}^{n+1}$ so that the Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P^n$ is a Riemannian submersion with linear isomorphism $\pi_\star : H_z \to T_{\pi(z)}\mathbb{C}P^n$ for each $z \in S^{2n+1}$, where H_z denotes the tangent subspace $\{\mathbb{R}z\}^\perp \cap \{\mathbb{R}iz\}^\perp$ of $T_z S^{2n+1}$. The natural complex structure on H_z, given by the complex multiplication by i, induces via $\pi_\star$ the standard complex structure J on $\mathbb{C}P^n$.

The Hopf fibration can be used to describe the geodesics of $\mathbb{C}P^n$ as projection of horizontal geodesics of S^{2n+1}, in other words, given $\zeta \in T_p(\mathbb{C}P^n)$, as π is a Riemannian submersion, then the geodesic $\gamma_{(p,\zeta)}(s)$ of $\mathbb{C}P^n$ is the projection of the horizontal geodesic

$$\tilde{\gamma}_{(\tilde{p},\tilde{\zeta})}(s) = \cos(s)\tilde{p} + \sin(s)\tilde{\zeta}, \tag{70}$$

where the tilde notation is used here to denote corresponding points and horizontal vectors under the maps π and $\pi_\star$ respectively, that is

$$\gamma_{(p,\zeta)}(s) = \pi(\tilde{\gamma}_{(\tilde{p},\tilde{\zeta})}(s)), \qquad \text{with} \qquad \begin{cases} \pi(\tilde{p}) & = p \\ \pi_\star(\tilde{\zeta}) & = \zeta. \end{cases} \tag{71}$$

Let M be a Hopf hypersurface of $\mathbb{C}P^n$ and as usual let ξ denote a local normal field on M.

By using similar procedure here, as in the case of the 6-sphere, we can also describe the integral curves of the Hopf vector field explicitly.

However, we shall recall first the fundamental calculation done by Cecil-Ryan ([4]) for the derivative of the normal exponential map G of M. They have shown that given $q \in M$ and a vector $X \in T_q M$, if we denote by $\tilde{X}$ its horizontal lift to H_z where z is a point in the fibre $\pi^{-1}(q)$, then

$$G_* |_{(q,r\xi)} (X) = d\pi_w\{\cos(r)\tilde{X} - \sin(r)(\tilde{Y} - \langle \tilde{X}, i\tilde{\xi}\rangle iz)\}, \tag{72}$$

Where $\xi = d\pi_z(\tilde{\xi})$, $\quad w = \cos(r)z + \sin(r)\tilde{\xi} \in \pi^{-1}(F(q, xi))$, $\quad Y = A_\xi X$, and the vector on the right hand side belongs to $T_z S^{2n+1}$ but not necessarily to H_z.

We call attention to the difference between our notation for the points z and w and that of Cecil-Ryan, which unfortunately is swapped.

Using (72), Cecil-Ryan located the focal points of a Hopf hypersurface of $\mathbb{C}P^n$, as we summarize in the following

Lemma 3. *Let M be a Hopf hypersurface of $\mathbb{C}P^n$. If $U = J\xi$ denotes the Hopf vector field of M and $\alpha = -2\cot(2r)$ is the Hopf principal curvature of M, then given $q \in M$*

(i) $G_* \,|_{(q,r\xi)}\, (U) = 0$

(ii) $G_* \,|_{(q,r\xi)}\, (X) = 0$ *whenever $X \in T_qM$ is a principal vector of (M, ξ) corresponding to the principal value* $-\cot(r)$.

(iii) $G_* \,|_{(q,r\xi)}\, (X) \neq 0$ *otherwise.*

Now, to determine the integral curve σ of U through a given point $q \in M$, we first note from the lemma above that the focal map of M is constant along the integral curves of the Hopf vector field, that is, $G(\sigma, \xi) \equiv p$.

Next, we consider a geodesic $\gamma = \gamma_{(p,\eta)}$ of $\mathbb{C}P^n$ normal to M at q and connecting the points q and p, where η denotes the tangent vector to γ at the point p. We shall assume γ to be parametrized by the arclength s from p to q and so $\gamma(0) = p$ and $\gamma(r) = q$.

Let $\tilde{\sigma}$ be the curve in S^{2n+1} obtained as the end points of the geodesics $\widetilde{\gamma}_{(\tilde{p},\tilde{\delta})}$ where

$$\tilde{\delta} = \tilde{\delta}(t) = \cos(\bar{t})\tilde{\eta} + i\sin(\bar{t})\tilde{\eta} \qquad \text{and} \qquad \bar{t} = \frac{t}{\sin(r)\cos(r)}\ ,$$

in other words, $\tilde{\sigma}(t) = \widetilde{\gamma}_{(\tilde{p},\tilde{\delta}(t))}(r)$. Let us define the vector $\delta = \pi_*\tilde{\delta}$ and the curve $\sigma(t) = \widetilde{\gamma}_{(p,\delta(t))}(r)$. Then the following calculations show that σ is indeed the integral curve of U through q.

In equations (73) and (74) below we must consider carefully along the curve $\tilde{\sigma}$ only the horizontal components of the vectors A and B, that is, their projections on $H_{\tilde{\sigma}}$.

$$\sigma' = (\pi_\star|_{\tilde{\sigma}})(\tilde{\sigma}') = (\pi_\star|_{\tilde{\sigma}})(\sin(r)\tilde{\delta}') = (\pi_\star|_{\tilde{\sigma}})(A). \tag{73}$$

Where $A = \frac{1}{\cos(r)}\{\cos(\bar{t})i\tilde{\eta} - \sin(\bar{t})\tilde{\eta}\}$.

$$\begin{aligned} U(\sigma) &= J\dot{\gamma}_{(p,\delta)}(r) = J\pi_\star|_{\tilde{\sigma}}\ (\dot{\tilde{\gamma}}_{(\tilde{p},\tilde{\delta})}(r)) \\ &= \pi_\star|_{\tilde{\sigma}}\ (-\sin(r)i\tilde{p} + \cos(r)i\tilde{\delta}) = \pi_\star|_{\tilde{\sigma}}\ (B). \end{aligned} \tag{74}$$

Where $B = (-\sin(r)i\tilde{p} + \cos(r)\cos(\bar{t})i\tilde{\eta} - \cos(r)\sin(\bar{t})\tilde{\eta})$. Now, observing that

$$\langle B, \tilde{\sigma}\rangle = \langle B, i\tilde{\sigma}\rangle = 0 = \langle A, \tilde{\sigma}\rangle \quad \text{and} \quad \langle A, i\tilde{\sigma}\rangle = \tan(r),$$

we can see that the projections of the vectors A and B on the space $H_{\tilde{\sigma}}$ are exactly the same and hence $\sigma' = U(\sigma)$. Therefore, using all the notation above we have

Proposition 4. *The flow of the Hopf vector field of a Hopf hypersurface of* $\mathbb{C}P^n$ *can be described as:*

$$\mathbb{C}F_t(\gamma_{(p,\eta)}(r)) = \gamma_{(p,\delta)}(r) = \pi(\cos(r)\tilde{p} + \sin(r)\tilde{\delta}). \tag{75}$$

We give now an application of the holomorphic congruence of hypersurfaces in $\mathbb{C}P^n$ discussed above. The following result shows that in CP^n there is a broader category of hypersurfaces whose Hopf vector fields are also Killing.

Theorem 6. *Let M be a connected real hypersurface of $\mathbb{C}P^n$. Then the Hopf vector field U of M is a Killing vector field if and only if M lies on a tube of constant radius around a totally geodesic complex submanifold.*

Proof: Let us assume first that U is a Killing vector field. This implies, using Lemma 2, that M is a Hopf hypersurface. Let $\mathbb{C}F_t$ denote the flow of U on an open subset of M. It follows from Theorem (5) that for each t the map $\mathbb{C}F_t$ can be extended to a holomorphic isometry of $\mathbb{C}P^n$ which we shall also name as $\mathbb{C}F_t$. Thus, we obtain a 1-parameter subgroup B_t of $SU(n+1)$ such that

$$\mathbb{C}F_t(\pi(z)) = \pi(B_t(z)). \tag{76}$$

As in the case of S^6 (cf. Theorem (2)), we can also show here that the focal set N of M is a connected component of the fixed point set of $\mathbb{C}F_t$ and the proof is exactly the same as in that proposition since, as we mentioned above, Lemma (3) shows that the focal map is constant along the integral curves of U.

On the other hand, it follows from (76) that the inverse image $\widetilde{N} = \pi^{-1}(N)$ consists only of points in $\mathbb{C}^{n+1}$ which are eigenvectors for the linear operator B_t. Therefore, $\widetilde{N}$ is a disjoint union of eigenspaces of B_t, say

$$\widetilde{N} = V_{\lambda_1} \cup \ldots \cup V_{\lambda_k}. \tag{77}$$

However, if $\widetilde{N}$ is not just a single eigenspace V_λ, we would have a contradiction to the connectivity of N since in this case we would write N as a disjoint union of closed sets

$$N = \pi(V_{\lambda_1}) \cup \ldots \cup \pi(V_{\lambda_k}).$$

Therefore, the focal set of M is the totally geodesic complex submanifold of $\mathbb{C}P^n$ given by the projectivisation of the complex linear subspace $\widetilde{N} = V_\lambda$.

Conversely, let M be an open subset of a tube $\Phi_r(N)$ of radius r around a totally geodesic complex submanifold. If we make use here of some properties of the Hopf hypersurfaces of $\mathbb{C}P^n$, then we can give a short proof of the fact that U is a Killing vector field. In accordance with the calculations of the eigenvectors of a tubular hypersurface (see [7] for details), the only possible eigenvalues for M are $\lambda_0 = \alpha = -2\cot(2r)$, $\lambda_1 = -\cot(r)$ and $\lambda_2 = \tan(r)$. Using this we can verify that U satisfies (41) as follows.

Since U has unit length and $\nabla_U U = 0$, we just need to verify (41) for vectors Y and Z orthogonal to U. Moreover, because of the linearity of $\langle \nabla_Y U, Z\rangle$ with respect to these variables, we just need to prove that equation for any pair of eigenvectors Y and Z. Thus, we have a few cases to consider.

If Y and Z lie in the same eigenspace V_λ then it follows from (52) and (8) that

$$\langle \nabla_Y U, Z\rangle = -\langle \phi AY, Z\rangle = -\lambda\langle \phi Y, Z\rangle = \lambda\langle Y, \phi Z\rangle = -\langle Y, \nabla_Z U\rangle.$$

For the other possibility we use the fact proved by Maeda ([6]) that the eigenspaces $V_1 = V_{\lambda_1}$ and $V_2 = V_{\lambda_2}$ are invariant under the operator ϕ. Thus, for each $i \in \{0, 1, 2\}$, the space ϕV_i is orthogonal to the spaces $\{V_j\}_{j\neq i}$. Therefore, given $i \in \{1, 2\}$, $Y \in V_i$ and $Z \in V_j$ with $i \neq j$, we have

$$\langle \nabla_Y U, Z\rangle = -\lambda_i\langle \phi Y, Z\rangle = 0 = \lambda_j\langle Y, \phi Z\rangle = -\langle Y, \nabla_Z U\rangle. \qquad \square$$

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