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An inequality of Clifford indices for a finite covering of curves

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Abstract. We prove that for a finite covering of curves the Clifford index of the source is at least that of the target.

Keywords: Clifford index, finite covering of curves.

Introduction

The *Clifford index* c_X of a smooth curve X is, by definition, the smallest possible value of the expression

$$\deg D - 2h^0(X, \mathcal{O}_X(D)) + 2$$

for a divisor D with $h^0(X, \mathcal{O}_X(D)) \geq 2$ and $h^1(X, \mathcal{O}_X(D)) \geq 2$. The notion was introduced by H. H. Martens [M], and has been studied by a number of authors from various points of view.

Let $f: X \to Y$ be a finite covering of smooth curves over an algebraically closed field. It seems natural to expect that $c_X \geq c_Y$. In this note, we prove it. Our proof is based on a result of Coppens and Martens [CM, Cor. 3.2.5], where the ground field is the complex numbers. So our proof works only in characteristic 0.

The Clifford index makes sense only when X is of genus g_X at least 4 or is hyperelliptic with $g_X = 3$. When Y is hyperelliptic, the inequality $c_X \geq c_Y$ is clear because $c_Y = 0$. So we may assume $g_X \geq 4$ and $g_Y \geq 4$.

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Theorem. *For a finite covering $f: X \to Y$ of smooth curves whose genera are at least 4 over an algebraically closed field of characteristic 0, we have $c_X \geq c_Y$.*

Proof. Choose a linear system g_d^r on X which computes the Clifford index c_X of X. Hence $d = c_X + 2r$. For a canonical divisor K, the linear system $|K - g_d^r|$ also computes the Clifford index of X. Hence we may assume that $d \leq g_X - 1$. Let us consider the complete linear system $|f_* D|$ for $D \in g_d^r$. Since $\dim |f_* D| \geq r$ and $\deg |f_* D| = d$, the proof is done if $h^1(Y, \mathcal{O}_Y(f_* D)) \geq 2$. When X is either hyperelliptic or trigonal, the inequality $h^1(Y, \mathcal{O}_Y(f_* D)) \geq 2$ holds by the Riemann–Roch theorem because the genus g_Y of Y is at least 4.

Now we consider the case when X is bi-elliptic, whose Clifford index is 2 and computed by a pencil g_4^1. If $g_Y \geq 5$, then we have $h^1(Y, \mathcal{O}_Y(f_* D)) \geq 2$, and get the inequality $c_X \geq c_Y$. If $g_Y = 4$, then Y is trigonal, and so $c_Y = 1$, which means that the inequality $c_X \geq c_Y$ is true.

Next we handle the case where X is a plane quintic curve, which is the only remaining case for $c_X \leq 1$. (For the classification of curves X with $c_X = 1$, see [M, (2.51)].) Since X is of genus 6, we have $g_Y \leq 3$ by the Hurwitz formula, which is out of our consideration.

Thus we may assume that $c_X \geq 2$ and X is not bi-elliptic. First we assume that $g_X > 2c_X + 5$, where g_X is the genus of X. Then by [CM, Cor. 3.2.5], we have

$$d \leq 3c_X/2 + 3. \tag{1}$$

Suppose that $h^1(Y, \mathcal{O}_Y(f_* D)) \leq 1$. Then we have

$$\dim |f_* D| \leq d - g_Y + 1$$

by the Riemann–Roch theorem. Hence we have

$$g_Y - 1 \leq (c_X + d)/2 \tag{2}$$

because $\dim |f_* D| \geq r$ and $d = c_X + 2r$. Recall that $c_Y \leq (g_Y - 1)/2$ by the existence theorem of Brill–Noether theory (see for example [ACGH, p. 206]). Therefore,

$$\begin{aligned} c_Y \leq (g_Y - 1)/2 &\leq (c_X + d)/4 && \text{by (2)} \\ &\leq 5c_X/8 + 3/4 && \text{by (1)} \\ &\leq c_X && \text{because } c_X \geq 2. \end{aligned}$$

Finally we consider the remaining case, that is, $g_X \leq 2c_X + 5$. By the Hurwitz formula, we have $g_X - 1 \geq 2(g_Y - 1)$. So we have

$$\begin{aligned} c_Y \leq (g_Y - 1)/2 &\leq (g_X - 1)/4 & \\ &\leq c_X/2 + 1 & \text{by our assumption} \\ &\leq c_X & \text{because } c_X \geq 2. \end{aligned}$$

The proof is now complete.

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