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The Weierstrass Semigroup of a pair and moduli in $\mathcal{M}_3$

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Abstract. We classify all the Weierstrass semigroups of a pair of points on a curve of genus 3, by using its canonical model in the plane. Moreover, we count the dimension of the moduli of curves which have a pair of points with a specified Weierstrass semigroup.

Keywords: Weierstrass semigroup of a pair, plane quartic curve, moduli of curves of genus 3.

Mathematical subject classification: Primary 14H55; Secondary 14H10, 14H45, 14H50.

1. Introduction

The theory of the Weierstrass semigroup of a pair of points on a curve was initiated by Arbarello, Cornalba, Griffiths and Harris [1, VIII Exercises B, p. 365], and it has been pushed forward by Kim [3] and Homma [2].

For us, a *curve* is always complete, non-singular and defined over an algebraically closed field $\mathbf{K}$ of characteristic zero. Let C be a curve of genus greater than one and $\mathbf{K}(C)$ be the field of rational functions on C. Let P and Q be distinct points of C. We define the Weierstrass semigroup $H(P, Q)$ of the pair (P, Q) by

$$H(P, Q) = \{(\alpha, \beta) \in \mathbb{N}\times\mathbb{N} \mid \text{there exists } f \in \mathbf{K}(C) \text{ with } (f)_\infty = \alpha P + \beta Q\},$$

where $\mathbb{N}$ denotes the additive semigroup of non-negative integers. For hyperelliptic curves, Kim[3] determined explicitly the semigroup $H(P, Q)$.

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In Section 2 we determine the candidates H for Weierstrass semigroups of a pair of points on a non-hyperelliptic curve of genus 3. In Section 3, for any semigroup H obtained in Section 2, we give an explicit example of a curve C with a pair (P, Q) of points satisfying $H(P, Q) = H$. Moreover, in Section 4 we count the dimension of the moduli of curves which have a pair of points with a specified semigroup.

2. Possible Weierstrass semigroups of genus 3

First, let us review some results in Kim [3]. Let C be a curve of genus g and P its point. We define the semigroup $H(P)$ by

$$H(P) = \{\alpha \in \mathbb{N} \mid \text{there exists } f \in \mathbf{K}(C) \text{ with } (f)_\infty = \alpha P\},$$

and we set $G(P) = \mathbb{N} \backslash H(P)$. Let Q be another point of C which is distinct from P, and let

$$G(P) = \{l_1 < l_2 < \dots < l_g\} \quad \text{and} \quad G(Q) = \{l'_1 < l'_2 < \dots < l'_g\}.$$

For each l_i with $1 \le i \le g$, the integer $\min\{\beta \mid (l_i, \beta) \in H(P, Q)\}$ must be equal to some element in $G(Q)$, say $l'_{\sigma(i)}$, and this correspondence gives a bijective map between the sets $G(P)$ and $G(Q)$ (Kim [3], Lemma 2.6). Thus σ gives a permutation of the set $\{1, 2, \dots, g\}$. We denote the graph of this bijective map by $\Gamma(P, Q)$, that is,

$$\Gamma(P, Q) = \{(l_i, l'_{\sigma(i)}) \mid i = 1, 2, \dots, g\}. \tag{1}$$

The semigroup $H(P, Q)$ is completely determined by the set $\Gamma(P, Q)$, that is,

$$G(P, Q) = \bigcup_{i=1}^{g} \Big(\big\{(l_i, \beta) \mid \beta = 0, 1, \dots, l'_{\sigma(i)} - 1\big\} \cup \big\{(\alpha, l'_{\sigma(i)}) \mid \alpha = 0, 1, \dots, l_i - 1\big\} \Big),$$

where we set $G(P, Q) = \mathbb{N} \times \mathbb{N} \backslash H(P, Q)$. Thus, it suffices to determine the set $\Gamma(P, Q)$ for describing the semigroup $H(P, Q)$.

In this section we consider the case of a non-hyperelliptic curve C of genus 3. For any point P on C, we know that $G(P)$ is either $\{1, 2, 3\}$ or $\{1, 2, 4\}$ or $\{1, 2, 5\}$. Hence, for a pair of distinct points $P, Q \in C$, there are six possibilities for a pair $(G(P), G(Q))$, up to permutation of coordinates. We denote them by type I, II, III, IV, V, and VI. (See the table below.) We will show that the set $\Gamma(P, Q)$ is uniquely determined when $G(P) = G(Q) = \{1, 2, 5\}$, but it is not determined in the other cases. In the following table, we give all possible sets for $\Gamma(P, Q)$, which we classify according to the relationship between P and Q. Recall that two divisors D and E are linearly equivalent, denoted by $D \sim E$, if there exists a rational function f on C such that $(f)_\infty = D - E$.

Theorem 1. *Any Weierstrass semigroup of a pair of points P and Q on a non-hyperelliptic curve C of genus 3 corresponds to one of the following 13 $\Gamma(P, Q)$'s, up to permutation of coordinates. In the table, K means a canonical divisor.*

Type	$G(P)$	$G(Q)$	Relations between P and Q	$\Gamma(P, Q)$
I	$1, 2, 5$	$1, 2, 5$		$(1,5), (2,2), (5,1)$
IIa	$1, 2, 5$	$1, 2, 4$	$3P \sim 3Q$	$(1,2), (2,4), (5,1)$
IIb			$3P \not\sim 3Q$	$(1,4), (2,2), (5,1)$
IIIa	$1, 2, 5$	$1, 2, 3$	$h^0(3P - 2Q) = 1$	$(1,2), (2,3), (5,1)$
IIIb			$h^0(3P - 2Q) = 0$	$(1,3), (2,2), (5,1)$
IVa	$1, 2, 4$	$1, 2, 4$	$P + 3Q \sim K$	$(1,2), (2,4), (4,1)$
IVb			$P + 3Q \not\sim K, 3P + Q \not\sim K$	$(1,4), (2,2), (4,1)$
Va	$1, 2, 4$	$1, 2, 3$	$3P + Q \sim K$	$(1,3), (2,1), (4,2)$
Vb			$3P + Q \not\sim K, h^0(K - P - 2Q) = 1$	$(1,2), (2,3), (4,1)$
Vc			$3P + Q \not\sim K, h^0(K - P - 2Q) = 0$	$(1,3), (2,2), (4,1)$
VIa	$1, 2, 3$	$1, 2, 3$	$2P + 2Q \sim K$	$(1,2), (2,1), (3,3)$
VIb			$2P + 2Q \not\sim K, h^0(K - 2P - Q) = 1$	$(1,3), (2,1), (3,2)$
VIc			$h^0(K - 2P - Q) = h^0(K - P - 2Q) = 0$	$(1,3), (2,2), (3,1)$

Proof. The following equivalence is used several times in the proof: $(\alpha, \beta) \in H(P, Q)$ if and only if

$$h^0(\alpha P + \beta Q) = h^0((\alpha - 1)P + \beta Q) + 1 = h^0(\alpha P + (\beta - 1)Q) + 1.$$

Moreover, we prove some facts which will be used frequently in the proof.

(a) $(1, 1) \notin H(P, Q)$.

(b) If $D = \sum_{R \in C} m_R R$ and $E = \sum_{R \in C} n_R R$ are distinct effective divisors such that $D \sim E \sim K$, then $\deg(D \wedge E) \leq 1$, where $D \wedge E = \sum_{R \in C} \min\{m_R, n_R\} R$.

(c) If $5 \in G(P)$ [resp. $5 \in G(Q)$], then $(5, 1) \in \Gamma(P, Q)$ [resp. $(1, 5) \in \Gamma(P, Q)$].

(a) and (b) are obvious since C is a non-hyperelliptic curve of genus 3. Since $h^0(4P+Q) = h^0(5P) = 3$ and $h^0(5P+Q) = 4$ by the Riemann-Roch theorem, (c) is proved.

Type I: If $G(P) = G(Q) = \{1, 2, 5\}$, then, by (c) and (1), $\Gamma(P, Q)$ is uniquely determined as $\{(1, 5), (2, 2), (5, 1)\}$.

Type II: We divide this type into two sub-types.

Type IIa: $3P \sim 3Q$. Since $P + 3Q \sim 4P \sim K$, we get $h^0(P + 2Q) = h^0(K - Q) = 2$. By (a), we have $(1, 2) \in \Gamma(P, Q)$. Now (c) and (1) determine the set $\Gamma(P, Q)$.

Type IIb: $3P \not\sim 3Q$. We have $K \sim 3Q + R$ for some $R \in C$ which is distinct from P. If $h^0(P + 2Q) = 2$, then $K \sim P + 2Q + R'$ for some $R' \in C$, which contradicts (b). Thus $h^0(P + 2Q) = 1$ and $(1, 2) \notin \Gamma(P, Q)$. By (1) and (a), $(1, 4) \in \Gamma(P, Q)$, and by (c), $\Gamma(P, Q)$ is determined.

Type III: By the Riemann-Roch theorem, $h^0(3P - 2Q) \leq 1$.

Type IIIa: $h^0(3P - 2Q) = 1$. We have $3P \sim 2Q + R$ for some point R of C. Since $K \sim 4P \sim P + 2Q + R$, we have $h^0(P + 2Q) = 2$. By (a), $(1, 2) \in \Gamma(P, Q)$. Then (c) determines the set $\Gamma(P, Q)$.

Type IIIb: $h^0(3P - 2Q) = 0$. We have $3P \not\sim 2Q + R$ for any point R of C. If $h^0(P + 2Q) = 2$, then $P + 2Q + R \sim K \sim 4P$ for some $R \in C$, which contradicts the assumption. Hence $(1, 3) \in \Gamma(P, Q)$ by (a) and (1). Now the set $\Gamma(P, Q)$ is determined by (c).

Type IV: We consider two cases, according to $P + 3Q \sim K$ or not. Note that we are determining $\Gamma(P, Q)$ up to permutation of coordinates.

Type IVa: $P + 3Q \sim K$. We have $h^0(P + 2Q) = 2$. By (a), $(1, 2) \in \Gamma(P, Q)$. If $h^0(2P + Q) = 2$, then $2P + Q + R \sim K$ for some $R \in C$, which contradicts (b). Thus $h^0(2P + Q) = 1$, hence $(2, 1) \notin \Gamma(P, Q)$. Now we conclude that $(2, 4) \in \Gamma(P, Q)$ by (1), which determines $\Gamma(P, Q)$.

Type IVb: $P + 3Q \not\sim K$ and $3P + Q \not\sim K$. Since $h^0(3P) = h^0(3Q) = 2$, we obtain $h^0(P + 2Q) = h^0(2P + Q) = 1$ from (b), which implies that $(1, 2) \notin H(P, Q)$ and $(2, 1) \notin H(P, Q)$. Hence the set $\Gamma(P, Q)$ is determined by (a) and (1).

Type V: We consider two cases, according to $3P + Q \sim K$ or not. We divide the latter into two sub-types, according to $h^0(K - P - 2Q)$ is 1 or 0.

Type Va: $3P + Q \sim K$. We have $h^0(2P + Q) = 2$, which implies that $(2, 1) \in \Gamma(P, Q)$. Since $\deg(4P + 2Q) = 2g$, $(4, 2) \in H(P, Q)$, and hence $(4, 3) \notin \Gamma(P, Q)$. By (1), we obtain the set $\Gamma(P, Q)$.

Type Vb: $3P + Q \not\sim K$ and $h^0(K - P - 2Q) = 1$. We have $P + 2Q + R \sim K$ for some point R of C. Then $h^0(P + 2Q) = 2$, which implies that $(1, 2) \in \Gamma(P, Q)$. As in Type Va, $(4, 3) \notin \Gamma(P, Q)$. Hence the set $\Gamma(P, Q)$ is determined by (1).

Type Vc: $3P + Q \not\sim K$ and $h^0(K - P - 2Q) = 0$. We have $P + 2Q + R \not\sim K$ for every point R of C, hence $h^0(P + 2Q) = 1$, which implies $(1, 2) \notin \Gamma(P, Q)$. Then by (a), $(1, 3) \in \Gamma(P, Q)$. Since $h^0(3P) = 2$, $3P + P' \sim K$ for some $P' \in C$ distinct from Q. Then, by (b), $h^0(2P + Q) = 1$, and hence $(2, 1) \notin \Gamma(P, Q)$. By (1), we determine $\Gamma(P, Q)$.

Type VI: We consider two cases, according to $2P + 2Q \sim K$ or not. We divide the latter into two sub-types, according to $h^0(K - 2P - Q)$ is 1 or 0. In this type, we also note that we are determining $\Gamma(P, Q)$ up to permutation of coordinates.

Type VIa: $2P + 2Q \sim K$. We have $h^0(P + 2Q) = h^0(2P + Q) = 2$, which determine the set $\Gamma(P, Q)$.

Type VIb: $2P + 2Q \not\sim K$ and $h^0(K - 2P - Q) = 1$. We have $2P + Q + R \sim K$ for some point R of C which is distinct from Q. Then $h^0(2P + Q) = 2$, which implies that $(2, 1) \in \Gamma(P, Q)$. Moreover, $h^0(P + 2Q) = 1$. If not, $h^0(P + 2Q) = 2$, which contradicts (b). Hence the set $\Gamma(P, Q)$ is determined by (1).

Type VIc: $h^0(K - 2P - Q) = h^0(K - P - 2Q) = 0$. Since $h^0(P + 2Q) = h^0(2P + Q) = 1$ by the Riemann-Roch theorem, the set $\Gamma(P, Q)$ is determined by (1). □

3. Some examples of curves with a pair of points

Every semigroup appeared in Theorem 1 actually occurs as a Weierstrass semigroup of a pair of points on some curve of genus 3. Indeed, for each semigroup H, we give the explicit equation of a plane curve C and the coordinates of points P and Q on C with $H = H(P, Q)$ in the table below. We note that every non-hyperelliptic curve of genus 3 can be embedded as a plane curve of degree 4 via its canonical map. The type VIc is general, see for example Arbarello, Cornalba, Griffiths and Harris [1, VIII Exercises B.7, p. 366]. Using the Bertini's theorem and elementary calculation, we can easily prove that each curve is nonsingular for general constants a and b, and that the given points P and Q satisfy the given relation in the table in Theorem 1 and hence $H(P, Q)$ is the semigroup of the given type. Note that the canonical series on each curve in the table are cut out by lines on the plane.

Type	C	P	Q
I	$y^3z - yz^3 - x^4 = 0$	$(0:0:1)$	$(0:1:0)$
IIa	$-x^4 + xy^3 + 2yz^3 = 0$	$(0:0:1)$	$(0:1:0)$
IIb	$-(x-z)^4 + xy^3 + 2yz^3 = 0$	$(1:0:1)$	$(0:1:0)$
IIIa	$yz^3 - x^4 + xy^3 - 2y^2z^2 = 0$	$(0:0:1)$	$(0:1:0)$
IIIb	$a(yz^3 - (x-z)^4) + b(xy^3 + y^2z^2) = 0$	$(1:0:1)$	$(0:1:0)$
IVa	$-x^3z + xy^3 + 2yz^3 = 0$	$(0:0:1)$	$(0:1:0)$
IVb	$-(x-z)^3z + xy^3 + 2yz^3 = 0$	$(1:0:1)$	$(0:1:0)$
Va	$a(yz^3 - x^3(x-z)) + by^4 = 0$	$(0:0:1)$	$(1:0:1)$
Vb	$a(yz^3 - x^3z) + b(xy^3 + y^2z^2) = 0$	$(0:0:1)$	$(0:1:0)$
Vc	$a(yz^3 - (x-z)^3z) + b(xy^3 + y^2z^2) = 0$	$(1:0:1)$	$(0:1:0)$
VIa	$a(yz^3 - x^2(x-z)^2) + by^4 = 0$	$(0:0:1)$	$(1:0:1)$
VIb	$a(yz^3 - x^2(x-z)(x-2z)) + by^4 = 0$	$(0:0:1)$	$(1:0:1)$
VIc	any curve	general	general

4. The dimension of the moduli

Let $\mathcal{M}_3$ be the moduli space of curves of genus 3 and $\mathcal{H}_3 \subset \mathcal{M}_3$ the hyperelliptic locus. It is well known that $\dim \mathcal{M}_3 = 6$ and $\dim \mathcal{H}_3 = 5$. We have an isomorphism

$$\left(\mathbb{P}^{14} - \Delta\right)/\mathrm{PGL}(3;\mathbb{C}) \cong \mathcal{M}_3 - \mathcal{H}_3, \tag{2}$$

where Δ is the closed subvariety corresponding to the forms which define singular plane curves of degree 4. In this section we count the dimension of the subscheme

$$\mathcal{M}_H = \{[C] \in \mathcal{M}_3 \mid \text{there exist two distinct points } P \text{ and } Q \text{ of } C \text{ such that } H(P,Q) = H\}$$

for a Weierstrass semigroup H of each type in Theorem 1. We use the notations $\mathcal{M}_I, \mathcal{M}_{IIa}, \cdots, \mathcal{M}_{VIc}$ for subschemes corresponding to given types, respectively.

Theorem 2. *For each semigroup H appeared in Theorem 1, the dimension of the corresponding subscheme $\mathcal{M}_H$ is given as follows:*

Type of H	Dimension of Moduli in $\mathcal{M}_3$
I	4
IIa	4
IIb	5
IIIa	5
IIIb	5
IVa	5
IVb	6
Va	6
Vb	6
Vc	6
VIa	6
VIb	6
VIc	6

Proof. **Type I:** Since the Weierstrass semigroup of any pair of two hyperflexes is of this type, this dimension is equal to that of moduli of curves with two or more hyperflexes. Thus we have dim $\mathcal{M}_I = 4$ by Vermeulen [4, II.9.4].

Type IIa: Let C be any nonsingular plane quartic curve with a hyperflex P and a flex Q such that $3P \sim 3Q$. Then the tangent line at Q passes through P. After a suitable projective transformation we may assume that P has the coordinate $(0 : 0 : 1)$ with tangent line $y = 0$ and Q has the coordinate $(0 : 1 : 0)$ with tangent line $x = 0$. Since $(0 : 0 : 1)$ is a hyperflex with tangent line $y = 0$, the curve C can be expressed by the equation $x^4 + yF = 0$ where

$$F = a_1x^3+a_2y^3+a_3z^3+a_4x^2y+a_5x^2z+a_6xy^2+a_7y^2z+a_8xz^2+a_9yz^2+a_{10}xyz.$$

Moreover, since $(0 : 1 : 0)$ is a flex with tangent line $x = 0$, we have $a_2 = 0$, $a_7 = 0$ and $a_9 = 0$. Thus we obtain 7-dimensional irreducible closed subvariety of $\mathbb{P}^{14} - \Delta$. On the other hand, let $B = (b_{ij})$ be an element of PGL(3; $\mathbb{C}$). If B fixes the point $(0 : 0 : 1)$, we must have $b_{13} = 0$ and $b_{23} = 0$. Moreover, we get $b_{21} = 0$ when B sends the point $(1 : 0 : 1)$ to a point on the line $y = 0$. We obtain $b_{12} = 0$ and $b_{32} = 0$ if B fixes the point $(0 : 1 : 0)$. Thus, from (2), we get dim $\mathcal{M}_{IIa} = 4$.

Type IIb: Let V be the subscheme of $\mathcal{M}_3 - \mathcal{H}_3$ consisting of curves with at least one hyperflex. Every curve in V has another hyperflex or flex, hence V is the union of three subschemes $\mathcal{M}_I$, $\mathcal{M}_{IIa}$ and $\mathcal{M}_{IIb}$. On the other hand, V is an irreducible 5-dimensional variety (Vermeulen [4, I.4.9]). Now the fact dim $\mathcal{M}_I$ = dim $\mathcal{M}_{IIa} = 4$ implies that dim $\mathcal{M}_{IIb} = 5$.

Type IIIa: Let C be any curve with a hyperflex P and an ordinary point Q such that $h^0(3P - 2Q) = 1$. Note that $h^0(3P - 2Q) = 1$ if and only if the tangent line at Q passes through P. We use a similar argument as in Type IIa. But, in this case, Q is not a Weierstrass point, hence a_9 does not need to be zero in the similar calculation. Hence we get $\dim \mathcal{M}_{IIIa} = 5$.

Type IIIb: Let C be any curve with a hyperflex P. Obviously the tangent line at a general point does not pass through P. Hence we may let Q be such a general ordinary point. Thus, we get $\dim \mathcal{M}_{IIIb} = \dim V = 5$.

Type IVa: Let C be any curve with two flexes P and Q such that $P + 3Q \sim K$. Note that $P + 3Q \sim K$ means that the tangent line at Q passes through P. After a suitable projective transformation we may assume that P has the coordinate $(0 : 0 : 1)$ with tangent line $y = 0$ and Q has the coordinate $(0 : 1 : 0)$ with tangent line $x = 0$. Then the curve C can be expressed by the equation $a_{11}x^4 + a_{12}x^3z + yF = 0$ where F is as in Type IIa. Comparing with the equation in Type IIa, we have only one more term x^3z in this equation. By a similar calculation, we get $\dim \mathcal{M}_{IVa} = 5$.

Type IVb: By Vermeulen [4, I.1.19 and I.4.9], a general curve in $\mathcal{M}_3 - \mathcal{H}_3$ contains 24 flexes. Since $\mathcal{M}_{IVa}$ has codimension 1 in $\mathcal{M}_3 - \mathcal{H}_3$, we have $\dim \mathcal{M}_{IVb} = 6$.

Type Va: Let C be a general curve with a flex P. There is a unique point Q of C such that $K \sim 3P + Q$. If Q is a flex or a hyperflex, then C belongs to $\mathcal{M}_{IIa'} \cup \mathcal{M}_{IVa'}$, where Type IIa′ [resp. Type IVa′] is the semigroup obtained by changing the first and second coordinates of elements in the semigroup of Type IIa [resp. Type IVa]. Since $\dim \mathcal{M}_{IIa'} \cup \mathcal{M}_{IVa'} = 5$, we obtain $\dim \mathcal{M}_{Va} = 6$.

Type Vb: Note that $h^0(K - P - 2Q) = 1$ means that the tangent line at Q passes through P. We use a similar argument with Type IIa, IIIa, and IVa. Comparing with Type IIIa, we only change the equation $x^4 + yF$ to $a_{11}x^4 + a_{12}x^3z + yF$. Thus we get $\dim \mathcal{M}_{Vb} = 6$.

Type Vc: Let C be any curve with a flex P. Choose a point Q such that Q is not contained in the tangent line at P and the tangent line at Q does not pass through P. In fact, such a point Q is general one. Then we proved that $[C] \in \mathcal{M}_{Vc}$. Thus, we obtain $\dim \mathcal{M}_{Vc} = 6$.

Type VIa: Note that any nonsingular plane quartic curve has 28 bitangents. (For example, see [4, I.2.2].) If we choose ordinary points P and Q such that two tangent lines at P and Q coincide, then $H(P, Q)$ is the semigroup of Type VIa. Thus $\dim \mathcal{M}_{VIa} = 6$.

Type VIb: For a general point P on any curve C, the tangent line at P meets C at two distinct points other than P. If we let Q be one of these points, then $H(P, Q)$ is of Type VIb. Hence, we get dim $\mathcal{M}_{VIb} = 6$.

Type VIc: By Arbarello, Cornalba, Griffiths and Harris [1, VIII Exercises B.7, p. 366] we have dim $\mathcal{M}_{VIc} = 6$. □

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