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Rigidity of Spheres in Riemannian Manifolds and a Non-Embedding Theorem

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Abstract. Let $f : M \longrightarrow \overline{M}$ be an isometric immersion between Riemannian manifolds. The purpose of this paper is to find the minimum possible conditions on M and $\overline{M}$ (in the terms of curvatures and external diameter) in order to the image of f be contained in a sphere. Our results generalize the other authors work in three major steps, domain, range and the codimension of immersions. As a byproduct, we obtain the non-embedding theorems Chern-Kuiper, Moore and Jacobowitz. The proofs are based on the maximum (comparison) principle.

Keywords: isometric immersion, rigidity, embedding, pinching, maximum principle, l-mean curvature.

Mathematical subject classification: Primary 53C24, 53C42; Secondary 53C40.

1 Introduction

The aim of this paper is to establish a pinching theorem and a non-embedding theorem for submanifolds of a Riemannian manifold.

By using the strong maximum (comparison) principle applied to the elliptic operators, Koutroufiotis [K] proved the following theorem:

Let $f : S \longrightarrow \mathbb{R}^3$ be an isometric immersion from a compact two dimensional manifold S into $\mathbb{R}^3$. Suppose that there is $R > 0$ such that either the sectional curvature of S or the square of the mean curvature of $f(S)$ is bounded from above by R^{-2}. Then, the smallest sphere enclosing $f(S)$ has radius larger than R, unless $f(S)$ is a sphere.

Also, Markvorsen [M] generalized the Koutroufiotis results to the isometric immersions $f : M^n \longrightarrow \overline{M}^{n+1}$ such that the absolute value of the mean cur-

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vature of $f(M)$ is bounded and the sectional curvature of $\overline{M}$ is bounded from above by a constant.

Recently, Fontenele and Silva [FS] generalized the Koutroufiotis results to the isometric immersions $f : M^n \longrightarrow \overline{M}^{n+1}$ such that the scalar curvature of compact manifold M is bounded from above and the target space $\overline{M}$ is a space form (a complete and simply connected space with constant sectional curvature) of non-positive curvature. Also, Vlachos [V] proved a rigidity type theorem for geodesic spheres of space forms in term of l-mean curvatures.

In this paper, we generalize the above results to the isometric immersions $f : M^n \longrightarrow \overline{M}^{n+k}$ with the weaker assumptions. Moreover, we obtain the non-embedding theorems Chern-Kuiper [CK], Moore [Mo] and Jacobowitz [J].

The proofs are based on the (strong and weak) maximum principle. For more works on this topic see [CI], [I], [J], [JK] and [L].

2 A Rigidity Theorem for Curves

In this section, we prove a generalization of the main theorem of this paper for curves in a quite general setting.

Let N be a Riemannian manifold (of class C^3) and let $\langle \cdot , \cdot \rangle$ denote the Riemannian metric on N. We denote the associated covariant derivative of N by D. For $p \in N$, we denote the distance from p to x by $r(x) = r_p(x)$. The function $r_p(x)$ is smooth on $N \setminus (\{p\} \cup C_p)$, where C_p denotes the cut locus of p. Also, we denote the Hessian of $r(x)$ by $Hess(r)(v, w) := \langle D_v^{\nabla r}, w \rangle$, for all vectors v and w in the tangent bundle of N. We denote the closed ball with the center at $q \in N$ and the radius $R > 0$ by $B(q, R)$.

Theorem 2.1. *Let $\gamma :]a, b[\longrightarrow \overline{M}$ be a regular curve (of class C^2) on the Riemannian manifold $\overline{M}$. Suppose that the image of γ is contained in $B(p, R) \setminus C_p$ and there is $s_0 \in]a, b[$ such that $\gamma(s_0)$ belongs to $\partial B(p, R)$. Suppose that the Hessian of distance function on $\overline{M}$, $r(y) = r_p(y)$, is bounded from below by $m(r) \geq 0$ on the tangent bundle of $\partial B(p, r)$, i.e. $Hess(r)(v, v) \geq m(r)\,||v||^2$ for all vectors v in the tangent bundle of $\partial B(p, r)$. Let the curvature of γ be bounded from above by $m(r)$, i.e. $0 \leq k(\gamma(s)) \leq m(r(\gamma(s)))$, for all $s \in]a, b[$. Then, the image of γ is contained in the sphere $\partial B(p, R)$.*

Proof. Without loss of generality, we can assume that γ is parametrized by arc-length, i.e. $||\gamma'(s)|| = 1$. Now, consider the function

$$h(s) := r(\gamma(s)).$$

Then, we have (note that $||\nabla r|| = 1$)

$$
\begin{aligned}
h'(s) &= \langle \nabla r \,,\, \gamma'(s) \rangle, \\
h''(s) &= \langle D^{\nabla r}_{\gamma'(s)} \,,\, \gamma'(s) \rangle + \langle \nabla r \,,\, D^{\gamma'(s)}_{\gamma'(s)} \rangle, \\
h''(s) &= \langle D^{\nabla r}_{\gamma'_T(s)} \,,\, \gamma'_T(s) \rangle + \langle \nabla r \,,\, D^{\gamma'(s)}_{\gamma'(s)} \rangle,
\end{aligned}
$$

where $\gamma'_T(s)$ denotes the projection of $\gamma'(s)$ on the tangent bundle of $\partial B(p, r(\gamma(s)))$. So, we have

$$
\begin{aligned}
h''(s) &\geq \Big[1 - |\langle \nabla r \,,\, \gamma'(s) \rangle|^2 \Big] m(r(\gamma(s))) - k(\gamma(s)) \\
&\geq -|\langle \nabla r \,,\, \gamma'(s) \rangle|^2 \, m(r(\gamma(s))).
\end{aligned}
$$

Let $\alpha \geq 2$. Define the function h_α as the following:

$$
h_\alpha(s) := [h(s)]^\alpha .
$$

Then, we have

$$
h'_\alpha(s) = \alpha \, [h(s)]^{(\alpha-1)} \, h'(s),
$$

and

$$
\begin{aligned}
h''_\alpha(s) &= \alpha(\alpha-1) \, [h(s)]^{(\alpha-2)} \left[h'(s) \right]^2 + \alpha \, [h(s)]^{(\alpha-1)} \, h''(s) \\
&\geq \alpha \, [h(s)]^{(\alpha-2)} \Big[(\alpha-1) \, |\langle \nabla r \,,\, \gamma'(s) \rangle|^2 - r(\gamma(s)) \\
&\qquad \cdot \langle \nabla r \,,\, \gamma'(s) \rangle|^2 m(r(\gamma(s))) \Big] \\
&\geq \alpha \, [h(s)]^{(\alpha-2)} \, |\langle \nabla r \,,\, \gamma'(s) \rangle|^2 \Big[\alpha - 1 - r(\gamma(s)) \, m(r(\gamma(s))) \Big].
\end{aligned}
$$

Now, let α be large enough such that $\left[\alpha - 1 - r(\gamma(s)) \, m(r(\gamma(s)))\right] \geq 0$, for s close enough to s_0. Therefore $h_\alpha(s)$ is a convex function, for s close enough to s_0. Since that $h_\alpha(s)$ attains its maximum at the interior point $s_0 \in]a, b[$, by the (strong) maximum principle, $h_\alpha(s)$ is a constant function. This completes the proof of theorem. □

Corollary 2.2. *Let notations and assumptions be as in Theorem 2.1. Let $\overline{M}$ be a space form, i.e. a complete and simply connected Riemannian manifold with constant sectional curvature. Then, the image of γ is contained in a circle (with center at p and radius R).*

Proof. By Theorem 2.1, we know that the image of γ is contained in the sphere $\partial B(p, R)$. Then, by the Meusnier theorem [H, p. 77] and the fact that $k(\gamma(s)) \leq m(r(\gamma(s)))$, we see that $k(\gamma(s)) = m(r(\gamma(s))) = m(r(\gamma(s_0)))$; and γ is a geodesic in the sphere $\partial B(p, R)$. This completes the proof of corollary. $\square$

Remark 2.3. In the Theorem 2.1, when $\overline{M} = \mathbb{R}^{n+1}$, we can instead of the condition $k(\gamma(s)) \leq m(r(\gamma(s)))$, replace the following weaker condition:

$$k(\gamma(s)) \leq \frac{1}{|\langle \gamma(s) - p\, ,\, N(\gamma(s))\rangle|},$$

where $N(\gamma(s))$ is a unit vector which is in the direction of $\gamma''(s)$.

Remark 2.4. Suppose that in Theorem 2.1, we replace the assumption $f(s_0) \in \partial B(p, R)$ (for some interior point $s_0 \in]a, b[$) by the following condition:

Either $a = -\infty$ and $h(s)$ is non-increasing, or $b = +\infty$ and $h(s)$ is non-decreasing.

Then, the conclusion of the Theorem 2.1 remains valid by replacing the sphere $\partial B(p, R)$ by the sphere $\partial B(p, R_0)$, for some $R_0 \leq R$.

3 The Maximum Principle

In this section, we introduce two operators on the Riemannian manifolds in such a way the (weak and strong) maximum principle remains valid.

Definition 3.1. Let N be an n-dimensional Riemannian manifold with the Riemannian metric $\langle \cdot\, ,\, \cdot \rangle$. Let $h : N \longrightarrow \mathbb{R}$ be a C^2-function. We define the *upper Laplacian* of h at the point $x \in N$ as the following:

$$\Delta_{\mathcal{U}} h(x) := \sup_{\gamma} \left[\frac{d^2}{ds^2}(h \circ \gamma)(s) \right]_{s=0},$$

where the supremum is taken over all geodesics γ such that $\gamma(0) = x$. Similarly, we can define the *lower Laplacian* of h at the point x (by replacing inf instead of sup), and we denote it by $\Delta_{\mathcal{L}} h(x)$. Let Ω be an open subset of N. We say that h is *generalized subharmonic* on Ω, if there are orthonormal vector fields $\{e_1(x), e_2(x), ..., e_n(x)\}$ on Ω such that the following condition holds:

$$\sum_{i=1}^{n} a_i^2(x) \left[\frac{d^2}{ds^2}(h \circ \gamma_i)(s) \right]_{s=0} \geq 0,$$

where γ_i is a geodesic such that $\gamma_i(0) = x$ and $\gamma_i'(0) = e_i(x)$, and $a_i(x)$'s are positive numbers. Moreover, there is $A > 0$ such that $a_i(x) \geq A$, for all $x \in \Omega$ and $1 \leq i \leq n$.

It is not hard to see that

$$\Delta_{\mathcal{L}} h(x) \leq \frac{1}{n}\ \Delta h(x) \leq \Delta_{\mathcal{U}} h(x).$$

Also, if h is generalized subharmonic on Ω, then $\Delta_{\mathcal{U}} h(x) \geq 0$, for $x \in \Omega$.

Proposition 3.2. **(Weak Max)** *Let N be a Riemannian manifold and let Ω be an open subset of N (with smooth boundary). Let $h : N \longrightarrow \mathbb{R}$ be a C^2-function. Then, we have*

(i) Suppose h attains its (local) maximum at $x_0 \in \Omega$, then $\Delta_{\mathcal{U}} h(x_0) \leq 0$.

(ii) Suppose that $\Delta_{\mathcal{U}} h(x) \geq 0$, for all $x \in \Omega$, then $\max_{z\in\overline{\Omega}} h(z) = \max_{z\in\partial\Omega} h(z)$.

Proof. It is similar to the proof of [GT, Thm 3.1] with minor changes. □

Proposition 3.3. **(Strong Max)** *Let N be a Riemannian manifold and let $h : N \longrightarrow \mathbb{R}$ be a C^2-function. Let Ω be an open subset of N (with smooth boundary) and let h be generalized subharmonic on Ω. Suppose that $\max_{z\in\overline{\Omega}} h(z) = h(x_0)$, for some $x_0 \in \Omega$, then h is a constant function on Ω.*

Proof. Let notations be as in Definition 3.1. Then, we have

$$\frac{d}{ds}(h \circ \gamma_i)(s) = \langle \nabla h\ ,\ \gamma_i'(s)\rangle,$$

$$\frac{d^2}{ds^2}(h \circ \gamma_i)(s) = \langle D^{\nabla h}_{\gamma_i'(s)}\ ,\ \gamma_i'(s)\rangle + \langle \nabla h\ ,\ D^{\gamma_i'(s)}_{\gamma_i'(s)}\rangle.$$

Since that γ_i is a geodesic, we have

$$\frac{d^2}{ds^2}(h \circ \gamma_i)(s) = \langle D^{\nabla h}_{\gamma_i'(s)}\ ,\ \gamma_i'(s)\rangle.$$

Therefore, we obtain

$$\begin{aligned}\sum_{i=1}^{n} a_i^2(x)\left[\frac{d^2}{ds^2}(h\circ\gamma_i)(s)\right]_{s=0} &= \sum_{i=1}^{n} a_i^2(x)\langle D_{e_i(x)}^{\nabla h},\, e_i(x)\rangle \\ &= \sum_{i=1}^{n}\langle D_{v_i(x)}^{\nabla h},\, v_i(x)\rangle,\end{aligned}$$

where $v_i(x) := a_i(x)\,e_i(x)$. Consider the orthogonal vector fields $\{v_1(x), v_2(x), ..., v_n(x)\}$, we can define the operator $\mathcal{T}$ as the following:

$$\mathcal{T}(h) := \sum_{i=1}^{n}\langle D_{v_i(x)}^{\nabla h},\, v_i(x)\rangle.$$

Since that $a_i(x) \geq A > 0$, we obtain that $\mathcal{T}$ is a uniformly elliptic operator. Now, the proposition follows from [GT, Thm 3.5]. □

4 The Rigidity Theorem

In this section, by applying the strong maximum principle to generalized subharmonic functions, we extend the results [FS], [M] and [V] to more general setting.

We start this section with the following lemma which is a generalization of this fact that on every compact hypersurface of $\mathbb{R}^n$ there is at least one point with the positive (sectional) curvature.

Lemma 4.1. *Let $f : M^n \longrightarrow \overline{M}^{n+k}$ be an isometric C^2-immersion between Riemannian manifolds. Suppose that the image of f is contained in $B(p, R)\backslash C_p$ and there is $x_0 \in M$ such that $f(x_0) \in \partial B(p, R)$. Suppose that the Hessian of distance function on $\overline{M}$, $r(y) = r_p(y)$, is bounded from below by $m(r) > 0$ on the tangent bundle of $\partial B(p, r)$, i.e. $Hess(r)(v, v) \geq m(r)\,||v||^2$ for all vectors v in the tangent bundle of $\partial B(p, r)$. Then, for any unit normal vector field $N(f(x))$ such that $N(f(x_0)) = -\nabla r(x_0)$, there is a small neighborhood of x_0 such that all principal curvatures of M in the normal direction of $N(f(x))$ are positive.*

Proof. It is clear that $\partial B(p, R)$ is tangent to $f(M)$ at the point $f(x_0)$. Let λ be a principle curvature of M in the normal direction of $N(f(x))$ at the point x_0 with the corresponding (unit) principle vector e. Let γ be the geodesic $\gamma(0) = x_0$

and $\gamma'(0) = e$. Define $h(s) := r(\gamma(s))$. Then, by using the proof of Theorem 2.1, we have

$$h''(s) = \langle D^{\nabla r}_{\gamma'(s)}\,,\ \gamma'(s)\rangle + \langle \nabla r\,,\ D^{\gamma'(s)}_{\gamma'(s)}\rangle.$$

Since that h attains its maximum at $s = 0$, we have $h''(0) \leq 0$. Then, we have

$$\langle D^{\nabla r}_{\gamma'(0)}\,,\ \gamma'(0)\rangle \leq \langle -\nabla r\,,\ D^{\gamma'(0)}_{\gamma'(0)}\rangle$$

Hence

$$0 < m(r(x_0)) \leq \lambda.$$

This completes the proof of lemma. □

Theorem 4.2. *Let $f : M^n \longrightarrow \overline{M}^{n+k}$ be an isometric C^2-immersion between Riemannian manifolds. Denote the l-mean curvature vector of $f(M)$ in $\overline{M}$ by H_l (see [Ch] for the basic definitions). Suppose that the image of f is contained in $B(p, R)\backslash C_p$ and there is $x_0 \in M$ such that $f(x_0) \in \partial B(p, R)$. Suppose that the Hessian of distance function on $\overline{M}$, $r(y) = r_p(y)$, is bounded from below by $m(r) > 0$ on the tangent bundle of $\partial B(p, r)$, i.e. $Hess(r)(v, v) \geq m(r)\,||v||^2$ for all vectors v in the tangent bundle of $\partial B(p, r)$. Moreover, suppose that for all $x \in M$ and any unit normal vector field $\eta(f(x))$ on M, we have*

$$|\langle H_l(f(x))\,,\ \eta(f(x))\rangle| \leq m(r(f(x)))\,|\langle H_{l-1}(f(x))\,,\ \eta(f(x))\rangle|,$$

where $l \geq 1$ is an integer (define $|\langle H_0(f(x))\,,\ \eta(f(x))\rangle| := 1$). If M is connected, then the image of f is contained in the sphere $\partial B(p, R)$.

Proof. Without loss of generality, we can assume that M is a submanifold of $\overline{M}$ (at least locally). Suppose that $x \in M$ is an arbitrary point which is close to x_0. Consider the second fundamental form of M (in $\overline{M}$) in the direction of the unit normal vector field $N(f(x))$ such that $N(f(x_0)) = -\nabla r(x_0)$. Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ denote the principal curvatures (in the direction $N(f(x))$) at the point $y = f(x)$ with the corresponding principal (unit) vectors $e_1, e_2, \ldots, e_n$ (note that by Lemma 4.1 we know that λ_i's are positive at x). Let γ_i be a geodesic in M such that $\gamma_i(0) = x$ and $\gamma_i'(0) = e_i$. Define

$$h_i(s) := r(\gamma_i(s)),$$

$$h_{i,\alpha}(s) := [h_i(s)]^{\alpha}.$$

By using the proof of Theorem 2.1, we have

$$h''_{i,\alpha}(s) \geq \alpha\, [h_i(s)]^{(\alpha-2)} \Big(m(r(\gamma(s))) - \lambda_i(\gamma(s)) \Big),$$

for $\alpha \geq 2$ large enough. Then, we have

$$h''_{i,\alpha}(0) \geq \alpha\, r(x) \Big(m(r(x)) - \lambda_i(x) \Big).$$

By multiplying the above inequality by $\left(\prod_{j_k \neq i\,,\, k=1}^{l-1} \lambda_{j_k}\right) > 0$ and using the assumptions of theorem, we obtain

$$\sum_{i=1}^{n} \left[\left(\prod_{j_k \neq i\,,\, k=1}^{l-1} \lambda_{j_k} \right) h''_{i,\alpha}(0) \right] \geq 0.$$

Therefore, the function $\rho(x)$, is a generalized subharmonic function for all x in a small neighborhood of x_0, where $\rho := r^\alpha \circ f$, for some $\alpha \geq 2$. Now, Proposition 3.3 implies that ρ is constant on a neighborhood of x_0. This implies the theorem. □

Corollary 4.3. *Let notations and assumptions be as in Theorem 4.2. Let $\overline{M}$ be a space form, i.e. a complete and simply connected Riemannian manifold with constant sectional curvature. Then, the image of γ is contained in an n dimensional sphere (with center at p and radius R).*

Proof. It is similar to the proof of Corollary 2.2. □

Remark 4.4. In the Theorem 4.2, suppose that $\overline{M} = \mathbb{R}^{n+1}$. Then we can replace the condition:

$$|\langle H_l(f(x))\,,\ \eta(f(x))\rangle| \leq m(r(f(x)))\, |\langle H_{l-1}(f(x))\,,\ \eta(f(x))\rangle|,$$

by the following condition:

$$|\langle H_l(f(x))\,,\ \eta(f(x))\rangle| \leq \frac{|\langle H_{l-1}(f(x))\,,\ \eta(f(x))\rangle|}{|\langle f(x) - p\,,\ N(f(x))\rangle|},$$

where $N(f(x))$ is a unit normal vector field on $f(M)$. Compare [FS, Thm B].

Remark 4.5. In Theorem 4.2, we can replace the condition:

$$|\langle H_l(f(x)), \eta(f(x))\rangle| \leq m(r(f(x)))\, |\langle H_{l-1}(f(x)), \eta(f(x))\rangle|,$$

with the following (stronger) condition:

$$|\langle H_l(f(x)), \eta(f(x))\rangle| \leq m^l(r(f(x))).$$

Note that by the Hessian comparison theorem (see for instance [SY, p. 4]), we can obtain the Markvorsen result [M]. Moreover, by Remark 4.4 and Remark 4.5, we can recover the results [FS] and [V].

Question 4.6. Let M be an orientable and compact (without boundary) hypersurface in $\overline{M} = \mathbb{R}^{n+1}$. Suppose that Ricci curvature of M is bounded from below by R^{-2}, for some $R > 0$. Suppose there is a ball $B(p, R) \subset \overline{M}$ which is inside M. Then, we have

- For $n = 1$; M is the circle $\partial B(p, R)$, by the Fenchel theorem (see [C1, p. 399]).
- For $n = 2$; M is the sphere $\partial B(p, R)$, by the Gauss-Bonnet theorem (see [H, p. 111]). Compare [K, Thm 1].
- For $n \geq 2$; M is the sphere $\partial B(p, R)$, by the Bonnet-Myers and Cheng theorems (see [C2, p. 201] and [Chg]).

Is it possible to generalize the above theorem with weaker assumptions similar to Theorem 4.2 and Remark 4.4?

5 A Non-Embedding Theorem

The aim of this section is to generalize the non-embedding theorems Chern-Kuiper [CK], Moore [Mo] and Jacobowitz [J]. See also [I] and [JK].

We start this section by the following algebraic lemma which is due to Otsuki.

Lemma 5.1. *Let $L : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be a symetric bi-linear form. Suppose that there is $\beta \geq 0$ such that*

$$\langle L(v, v), L(w, w)\rangle - ||L(v, w)||^2 \leq \beta^2,$$

for all orthonormal vectors $v, w \in \mathbb{R}^n$ with Euclidean inner product $\langle \cdot, \cdot\rangle$ on $\mathbb{R}^n$. If $k < n$, there is a unit vector $e \in \mathbb{R}^n$ such that

$$||L(e, e)|| \leq \beta.$$

Proof. See [C2, p. 224]. □

Theorem 5.2. *Let $f : M^n \longrightarrow \overline{M}^{n+k}$ be an isometric C^2-immersion between Riemannian manifolds with codimension $k < n$. Denote the sectional curvature of M and $\overline{M}$ by $K(\cdot,\cdot)$ and $\overline{K}(\cdot,\cdot)$, respectively. Suppose that the image of f is contained in $B(p, R)\backslash C_p$ and there is $x_0 \in M$ such that $f(x_0) \in \partial B(p, R)$. Suppose that the Hessian of distance function on $\overline{M}$, $r(y) = r_p(y)$, is bounded from below by $m(r) \geq 0$ on the tangent bundle of $\partial B(p, r)$, i.e. $Hess(r)(v, v) \geq m(r)\,||v||^2$ for all vectors v in the tangent bundle of $\partial B(p, r)$. Moreover, suppose that for all $x \in M$, there is a subspace P_x of dimension $k + 1$ in tangent space T_xM such that for any orthonormal vectors $v_x, w_x \in P_x$, we have*

$$K(v_x, w_x) - \overline{K}(v_x, w_x) < m^2(r(f(x))).$$

Then, the isometric immersion f with the above properties cannot exist.

Proof. Let $B(\cdot,\cdot)$ denote the second fundamental form M in $\overline{M}$, i.e. $B(v, w) := \overline{D}_w v - D_w v$, for all v and w in the tangent bundle of M. By the Gauss formula, we have

$$K(v_x, w_x) - \overline{K}(v_x, w_x) = \langle B(v_x, v_x)\,,\ B(w_x, w_x)\rangle - ||B(v_x, w_x)||^2,$$

where orthonormal vectors $v_x, w_x \in P_x$. By the Otsuki's lemma, for any $x \in M$ there is a unit vector $e_x \in P_x$ such that

$$||B(e_x, e_x)|| < m(r(x)).$$

Consider the geodesic γ in M such that $\gamma(0) = x$ and $\gamma'(0) = e_x$. Define the function h as the following:

$$h(s) := r(\gamma(s)).$$

Then, similar to the proof of Theorem 2.1, we have

$$h''(s) = \langle \overline{D}^{\nabla r}_{\gamma'(s)}\,,\ \gamma'(s)\rangle + \langle \nabla r\,,\ \overline{D}^{\gamma'(s)}_{\gamma'(s)}\rangle,$$

Now, let $x = x_0$. Since that ∇r is orthogonal to M at x_0, we obtain

$$h''(0) \geq m(r(x_0)) - \lambda > 0,$$

where $\lambda =: ||\overline{D}^{\gamma'(s)}_{\gamma'(s)}||_{s=0}$ (note that since that γ is a geodesic in M, we have $D^{\gamma'(s)}_{\gamma'(s)} = 0$). Therefore, we have $\Delta_{\mathcal{U}}\rho(x_0) > 0$, where $\rho := r \circ f$. But, ρ attains its maximum at x_0, then, by Proposition 3.2 we have $\Delta_{\mathcal{U}}\rho(x_0) \leq 0$. It is a contradiction. This completes the proof of theorem. □

Remark 5.3. In Theorem 5.2, if we replace the condition:

$$K(v_x, w_x) - \overline{K}(v_x, w_x) < m^2(r(f(x))),$$

with the following (weaker) assumption:

$$K(v_x, w_x) - \overline{K}(v_x, w_x) \leq m^2(r(f(x))),$$

we can show that $f(M)$ touches $\partial B(p, R)$ at infinitely many points (by Proposition 3.2 and the proof of Theorem 2.1).

Remark 5.4. In Theorem 5.2, we need the condition:

$$K(v_x, w_x) - \overline{K}(v_x, w_x) < m^2(r(f(x))),$$

only at the point $x = x_0$.

Remark 5.5. We can relax the assumptions of Theorem 5.2, similar to Remark 4.4 when $\overline{M} = \mathbb{R}^{n+k}$.

Note that the non-embedding theorems Chern-Kuiper [CK] and Jacobowitz [J] are an immediate consequence of Theorem 5.2 and the non-embedding theorem Moore [Mo] is followed from Theorem 5.2 and the Hessian comparison theorem ([SY, p. 4]).

Remark 5.6. By using Lemma 4.1 and [BS, Thm 2], we can prove Chern-Kuiper theorem [CK] for more general target space $\overline{M}$.

Remark 5.7. We can remove the condition $f(x_0) \in \partial B(p, R)$ in Theorem 5.2, and by adding other assumptions similar to [JK] and using Omori theorem (see [JK]) in order to obtain a lower bound for external diameter.

Question 5.8. In Theorem 5.2, is it possible to replace the condition $f(x_0) \in \partial B(p, R)$ with a different condition? Compare Remark 5.7 and also see [BZ, 28.2.7].

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