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Expression of curvature tensors and some applications

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Abstract. We get an explicit expression of curvature operators in terms of at most eight terms of sectional curvatures. Some applications of this result are also given, particularly we improve a result of Chen-Tian related to the first Chern class of admissible surfaces in pinched manifolds. We also characterize in a simple way all functions $k(x, y)$ which can be sectional curvatures of some curvature operator R.

Keywords: minimal surface, curvature tensor, sectional curvature.

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§1. Introduction

It is well known that the curvature tensor is determined uniquely by sectional curvatures and some expressions in terms of sectional curvatures have been given, see for example [K] and [GKM, p. 93]. In this paper we give an explicit expression of curvature tensors with at most eight terms and show that this expression also determines a curvature tensor algebraically. We can see that the number of the terms in this formula is sharp in some sense.

First we have a general result.

Theorem 1. *Let V be a linear vector space and $R : V \times V \times V \times V \to \mathbb{R}$ a 4-linear map satisfying*

$$R(x, y, z, t) + R(x, z, t, y) + R(x, t, y, z) = 0, \tag{1.1}$$

$$R(x, y, z, t) = -R(y, x, z, t), \tag{1.2}$$

$$R(x, y, z, t) = R(z, t, x, y), \tag{1.3}$$

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where $x, y, z, t \in V$. Denote $k(x, y) := R(x, y, x, y)$. Then

$$\begin{aligned} 24R(x, y, z, t) = {} & k(x+z, y+t) + k(x-z, y-t) \\ & - k(x+z, y-t) - k(x-z, y+t) \\ & - k(x+t, y+z) - k(x-t, y-z) \\ & + k(x+t, y-z) + k(x-t, y+z). \end{aligned} \tag{1.4}$$

Conversely let $k(x, y)$ be a real valued map with

$$k(\lambda x, y) = \lambda^2 k(x, y),\ k(x, y) = k(y, x),\ k(x, x) = 0,$$

and satisfying the linearity conditions below:

$$2\big(k(a, b) + k(a, c)\big) = k(a, b+c) + k(a, b-c), \tag{1.5}$$

$$k(a + \lambda b, c) - k(a - \lambda b, c) = \lambda\big(k(a+b, c) - k(a-b, c)\big). \tag{1.6}$$

Then the formula (1.4) defines a 4-linear operator satisfying (1.1), (1.2) and (1.3), and $k(x, y) = R(x, y, x, y)$. In particular k satisfies equation (2.4) of the second section.

Note that (1.5) and (1.6) are directly obtained from the 4-linearity of R. We think that Theorem 1 could be used to obtain new examples of manifolds with prescribed conditions of curvature. Note that theorem 1 is purely algebraic and we do not assume that V has finite dimension. If further V admits an inner product, we can define the sectional curvature $K(x, y) = k(x, y)/\left(|x|^2|y|^2 - \langle x, y\rangle^2\right)$, for any linearly independent vectors x, y. When V is a Hilbert space, the inner product allows us to define a 3-linear operator $R\colon V \times V \times V \to V$ as $\langle R(x, y, z), t\rangle = R(x, y, z, t)$. In the case of finite dimension, the Ricci curvature does not depend on the choice of orthonormal basis if k satisfies the above linearity conditions.

Note that for any four orthonormal vectors $x, y, z, t \in V$, (1.4) implies

$$\begin{aligned} 6R(x, y, z, t) = {} & K(x+z, y+t) + K(x-z, y-t) \\ & - K(x+z, y-t) - K(x-z, y+t) \\ & - K(x+t, y+z) - K(x-t, y-z) \\ & + K(x+t, y-z) + K(x-t, y+z). \end{aligned} \tag{1.7}$$

The following corollary is a direct consequence of (1.7). The inequality (1.8) is due to Berger([B]) when δ is positive and Karcher([K]) in general. As shown in many works this inequality is very useful. Besides we can give here the characterization of the equality by the formula (1.7).

Corollary 1. *Let $K(x, y) \in [\delta, \Delta]$ for any $x, y \in V$. Then for any four orthonormal vectors $x, y, z, t \in V$,*

$$|R(x, y, z, t)| \leq \frac{2}{3}(\Delta - \delta), \tag{1.8}$$

and the equality holds if and only if either

$$\begin{aligned} K(x+z, y+t) &= K(x-z, y-t) = K(x+t, y-z) = \\ &= K(x-t, y+z) = \Delta, \text{ and} \\ K(x+z, y-t) &= K(x-z, y+t) = K(x+t, y+z) = \\ &= K(x-t, y-z) = \delta; \end{aligned} \tag{1.9}$$

or

$$\begin{aligned} K(x+z, y+t) &= K(x-z, y-t) = K(x+t, y-z) = \\ &= K(x-t, y+z) = \delta, \text{ and} \\ K(x+z, y-t) &= K(x-z, y+t) = K(x+t, y+z) = \\ &= K(x-t, y-z) = \Delta. \end{aligned} \tag{1.10}$$

Note that if some formula could exist with less terms then (1.7), and the same factor 6 on the left side, the Berger inequality could be improved, and it is well-known that this is not possible. When there exists an almost complex structure $J : V \to V$, that is $J^2 = -I$, we have:

Corollary 2. *For any four orthonormal vectors $x, y, Jx, Jy \in H$,*

$$\begin{aligned} 6R(x, Jx, y, Jy) = {} & K(x+y, J(x+y)) + K(x-y, J(x-y)) \\ & - K(x+y, J(x-y)) - K(x-y, J(x+y)) \\ & - K(x+Jy, Jx+y) - K(x-Jy, Jx-y) \\ & + K(x-Jy, Jx+y) + K(x+Jy, Jx-y). \end{aligned} \tag{1.11}$$

Notice that the positive terms of the right hand side of (1.11) are sectional curvatures of complex planes while the negative ones are sectional curvatures of totally real planes. In particular when R is the curvature operator of Kähler space form $M(4c)$ of constant holomorphic sectional curvature $4c$, Then

$$R(x, Jx, y, Jy) = 2c.$$

This is another reason which convinces us that the number of terms in the right hand side of (1.4) is sharp.

Now let us state some applications of these formulas. We need to recall some notations and facts from [CT]. A surface Σ is *admissible* in a given 4-dimensional Riemannian manifold M if it is immersed except at finitely many singular points in Σ and satisfies

$$\int_{\tilde{\Sigma}} \pi^\star(|B|^2)dA < +\infty,$$

where $\pi : \tilde{\Sigma} \to \Sigma$ is the oriented covering of Σ and B is the second fundamental form of Σ in M. Denote by $T\Sigma$ and $N\Sigma$ the tangent and normal bundles of Σ in M respectively. We can define

$$\chi(T\Sigma) = \int_\Sigma K_T dA,$$

$$\chi(N\Sigma) = \int_\Sigma K_N dA,$$

where dA is the surface area element, and K_T and K_N are curvatures of the bundles $T\Sigma$ and $N\Sigma$, respectively. If Σ is embedded in M, then $\chi(N\Sigma)$ is the self intersection number of Σ in M. If Σ is immersed, then $\chi(T\Sigma)$ is the geometric genus of Σ, and $\chi(N\Sigma)$ is equal to the self intersection number of Σ minus twice the number of double points in Σ (counted with sign). Let Σ be an admissible surface in a 4-dimensional Riemannian manifold M. Around any $p \in \Sigma$, we choose a local orthonormal frame $\{e_1, e_2, e_3, e_4\}$ such that $\{e_1, e_2\} \subset T\Sigma$ and $\{e_3, e_4\} \subset N\Sigma$, and $\{w_1, w_2, w_3, w_4\}$ are coframe vectors coresponding to $\{e_1, e_2, e_3, e_4\}$ and (w_{ij}) is the connection matrix of the Levi-Civita connection. Then there is an almost complex structure J_Σ on M along Σ which is defined as

$$\begin{cases} J_\Sigma e_1 & = e_2, \\ J_\Sigma e_2 & = -e_1, \\ J_\Sigma e_3 & = e_4, \\ J_\Sigma e_4 & = -e_3. \end{cases}$$

Finally we have

$$\chi(T\Sigma) + \chi(N\Sigma) = c_1(f^*TM, J_\Sigma)(\tilde{\Sigma}),$$

where $c_1(.)$ denotes the first Chern class of f^*TM, and $f : \tilde{\Sigma} \to \Sigma$ is some normalization. We can improve Theorem 4.7 in [CT] as follows:

Theorem 2. *Let M be a Riemannian 4-manifold with sectional curvatures pinched as*

$$-\frac{5}{2}a^2(x) \le K_x \le -a^2(x) \left(\text{respectively, } a^2(x) \le K_x \le \frac{5}{2}a^2(x) \right),$$

for some measurable function $a(x)$. If Σ is an admissible oriented minimal surface in M (respectively, which satisfies $J_\Sigma \equiv 0$ along Σ), then

$$\chi(T\Sigma) + \chi(N\Sigma) \le 0 \quad (\text{ respectively, } \chi(T\Sigma) + \chi(N\Sigma) \ge 0), \qquad (1.12)$$

and, if equality holds, this implies that J_Σ is parallel along Σ (respectively, Σ is minimal), and either the ambient space is flat along Σ or the curvature pinching is $\frac{5}{2}$.

Remark. From Theorem 2 the inequality (1.12) is strict when for example M is the hyperbolic space, or the standard sphere, since the pinching constant is 1.

Let H be the mean curvature of Σ. Following the same idea we get:

Corollary 3. *Let Σ be an admissible surface in a Riemannian 4-manifold M. Let M be a Riemannian 4-manifold whose sectional curvature K satisfies*

$$-\frac{5}{2}a^2(x) \le K_x \le -a^2(x) \left(\text{respectively, } a^2(x) \le K_x \le \frac{5}{2}a^2(x) \right),$$

for some measurable function $a(x)$. Assume that one of the following conditions is satisfied

(a) $w_{13} = w_{24}$,

(b) $w_{14} = -w_{23}$,

(c) $2|H|^2 \le |\nabla J_\Sigma|^2$ *(respectively,* $2|H|^2 \ge |\nabla J_\Sigma|^2$*).*

Then

$$\chi(T\Sigma) + \chi(N\Sigma) \le 0 \quad (\text{ respectively, } \chi(T\Sigma) + \chi(N\Sigma) \ge 0)$$

and equality implies that either M is flat along Σ, or the pinching $\frac{5}{2}$ is attained.

Note that (a) together with (b) is equivalent to say $J_\Sigma \equiv 0$ (see Lemma 4.1 in [CT]). So our condition here is weaker than to say that J_Σ is parallel.

§2. The Proof of Theorem 1

Let us collect the most useful formulas we can use, relating the operator R and $k(x, y) := R(x, y, x, y)$.

Proposition 2.1. *Let V be a linear vector space and $R : V \times V \times V \times V \to \mathbb{R}$ a 4-linear map satisfying (1.1), (1.2) and (1.3) for any four vectors $x, y, z, t \in V$. Then for any vectors $x, y, z, t \in V$, we have (1.4), (1.5), (1.6) and*

$$4R(x, y, z, y) = k(x + z, y) - k(x - z, y) \quad \text{(due to Karcher)} \tag{2.1}$$

$$2R(x, y, z, y) = k(x + z, y) - k(x, y) - k(z, y) \tag{2.2}$$

$$\begin{aligned} 6R(x, y, z, t) =& R(x, y + t, z, y + t) - R(x, y - t, z, y - t) \\ & - R(y, x + t, z, x + t) + R(y, x - t, z, x - t) \\ & \text{(due to Karcher)} \end{aligned} \tag{2.3}$$

and the Bianchi Identity expressed by sectional curvatures:

$$\begin{aligned} & k(x + y, z + t) + k(x - y, z - t) - k(x + y, z - t) - \\ & k(x - y, z + t) + k(x + z, y + t) + k(x - z, y - t) - \\ & k(x + z, y - t) - k(x - z, y + t) + k(x + t, y + z) + \\ & k(x - t, y - z) - k(x + t, y - z) - k(x - t, y + z) = 0, \end{aligned} \tag{2.4}$$

Note that the formula (1.4) can be obtained from (2.1) and (2.3). We will give a different proof of (1.4), because this computation will allow us to obtain (2.4). Note also that (2.1), (2.2) and (2.3) are trivially obtained from the 4-linearity of R.

Proof of Proposition 2.1. Since

$$R(x, y+t, z, y+t) = R(x, y, z, y) + R(x, t, z, t) + R(x, y, z, t) + R(x, t, z, y),$$

then

$$\begin{aligned} R(x, y, z, t) + R(x, t, z, y) = & R(x, y + t, z, y + t) \\ & - R(x, y, z, y) - R(x, t, z, t) \end{aligned} \tag{2.5}$$

In (2.5) change $y \to t, t \to z, z \to y$, and we get

$$R(x,t,y,z) + R(x,z,y,t) = R(x,t+z,y,t+z) - R(x,t,y,t) - R(x,z,y,z). \tag{2.6}$$

Thus 2(2.5)+(2.6)+(1.1) gives:

$$\begin{aligned}3R(x,y,z,t) = {} & 2[R(x,y+t,z,y+t) - R(x,y,z,y) - R(x,t,z,t)] \\ & + R(x,t+z,y,t+z) - R(x,t,y,t) - R(x,z,y,z).\end{aligned} \tag{2.7}$$

Using (2.1) in (2.7) we have

$$\begin{aligned}12R(x,y,z,t) = {} & 2[k(x+z,y+t) - k(x-z,y+t) - k(x+z,y) \\ & + k(x-z,y) - k(x+z,t) + k(x-z,t)] \\ & + k(x+y,t+z) - k(x-y,t+z) - k(x+y,t) \\ & + k(x-y,t) - k(x+y,z) + k(x-y,z)\end{aligned} \tag{2.8}$$

Now apply (1.5) to the first six terms in (2.8) and get

$$\begin{aligned}& 2[k(x+z,y+t) - k(x-z,y+t) - k(x+z,y) + k(x-z,y) \\ & \quad - k(x+z,t) + k(x-z,t)] \\ & = 2[k(x+z,y+t) - k(x-z,y+t)] - k(x+z,y+t) \\ & \quad - k(x+z,y-t) + k(x-z,y+t) + k(x-z,y-t) \\ & = k(x+z,y+t) + k(x-z,y-t) - k(x+z,y-t) \\ & \quad - k(x-z,y+t).\end{aligned} \tag{2.9}$$

Now applying (1.5) to the last six terms in (2.8) we obtain

$$\begin{aligned}& k(x+y,t+z) - k(x-y,t+z) - k(x+y,t) + k(x-y,t) \\ & \quad - k(x+y,z) + k(x-y,z) \\ & = k(x+y,t+z) - k(x-y,t+z) + \\ & \quad \frac{1}{2}[-k(x+y,t+z) - k(x+y,t-z) + k(x-y,t+z) \\ & \quad + k(x-y,t-z)] \\ & = \frac{1}{2}[k(x+y,t+z) + k(x-y,t-z) - k(x+y,t-z) \\ & \quad - k(x-y,t+z)].\end{aligned} \tag{2.10}$$

By replacing (2.9) and (2.10) in (2.8), we arrive at

$$\begin{aligned}24R(x, y, z, t) = 2[&k(x+z, y+t) + k(x-z, y-t)\\ &- k(x+z, y-t) - k(x-z, y+t)]\\ &+ k(x+y, z+t) + k(x-y, z-t)\\ &- k(x-y, z+t) - k(x+y, t-z),\end{aligned} \tag{2.11}$$

Now applying (1.1) to (2.11) we arrive directly to the Bianchi identity (2.4). By (2.4) and (2.11) we obtain (1.4). So Proposition 2.1 is proved. □

Proof of Theorem 1. It is immediate to verify that R defined by (1.4) satisfies (1.1), (1.2) and (1.3). Then we only need to show that $R(x, y, x, y) = k(x, y)$ and that R is a 4-linear operator.

Claim 1. $k(x+y, x-y) = 4k(x, y)$.

In fact, by (1.5) we obtain

$$k(x, x+y) + k(x, x-y) = \frac{1}{2}[k(x, 2x) + k(x, 2y)] = 2k(x, y), \tag{2.12}$$

and

$$k(y, x+y) + k(y, x-y) = 2k(x, y). \tag{2.13}$$

By summing (2.12) and (2.13) and using again (1.5) we have

$$\begin{aligned}&\frac{1}{2}[k(x+y, x+y) + k(x-y, x+y) + k(x+y, x-y)\\ &\qquad + k(x-y, x-y)] = k(x+y, x-y) = 4k(x, y),\end{aligned}$$

and Claim 1 is proved.

Claim 2. $R(x, y, x, y) = k(x, y)$.

In fact, by (1.4) and Claim 1 we have

$$24R(x, y, x, y) = 16k(x, y) + k(x+y, y-x) + k(x-y, y+x) = 24k(x, y).$$

Claim 3. *(2.1) holds.*

Indeed, by (1.4) we have

$$24R(x, y, z, y) = 4k(x+z, y) - 4k(x-z, y) - k(x+y, y+z) - k(x-y, y-z) + k(x+y, y-z) + k(x-y, y+z). \tag{2.14}$$

By applying Claim 1 to each one of the last four terms in (2.14) we get

$$24R(x, y, z, y) = 4k(x+z, y) - 4k(x-z, y) + \frac{1}{4}[-k(x+2y+z, x-z) - k(x-z, x-2y+z) + k(x+2y-z, x+z) + k(x+z, x-2y-z)].$$

Now apply (1.5) and obtain

$$\begin{aligned} 24R(x, y, z, y) &= 4k(x+z, y) - 4k(x-z, y) + \frac{1}{2}[-k(x-z, 2y) - k(x-z, x+z) + k(x+z, 2y) + k(x+z, x-z)] \\ &= 6k(x+z, y) - 6k(x-z, y), \end{aligned}$$

And (2.1) follows.

Claim 4. *(2.3) holds.*

It suffices to apply (2.1) to (1.4), obtaining immediately (2.3).

Claim 5. *$R(x, y, z, y)$ depends linearly on z.*

By (1.6) and (2.1) we have $R(x, y, \lambda z, y) = \lambda R(x, y, z, y)$. By (2.1) we have

$$4R(x, y, z, y) = k(x+z, y) - k(x-z, y) \tag{2.15}$$

and

$$4R(x, y, a, y) = k(x+a, y) - k(x-a, y). \tag{2.16}$$

We sum (2.15) and (2.16), apply (1.5) and (2.1), and get

$$\begin{aligned} 4[R(x, y, z, y) + R(x, y, a, y)] &= \frac{1}{2}[k(2x + z + a, y) + k(z - a, y) \\ &\quad - k(2x - z - a, y) - k(a - z, y)] \\ &= 2R(2x, y, z + a, y) = 4R(x, y, z + a, y), \end{aligned}$$

So Claim 5 is proved. Now Claim 5 together with (2.3) imply that $R(x, y, z, t)$ depends linearly on z. Thus Theorem 1 is proved. □

§3. The Proof of Theorem 2 and Corollary 3

The curvature tensor can be expressed as

$$\Omega_{ij} = \sum_{k\ell} R_{ijk\ell} w_k \wedge w_l.$$

It follows from Propostion 4.2 in [CT] that if Σ is minimal, then

$$\chi(T\Sigma) + \chi(N\Sigma) = \int_\Sigma \Omega_\Sigma + \int_\Sigma |H|^2 - \frac{1}{2}\int_\Sigma |\nabla J_\Sigma|^2, \tag{3.1}$$

where $\Omega_\Sigma = \Omega_{12} + \Omega_{34}$, and H is the mean curvature vector.

Proof of Theorem 2. We will prove for the case of nonpositive pinching. The other case is completely similar. Since

$$\int_\Sigma \Omega_\Sigma = \int_\Sigma \big(K(e_1, e_2) + R(e_1, e_2, e_3, e_4)\big) dA, \tag{3.2}$$

and

$$\begin{aligned} &K(e_1, e_2) + R(e_1, e_2, e_3, e_4) = \\ &K(e_1, e_2) + \frac{1}{6}[K(e_1 + e_3, e_2 + e_4) + K(e_1 - e_3, e_2 - e_4) + \\ &K(e_1 + e_4, e_2 - e_3) + K(e_1 - e_4, e_2 + e_3) - K(e_1 + e_3, e_2 - e_4) \\ &- K(e_1 - e_3, e_2 + e_4) - K(e_1 + e_4, e_2 + e_3) - K(e_1 - e_4, e_2 - e_3)] \\ &\le -a^2(x) - \frac{2}{3}a^2(x) + \frac{2}{3}\cdot\frac{5}{2}a^2(x) = 0, \end{aligned} \tag{3.3}$$

hence (1.1) is proved. Notice that the equality holds in (3.3) if and only if $a = 0$ or

$$\begin{aligned} K(e_1, e_2) &= K(e_1 + e_3, e_2 + e_4) \\ &= K(e_1 - e_3, e_2 - e_4) \\ &= K(e_1 - e_4, e_2 + e_3) \\ &= K(e_1 + e_4, e_2 - e_3) = -a^2, \end{aligned}$$

and

$$\begin{aligned} K(e_1 - e_3, e_2 + e_4) &= K(e_1 + e_3, e_2 - e_4) \\ &= K(e_1 + e_4, e_2 + e_3) \\ &= K(e_1 - e_4, e_2 - e_3) \\ &= -\frac{5}{2}a^2. \end{aligned}$$

So the pinching constant is $\frac{5}{2}$ and the proof is complete. □

Proof of Corollary 3. By [CT], p. 883, we have

$$\begin{aligned} &\chi(T\Sigma) + \chi(N\Sigma) \\ &= \int_\Sigma (\Omega_{12} + \Omega_{34} + w_{13} \wedge w_{23} + w_{14} \wedge w_{24} + w_{13} \wedge w_{14} + w_{23} \wedge w_{24}) \\ &= \int_\Sigma \big(\Omega_{12} + \Omega_{34} + w_{13} \wedge (w_{23} + w_{14}) + (w_{14} + w_{23}) \wedge w_{24}\big) \\ &= \int_\Sigma \big(\Omega_{12} + \Omega_{34} + w_{13} \wedge (w_{23} + w_{14}) - w_{24} \wedge (w_{14} + w_{23})\big) \\ &= \int_\Sigma \big(\Omega_{12} + \Omega_{34} + (w_{13} - w_{24}) \wedge (w_{14} + w_{23})\big). \end{aligned}$$

So any of the conditions of Corollary 3 implies that

$$\chi(T\Sigma) + \chi(N\Sigma) = \int_\Sigma (\Omega_{12} + \Omega_{34}),$$

and the same proof of Theorem 2 applies. □

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