Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol.32, No. 3, 303-319

On a one-dimensional version of the dynamical Marguerre-Vlasov system

G. Perla Menzala and Enrique Zuazua

— Dedicated to Constantine Dafermos on his 60th birthday

Abstract. A one-dimensional version of the so-called Marguerre-Vlasov system of equations describing the vibrations of shallow shells is considered. The system depends on a parameter $\epsilon \to 0$ in a singular way and undergoes the effect of damping mechanisms. We show that the system converges to a nonlinear beam equation while the energy decays exponentially uniformly (on $\epsilon \to 0$) as time goes to infinity.

Keywords: singular limit, Marguerre-Vlasov system, uniform stabilization.

Mathematical subject classification: 35B40, 73C50, 73K15.

1 Introduction

We consider a one-dimensional version of the so-called dynamical Marguerre-Vlasov system which describes the vibrations of shallow shells (see [8] and [9]).

The damped one-dimensional system reads as follows

$$\begin{cases} u_{tt} = \dfrac{2}{1-\mu}\left[u_x + \dfrac{1}{2}w_x^2 + k_1(x)w\right]_x - u_t \\ w_{tt} + w_{xxxx} - w_{xxtt} = [f(u,w)]_x - g(u,w) - w_t + w_{xxt} \end{cases} \tag{1.1}$$

where

$$f(u,w) = \frac{2}{1-\mu}\left[w_x\left(u_x + \frac{1}{2}w_x^2 + k_1(x)w\right)\right] \tag{1.2}$$

Received 29 October 2001.

and

$$g(u,w)=\frac{2k_1}{1-\mu}\left[u_x+\frac{1}{2}w_x^2+k_1(x)w\right]. \tag{1.3}$$

In (1.1), the space variable x runs in the interval $0<x<L$ and t denotes the (positive) time variable. The quantities $u=u(x,t)$ and $w=w(x,t)$ represent, respectively, the longitudinal and transversal displacements of the beam at the point x at time t. Additionally, μ is a constant, $0<\mu<1$ and $k_1=k_1(x)$ represents the curvature of the beam at the point x.

The terms $-u_t$ (resp. $-w_t+w_{xxt}$) of the first (resp. second equation) in (1.1) constitute damping mechanisms that dissipate the energy of solutions as time increases.

This work is devoted to analyze the following two questions:

a) Under a suitable perturbation of the system above in which the various constants are conveniently scaled, we investigate the proximity of the component w in (1.1) to the solution $z=z(x,t)$ of a scalar beam equation of Timoshenko's type.

b) The uniform (with respect to $\epsilon\to 0$) rate of decay of the total energy of the solutions of (1.1) as $t\to+\infty$.

To be more precise, given $\epsilon>0$ and $0\le\alpha\le 1$ we consider $u=u^\epsilon$, $w=w^\epsilon$ the solution of the coupled system of equations

$$\begin{cases} \epsilon u_{tt}=\dfrac{2}{1-\mu}\left[u_x+\dfrac{1}{2}w_x^2+k_1(x)w\right]_x-\epsilon^\alpha u_t \\ w_{tt}+w_{xxxx}-w_{xxtt}=[f(u,w)]_x-g(u,w)-w_t+w_{xxt} \end{cases} \tag{1.4}$$

where f and g are given as in (1.2) and (1.3).

Once again, in (1.4) the variable x runs in the interval $0<x<L$ and $t>0$. We consider (1.4) with Dirichlet boundary conditions on u and clamped ends for w:

$$u(0,t)=u(L,t)=0,\quad \forall t>0 \tag{1.5}$$
$$w(0,t)=w(L,t)=w_x(0,t)=w_x(L,t)=0,\quad \forall t>0$$

and initial conditions at $t=0$:

$$(u(0),u_t(0),w(0),w_t(0))=(u_0,v_0,w_0,w_1)\in H, \tag{1.6}$$

where H is the *energy space*

$$H = H_0^1(I) \times L^2(I) \times H_0^2(I) \times H_0^1(I),$$

with $I = (0, L)$.

Problem (1.4)-(1.6) is globally well posed in the above space provided $k_1 \in H^1(I)$.

Moreover, the total energy associated with (1.4)-(1.6) is given by

$$E_\epsilon(t) = \frac{1}{2}\int_0^L \left[\epsilon u_t^2 + \frac{2}{1-\mu}\left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)^2 + w_t^2 + w_{xt}^2 + w_{xx}^2\right]dx, \quad (1.7)$$

and it is dissipated according to the law

$$\frac{d}{dt}E_\epsilon(t) = -\int_0^L [\epsilon^\alpha u_t^2 + w_t^2 + w_{xt}^2]dx. \quad (1.8)$$

According to this, in particular, $w = w^\epsilon$ in uniformly bounded in $L^\infty(0, \infty; H_0^2(0, L))$.

The first result of this paper guarantees that, as $\epsilon \to 0$, the component w^ϵ of the solution converges in the weak-* topology of that space to the solution z of the equation

$$z_{tt} + z_{xxxx} - z_{xxtt} = h(t)z_{xx} - z_t + z_{xxt} - k_1 h(t) \quad (1.9)$$

where

$$h(t) = \frac{1}{1-\mu}\left[\frac{1}{L}\int_0^L \left(z_x^2 + 2k_1 z\right)dx\right] \quad (1.10)$$

together with the boundary and initial conditions

$$\begin{cases} z(0,t) = z(L,t) = z_x(0,t) = z_x(L,t) = 0 \quad \forall t > 0 \\ z(x,0) = w_0(x), z_t(x,0) = w_1(x), \quad 0 < x < L. \end{cases} \quad (1.11)$$

In what concerns the second question related to the uniform decay rate of solutions, we prove that there exist positive constants $c > 0$ and $\beta > 0$ such that.

$$E_\epsilon(t) \le CE_\epsilon(0)\exp\left(-\frac{\beta t}{1 + \epsilon^\alpha\left[E_\epsilon(0) + \|k_1\|_\infty^2\right]}\right) \quad (1.12)$$

for all $t \ge 0$ where $0 \le \alpha \le 1$.

These problems have been previously considered by the authors in [6] (together with A. Pazoto) and [7] in the context of the classical von Kármán system for the vibrations of a beam. There, it was proved that:

a) Timoshenko's beam model may be derived as a singular limit of the Von Kármán beam model,

b) A similar uniform (as $\epsilon \to 0$) exponential decay rate as $t \to \infty$ of the energy of solutions holds.

Therefore, in this paper we extend these results to the 1-D model of the so-called Marguerre-Vlasov system for shallow shells.

As far as we know, model (1.9), which is a ''perturbed'' Timoshenko's type equation, has not been studied before. However, it can be easily handeled by the by now classical methods, as a perturbation of the classical Timoshenko beam equation.

Our notations are standard and can be found in the book of J.L.Lions [4]

2 Global well-posedness

In this section, for the sake of completeness we analyse the problem of the existence and uniqueness of solutions of system (1.4)-(1.6).

Let $\epsilon > 0$, $0 < \mu < 1$ and $\alpha \geq 0$ and consider the Hilbert space

$$H = H_0^1(I) \times L^2(I) \times H_0^2(I) \times H_0^1(I)$$

where $I = \{0 < x < L\}$ endowed with the norm

$$\|(u, y, w, p)\|_H^2 = \frac{2}{1-\mu}\|u_x\|^2 + \varepsilon\|y\|^2 + \|w_{xx}\|^2 + \|p\|^2 + \|p_x\|^2$$

for any $(u, y, w, p) \in H$. Here $\| \cdot \|$ denotes the norm in $L^2(I)$.

We write problem (1.4)-(1.6) in the abstract form

$$\begin{cases} DU_t = AU + N(U) \\ U(0) = U_0 = (u_0, u_1, w_0, w_1) \in H \end{cases} \tag{2.1}$$

where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 - \frac{\partial^2}{\partial x^2} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{1-\mu}\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{\partial^4}{\partial x^4} & 0 \end{bmatrix}$$

$$N(U) = \begin{bmatrix} 0 \\ \dfrac{2}{1-\mu}\left(\dfrac{1}{2}w_x^2 + k_1 w\right)_x - \epsilon^\alpha y \\ 0 \\ m(U) \end{bmatrix}$$

where

$$m(U) = \frac{2}{1-\mu}\left[w_x\left(4_x + \frac{1}{2}w_x^2 + k_1 w\right)\right]_x - \frac{2k_1}{1-\mu}\left[u_x + \frac{1}{2}w_x^2 + k_1 w\right] - w_t + w_{xxt}$$

and $U = (u, y, w, p)^\tau$.

The operator $\widetilde{A} = D^{-1}A$ with domain $D(\widetilde{A}) = (H^2 \cap H_0^1)(I) \times H_0^1(I) \times (H^3 \cap H_0^2)(I) \times H_0^2(I)$ is the infinitesimal generator of a semigroup of operators in H.

A direct calculation shows that, for any $U \in D(\widetilde{A})$ we have that

$$\langle \widetilde{A}U, U \rangle_H = \frac{2}{1-\mu}(y_x, u_x) + \frac{2}{1-\mu}(u_{xx}, y) + (p_{xx}, w_{xx})$$
$$-\left(\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1} \frac{\partial^4}{\partial x^4} w, p\right) - \left(\frac{\partial}{\partial x}\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1} \frac{\partial^4}{\partial x^4} w, p_x\right)$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(I)$.

Integrating by parts and observing that the term (p_{xx}, w_{xx}) can be written as $\left(p, \left(I - \frac{\partial^2}{\partial x^2}\right)\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1} \frac{\partial^4}{\partial x^4} w\right)$ we get

$$\langle \widetilde{A}U, U \rangle_H = 0. \tag{2.2}$$

Now, given $G = (g_1, g_2, g_3, g_4)^\tau \in H$, we claim that the system

$$\widetilde{A}U = G \tag{2.3}$$

admits a unique solution $U \in D(\widetilde{A})$. This is equivalent to finding $(u, y, w, p) \in H$ such that

a) $y = g_1$,

b) $\dfrac{2}{(1-\mu)\epsilon} u_{xx} = g_2$ with $u(0) = u(L) = 0$,

c) $p = g_3$

and

d) $-\left(I - \frac{\partial^2}{\partial_x 2}\right)^{-1} \frac{\partial^4}{\partial_x^4} w = g_4$ with $w = w_x = 0$ at $x = 0, L$.

Clearly, b) admits a unique solution $u \in H^2 \cap H_0^1(I)$ since $g_2 \in L^2(I)$. Problem d) is equivalent to

$$\frac{\partial^4}{\partial x^4} w = -\left(I - \frac{\partial^2}{\partial x^2}\right) g_4 \text{ in } 0 < x < L, \qquad w = w_x = 0 \text{ at } x = 0, L$$

which admits a unique solution $w \in H^3 \cap H_0^1(I)$ because $\left(I - \frac{\partial^2}{\partial x^2}\right) g_4 \in H^{-1}(I)$.

Thus, $\widetilde{A}$ is indeed the infinitesimal generator of a semigroup of operators in H.

In order to prove local existence of problem (1.4)-(1.6) it is enough to prove that $D^{-1}N(U)$ is locally Lipschitz continuous in H.

Let $U = (u, y, w, p)^\tau$ and $\tilde{U} = (\tilde{u}, \tilde{y}, \tilde{w}, \tilde{p})^\tau$ be elements of H. A direct calculation shows that

$$D^{-1}[N(U) - N(\tilde{U})] = (0, \tilde{f}, 0, \tilde{g})^\tau$$

where

$$\tilde{f} = \frac{2}{(1-\mu)\epsilon}\left[\frac{1}{2}(w_x^2 - \tilde{w}_x^2) + k_1(w - \tilde{w})\right]_x + \epsilon^{\alpha-1}(\tilde{y} - y)$$

and

$$\tilde{g} = \left(I - \frac{\partial^2}{\partial x^2}\right)^{-1}\left\{\frac{2}{1-\mu}\left[(u_x + \frac{1}{2}w_x^2 + k_1 w)w_x - \left(\tilde{u}_x + \frac{1}{2}\tilde{w}_x^2 + k_1\tilde{w}\right)\tilde{w}_x\right]_x \right.$$
$$\left. - (p - \tilde{p}) + (p_{xx} - \tilde{p}_{xx}) + \frac{2k_1}{1-\mu}\left[\tilde{u}_x + \frac{1}{2}\tilde{w}_x^2 + k_1\tilde{w} - u_x - \frac{1}{2}w_x^2 - k_1 w\right]\right\}.$$

We have to estimate the expression

$$\|D^{-1}[N(U) - N(\tilde{U})]\|_H^2 = \epsilon\|\tilde{f}\|^2 + \|\tilde{g}\|^2 + \|\tilde{g}_x\|^2.$$

Assuming that $k_1 \in H^1(I)$ we can easily prove that

$$\begin{aligned}
\|\tilde{f}\|^2 \;\leq\; & \frac{C}{\epsilon^2}\{\|w_x - \tilde{w}_x\|_\infty^2(\|w_{xx}\| + \|\tilde{w}_{xx}\|)^2 \\
+ \; & \|w_{xx} - \tilde{w}_{xx}\|^2(\|w_x\|_\infty + \|\tilde{w}_x\|_\infty)^2\} + C\epsilon^{2(\alpha-1)}\|y - \tilde{y}\|^2 \\
+ \; & \frac{C}{(1-\mu)^2\epsilon^2}\|k_1\|_{H^1}^2\|w_x - \tilde{w}_x\|^2.
\end{aligned}$$

Using the embedding $H^1(I) \hookrightarrow L^\infty(I)$ we deduce from the above estimate that

$$\|\tilde{f}\| \leq C(1 + \|U\|_H + \|\tilde{U}\|_H)\|U - \tilde{U}\|_H, \tag{2.4}$$

where C is a positive constant depending on ϵ, μ, α and $\|k_1\|_{H^1}$.

Now, let us estimate $\|\tilde{g}\|_{H^1(I)}$. First, let

$$g_1 = \frac{2}{1-\mu}\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1}\left[\left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)w_x - \left(\tilde{u}_x + \frac{1}{2}\tilde{w}_x^2 + k_1\tilde{w}\right)\tilde{w}_x\right]_x$$

Taking into account that the operator $\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1}\frac{\partial}{\partial x}$ is bounded from $L^2(I)$ into $H_0^1(I)$ we deduce that

$$\|g_1\|_{H^1} \leq C\left\|\left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)w_x - \left(\tilde{u}_x + \frac{1}{2}\tilde{w}_x^2 + k_1\tilde{w}\right)\tilde{w}_x\right\|. \tag{2.5}$$

Adding and substracting the term $\left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)\tilde{w}_x$ inside the norm on the right hand side of (2.5) it is easy to see that

$$\|g_1\|_{H^1} \quad \leq \quad C(\mu, \epsilon, \|k_1\|_{H^1})(\|U\|_H + \|\tilde{U}\|_H)\|U - \tilde{U}\|_H.$$

Finally, let g_2 be given by

$$\begin{aligned} g_2 = \left(I - \frac{\partial^2}{\partial x^2}\right)^{-1}\Bigg\{ & -(p - \tilde{p}) + p_{xx} - \tilde{p}_{xx} + \\ & + \frac{2k_1}{1-\mu}\left[\tilde{u}_x + \frac{1}{2}\tilde{w}_x^2 + k_1\tilde{w} - u_x - \frac{1}{2}w_x^2 - k_1 w\right]\Bigg\}. \end{aligned}$$

A similar discussion allows to show that

$$\|g_2\|_{H^1} \leq C(\mu, \epsilon, \|k_1\|_{H^1})(\|U\|_H + \|\widetilde{U}\|_H)\|U - \widetilde{U}\|_H. \tag{2.6}$$

From (2.4), (2.5) and (2.6) we deduce that

$$\|D^{-1}[N(U) - N(\widetilde{U})]\|_H \leq C(1 + \|U\|_H + \|\widetilde{U}\|_H)\|U - \widetilde{U}\|_H$$

where C is a positive constant depending on ε, μ, α and $\|k_1\|_{H^1}$. This proves that $D^{-1}N(U)$ is locally Lipschitz continuous in H.

Consequently, one obtains local existence of a unique finite energy solution.

Global existence in our case is consequence of the energy identity (1.8) which provides a priori bounds in the energy space for all $t \geq 0$.

We have shown:

Theorem 2.1. *Let* $\epsilon > 0$, $0 \leq \alpha$, $0 < \mu < 1$, $k_1 \in H^1(I)$ *and* $(u_0, u_1, w_0, w_1) \in H$. *Then, problem* (1.4)–(1.6) *has a unique global (weak) solution*

$$(u^\epsilon, u_t^\epsilon, w^\epsilon, w_t^\epsilon) \in C([0, +\infty); H)$$

and the total energy $E_\epsilon(t)$ *given by* (1.7) *satisfies* (1.8) *for all* $t \geq 0$.

3 The asymptotic limit

In this section we study the asymptotic limit of the solution $\{u^\epsilon, w^\epsilon\}$ of (1.4)-(1.6) es $\epsilon \to 0^+$.

Let $\epsilon > 0$, $0 < \alpha$ and $0 < \mu < 1$.

From the energy dissipation law (1.8) that guarantees that $E_\epsilon(t) \leq E_\epsilon(0)$ for all $t \geq 0$ and all ϵ, we deduce that the sequences

$$\{\sqrt{\epsilon}u_t^\epsilon\}, \left\{u_x^\epsilon + \frac{1}{2}(w_x^\epsilon)^2 + k_1 w^\epsilon\right\}, \{w_t^\epsilon\}, \{w_{xt}^\epsilon\} \text{ and } \{w_{xx}^\epsilon\}$$

are bounded in $L^\infty(0, +\infty; L^2(I))$ and

$$\{\epsilon^{\alpha/2}u_t^\epsilon\}, \{w_t^\epsilon\} \text{ and } \{w_{xt}^\epsilon\}$$

are bounded in $L^2(0, +\infty; L^2(I))$.

Extracting subsequences (that we still denote by the index ε in order to simplify notations) we deduce that there exist $\xi(x, t)$, $\eta(x, t)$ and $z(x, t)$ such that

$$\sqrt{\epsilon}u_t^\epsilon \rightharpoonup \xi \quad \text{weakly * in } L^\infty(0, +\infty; L^2(I)) \tag{3.1}$$

$$u_x^\epsilon + \frac{1}{2}\left(w_x^\epsilon\right)^2 + k_1 w^\epsilon \rightharpoonup \eta \quad \text{weakly * in } L^\infty(0, +\infty; L^2(I)) \tag{3.2}$$

and

$$w^\varepsilon \rightharpoonup z \quad \text{weakly * in } L^\infty(0, +\infty; H^2(I)) \cap W^{1,\infty}(0, +\infty; H_0^1(I)) \tag{3.3}$$

as $\epsilon \to 0$.

Clearly, the weak convergence in (3.3) is enough to allow us to pass to the limit in the linear part of the equation for w^ϵ in (1.4) provided, say, $k_1 \in L^\infty(I)$.

It remains to identify the weak limit of the nonlinear terms $\left\{u_x^\epsilon + \frac{1}{2}\left(w_x^\epsilon\right)^2\right\}$ and $\left[\left(u_x^\epsilon + \frac{1}{2}\left(w_x^\epsilon\right)^2 + k_1 w^\epsilon\right)w_x^\epsilon\right]_x$ as $\epsilon \to 0$.

As we said above, the boundedness of $E_\epsilon(t)$ implies that $\{w^\epsilon\}_{\epsilon>0}$ is uniformly bounded in $L^\infty(0,\infty; H_0^2(I)) \cap W^{1,\infty}(0,+\infty; H_0^1(I))$. Then, we can use Aubin-Lions compactness lemma [4] to deduce that

$$w^\epsilon \to z \quad \text{strongly in } L^\infty(0,T; H^{2-\delta}(I)) \tag{3.4}$$

as $\varepsilon \to 0$ for any $\delta > 0$ and $T < +\infty$.

Combining (3.2) with (3.4) we deduce that

$$\left(u_x^\epsilon + \frac{1}{2}\left(w_x^\epsilon\right)^2 + k_1 w^\epsilon\right) w_x^\epsilon \rightharpoonup \eta z_x \quad \text{weakly in } L^2(I \times (0,T))$$

as $\epsilon \to 0$ for any $T < +\infty$.

Let us find out what the value of η is. We claim that

a) η is independent of x and

b) η is given by

$$\eta = \frac{1}{2L}\int_0^L z_x^2 dx + \frac{1}{L}\int_0^L k_1 z dx.$$

To see this we first observe that $\left\{u_x^\epsilon\right\}$ is bounded in $L^2(I \times (0,T))$ since

$$\begin{aligned}
\int_0^L \left(u_x^\epsilon\right)^2 dx &= \int_0^L \left[u_x^\epsilon + \frac{1}{2}\left(w_x^\epsilon\right)^2 + k_1 w^\epsilon - \frac{1}{2}\left(w_x^\epsilon\right)^2 - k_1 w^\epsilon\right]^2 dx \\
&\leq C\left[E_\epsilon(0) + \int_0^L \left(w_x^\epsilon\right)^4 dx + \int_0^L k_1^2 \left(w^\epsilon\right)^2 dx\right] \\
&\leq C\left[E_\epsilon(0) + \left(\int_0^L \left(w_{xx}^\epsilon\right)^2 dx\right)^2 + \|k_1\|_\infty^2 \int_0^L \left(w^\epsilon\right)^2 dx\right] \\
&\leq C E_\epsilon(0)
\end{aligned}$$

for some positive constant C depending on the initial energy $E_\epsilon(0)$ and k_1. Obviously, this constant is independent of ϵ.

Thus, there exists a subsequence such that

$$u_x^\epsilon \rightharpoonup \rho \quad \text{weakly in } L^2(I \times (0,T)) \tag{3.5}$$

as $\epsilon \to 0$ for some $\rho = \rho(x,t)$. Using (3.4) and (3.5) we deduce that

$$u_x^\epsilon + \frac{1}{2}\left(w_x^\epsilon\right)^2 + k_1 w^\epsilon \rightharpoonup \rho + \frac{1}{2} z_x^2 + k_1 z = \eta \tag{3.6}$$

as $\epsilon \to 0$, weakly in $L^2(I \times (0,T))$.

Since $\alpha > 0$, using Poincaré's inequality and (1.8) we can bound $\{u^\epsilon\}$ in $L^\infty(0,T;H_0^1(I))$ to obtain that

$$\epsilon^\alpha u_t^\epsilon \rightharpoonup 0 \quad \text{weakly in } H^{-1}(0,T;H_0^1(I)) \tag{3.7}$$

as $\epsilon \to 0$. Now, using (3.1) we also know that

$$\epsilon u_{tt}^\epsilon = \sqrt{\epsilon}\sqrt{\epsilon} u_{tt}^\epsilon \rightharpoonup 0 \quad \text{weakly in } H^{-1}(0,T;L^2(I)) \tag{3.8}$$

as $\epsilon \to 0$. Thus, from the first equation in (1.4), (3.2) (3.7) and (3.8) we obtain that

$$\eta_x = \left[\rho + \frac{1}{2}z_x^2 + k_1 z\right]_x = 0$$

therefore, $\eta = \eta(t)$ which proves claim a).

To prove item b) we integrate the identity $\eta = \rho + \frac{1}{2}z_x^2 + k_1 z$ in x from $x = 0$ up to $x = L$ to obtain

$$L\eta(t) = \int_0^L \rho dx + \frac{1}{2}\int_0^L z_x^2 dx + \int_0^L k_1 z dx = \frac{1}{2}\int_0^L z_x^2 dx + \int_0^L k_1 z dx$$

since $\int_0^L \rho dx = \lim_{\epsilon\to 0}\int_0^L u_x^\epsilon dx = 0$, because u^ϵ vanishes at the boundary $x = 0, L$. Consequently,

$$\eta(t) = \frac{1}{2L}\int_0^L z_x^2 dx + \frac{1}{L}\int_0^L k_1 z dx.$$

The above discussion indicates that

$$\left[\left(u_x^\epsilon + \frac{1}{2}\left(w_x^\epsilon\right)^2 + k_1 w^\epsilon\right)w_x^\epsilon\right]_x \rightharpoonup \left(\frac{1}{2L}\int_0^L z_x^2 dx + \frac{1}{L}\int_0^L k_1 z dx\right) z_{xx}$$

as $\epsilon \to 0$, weakly in $L^2(0,T;H^{-1}(I))$.

We conclude that the component w^ϵ in system (1.4)-(1.6) converges to the solution $z = z(x,t)$ of (1.9) weakly in $L^2(0,T;H_0^2(I))$ as $\epsilon \to 0$ for any $T < +\infty$.

Clearly z satisfies the boundary conditions in (1.11).

Finally we want to identify the initial data of the limit system. Since $w^\epsilon \to z$ in $C([0,T];H^{2-\delta}(I))$ as $\epsilon \to 0$ for any $T < +\infty$ then $w^\epsilon(x,0) \to z(x,0)$ as $\epsilon \to 0$ in $H^{2-\delta}(I)$. Hence $z(x,0) = w_0(x)$. Observing that

$$\left\{w_t^\varepsilon\right\} \text{ is bounded in } L^\infty(0,T;H_0^1(I))$$
$$\left\{w_{tt}^\varepsilon\right\} \text{ is bounded in } L^\infty(0,T;L^2(I))$$

for any $T < +\infty$ (the last bound is easily obtained using the equation in (1.4) that w^ϵ satisfies and our previous discussion), from (3.9) and using Aubin-Lions compactness lemma [4] it follows that $w_t^\epsilon \to z_t$ in $C([0, T]; L^2(I))$ as $\epsilon \to 0$. In particular, $w_t^\epsilon(x, 0) \to z_t(x, 0)$ as $\epsilon \to 0$ in $L^2(I)$. Hence $z_t(x, 0) = w_1(x)$.

The above results can be summarized as follows.

Theorem 3.1. *Let* $(u_0, u_1, w_0, w_1) \in H = H_0^1(I) \times L^2(I) \times H_0^2(I) \times H_0^1(I)$, $0 < \mu < 1, \alpha > 0$ *and* $k_1 \in H^1(I)$. *Consider the global solution* u^ϵ, w^ϵ *of system* (1.4)–(1.6) *obtained in Theorem* 2.1. *Then, as* $\epsilon \to 0^+$,

$$w^\epsilon \rightharpoonup z \quad \text{weakly in} \quad L^2(0, T; H_0^2(I))$$

Furthermore,

$$U_x^\epsilon \rightharpoonup \frac{1}{2L}\int_0^L (z_x^2 + 2k_1 z)dx - \frac{1}{2}z_x^2 - k_1 z$$

weakly in $L^2(I \times (0, T))$ *as* $\epsilon \to 0$ *for any* $T < +\infty$, *where* $z = z(x, t)$ *is the global (weak) solution of problem* (1.9)–(1.11).

4 Uniform stabilization as $\varepsilon \to 0$

The total energy of the limit system (1.9)–(1.11) is given by

$$G(t) = \frac{1}{2}\int_0^L \left(z_t^2 + z_{xx}^2 + z_{xt}^2\right) dx + \frac{1}{(1-\mu)L}\left(\frac{1}{2}\int_0^L z_x^2 dx + \int_0^L k_1 z dx\right)^2$$

and it is dissipated according to the law

$$\frac{dG(t)}{dt} = -\int_0^L (z_t^2 + z_{xt}^2)dx.$$

Then, it is not difficult to prove that $G(t)$ decays exponentially as $t \to +\infty$.

In this Section we prove that the energy $E_\epsilon(t)$ associated to problem (1.4)–(1.6) also decays exponentially as $t \to \infty$ and that the decay rate is uniform (as $\epsilon \to 0$) provided $0 \le \alpha \le 1$, recovering the rate of decay of the limit system.

More precisely, the following holds:

Theorem 4.1. *Let u^ϵ, w^ϵ be the global solution of system* (1.4)–(1.6) *obtained in Theorem* 2.1 *with* $0 \leq \alpha \leq 1$. *Then, there exist positive constants* $C > 0$ *and* $\beta > 0$ *such that*

$$E_\epsilon(t) \leq CE_\epsilon(0)\ \exp\left(-\frac{\beta t}{1+\epsilon^\alpha[E_\epsilon(0)+\|k_1\|_\infty^2]}\right)$$

for all $t \geq 0$ *and all* $0 < \epsilon < 1$.

Proof. Let $\epsilon > 0$. In order to simplify notations we write $u^\epsilon = u$, $w^\epsilon = w$. We consider the functional

$$F_\epsilon(t) = \epsilon \int_0^L uu_t dx + \int_0^L (ww_t + w_x w_{xt}) dx. \tag{4.1}$$

Direct calculations using the equations give us that

$$\begin{aligned}\frac{dF_\varepsilon}{dt} \leq &-\frac{8}{1-\mu}\int_0^L \left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)^2 dx - \varepsilon^\alpha \int_0^L uu_t dx \\ &+ \varepsilon \int_0^L u_t^2 dx - \int_0^L w_{xx}^2 dx - \int_0^L ww_t dx + \int_0^L ww_{xxt} dx \\ &+ \int_0^L [w_t^2 + w_{xt}^2] dx. \end{aligned} \tag{4.2}$$

In the following estimates C denotes a positive constant which may vary from line to line but is independent of ϵ.

For any $\gamma > 0$ we have

$$\left|\int_0^L ww_{xxt} dx\right| = \left|\int_0^L w_x w_{xt} dx\right| \leq C \int_0^L \left[\gamma w_{xx}^2 + \frac{1}{\gamma} w_{xt}^2\right] dx \tag{4.3}$$

$$\left|\int_0^L ww_t dx\right| \leq C \int_0^L \left[\gamma w_{xx}^2 + \frac{1}{\gamma} w_t^2\right] dx, \tag{4.4}$$

since $\|w_{xx}\|$ defines a norm in $H^2 \cap H_0^1(I)$ which is equivalent to the one induced by $H^2(I)$.

Also

$$\epsilon^\alpha \left|\int_0^L uu_t dx\right| \leq \frac{\epsilon^\alpha}{2\gamma}\int_0^L u_t^2 dx + \frac{\epsilon^\alpha \gamma}{2}\int_0^L u^2 dx. \tag{4.5}$$

Moreover

$$\begin{aligned}\int_0^L u^2 dx &\le C \int_0^L u_x^2 dx \\ \le C\bigg\{ \int_0^L \Big(u_x + \frac{1}{2}w_x^2 + k_1 w\Big)^2 dx &+ \Big(\int_0^L w_{xx}^2 dx\Big)^2 + \|k_1\|_\infty^2 \int_0^L w_{xx}^2 dx\bigg\} \quad (4.6)\\ \le C\bigg\{ \int_0^L \Big(u_x + \frac{1}{2}w_x^2 + k_1 w\Big)^2 dx &+ C\big[E_\epsilon(0) + \|k_1\|_\infty^2\big] \int_0^L w_{xx}^2 dx\bigg\}.\end{aligned}$$

Consequently, from (4.5) and (4.6) we obtain that

$$\begin{aligned}\epsilon^\alpha \left| \int_0^L u u_t dx \right| \le & \frac{\epsilon^\alpha}{2\gamma} \int_0^L u_t^2 dx + \frac{c\epsilon^\alpha \gamma}{2} \int_0^L \Big(u_x + \frac{1}{2}w_x^2 + k_1 w\Big)^2 dx \\ & + \frac{c\epsilon^\alpha\gamma}{2}\big[E_\epsilon(0) + \|k_1\|_\infty^2\big] \int_0^L w_{xx}^2 dx.\end{aligned} \tag{4.7}$$

Let $\delta > 0$ and define $G_{\varepsilon,\delta}(t)$ given by

$$G_{\epsilon,\delta}(t) = E_\epsilon(t) + \delta F_\epsilon(t)$$

Using (1.8) and (4.2) together with (4.3)–(4.7) we obtain that

$$\begin{aligned}\frac{dG_{\varepsilon,\delta}(t)}{dt} \le & -\bigg\{\varepsilon^{\alpha-1} - \frac{\varepsilon^{\alpha-1}\delta}{2\gamma} - \delta\bigg\}\varepsilon \int_0^L u_t^2 dx \\ & - \bigg\{1 - \delta - \frac{\delta C}{\gamma}\bigg\} \int_0^L [w_t^2 + w_{xt}^2] dx \\ & - \delta\bigg\{\frac{8}{1-\mu} - \frac{C\varepsilon^\alpha\gamma}{2}\bigg\} \int_0^L \Big(u_x + \frac{1}{2}w_x^2 + k_1 w\Big)^2 dx \\ & - \delta\bigg\{1 - \gamma C\Big[2 + \frac{\varepsilon^\alpha}{2}\{E_\varepsilon(0) + \|k_1\|_\infty^2\}\Big]\bigg\} \int_0^L w_{xx}^2 dx.\end{aligned} \tag{4.8}$$

Now let us choose $\gamma > 0$ as

$$\gamma = \lambda \Big[2 + \frac{\epsilon^\alpha}{2}\{E_\epsilon(0) + \|k_1\|_\infty^2\}\Big]^{-1}$$

where $\lambda > 0$ is small enough but independent of ε and $E_\epsilon(0)$. Then, (4.8) reads as follows

$$
\begin{aligned}
\frac{dG_{\epsilon,\delta}}{dt} \le & -\left\{\epsilon^{\alpha-1} - \frac{\epsilon^{\alpha-1}\delta}{2\lambda}\left(2+\frac{\epsilon^\alpha}{2}[E_\epsilon(0)+\|k_1\|_\infty^2]\right)-\delta\right\}\epsilon\int_0^L u_t^2dx \\
& - \left\{1-\delta-\frac{\delta C}{\lambda}\left(2+\frac{\epsilon^\alpha}{2}[E_\epsilon(0)+\|k_1\|_\infty^2]\right)\right\}\int_0^L [w_t^2+w_{xt}^2]dx \\
& - \delta\left\{\frac{8}{1-\mu}-\frac{c\epsilon^\alpha\lambda}{2\left(2+\frac{\epsilon^\alpha}{2}[E_\epsilon(0)+\|k_1\|_\infty^2]\right)}\right\} \\
& \int_0^L\left(u_x+\frac{1}{2}w_x^2+k_1w\right)^2dx - \delta\{1-\lambda C\}\int_0^L w_{xx}^2dx.
\end{aligned}
\tag{4.9}
$$

We want to impose suitable conditions on δ (and λ) so that the coefficients on the right hand side of (4.9) are all strictly less that $-\frac{\delta}{2}$. We will do this in case when $k_1 \not\equiv 0$ since the situation $k_1 \equiv 0$ was already treated in [6].

We choose $\lambda > 0$ small so that

$$\lambda < \min\left\{\frac{8\|k_1\|_\infty^2}{1-\mu}, \frac{1}{C}\right\}$$

which implies that $1-\lambda C > 0$ and

$$\frac{8}{1-\mu}-\frac{C\epsilon^\alpha\lambda}{2\left(2+\frac{\epsilon^\alpha}{2}[E_\epsilon(0)+\|k_1\|_\infty^2]\right)} > 0.$$

Once this choice of λ is done we need $\delta > 0$ to satisfy

$$\delta \le \frac{\epsilon^{\alpha-1}}{\frac{3}{2}+\frac{\epsilon^{\alpha-1}}{2\lambda}\left(2+\frac{\epsilon^\alpha}{2}[E_\epsilon(0)+\|k_1\|_\infty^2]\right)} \tag{4.10}$$

and

$$\delta \le \frac{1}{2}\left[1+\frac{C}{\lambda}\left(2+\frac{\epsilon^\alpha}{2}\{E_\epsilon(0)+\|k_1\|_\infty^2\}\right)\right]^{-1}. \tag{4.11}$$

We observe that (4.10) and (4.11) will be satisfied if we choose $\delta > 0$ of the form

$$\delta = C_1\{1+\epsilon^\alpha[E_\epsilon(0)+\|k_1\|_\infty^2]\}^{-1}$$

for some positive constant C_1 (that may depend of λ) but is independent of $0 < \epsilon < 1$. With this choice, the coefficients of $\epsilon \int_0^L u_t^2 dx$ and $\int_0^L [w_t^2 + w_{xt}^2] dx$ on the right hand side of (4.9) are, respectively, less than or equal than $-\delta/2$ and $-1/2$.

In conclusion, with the above choice of λ and $\delta > 0$, (4.9) implies that

$$\frac{dG_{\epsilon,\delta}}{dt} \leq -\min\left\{\frac{1}{2}, \frac{\delta}{2}\right\} E_{\varepsilon}(t). \tag{4.12}$$

Finally, we compare $E_\epsilon(t)$ with $G_{\epsilon,.\delta}(t)$. Using (4.1) together with (4.3), (4.4) and (4.7) we obtain that

$$\begin{aligned} |F_\epsilon(t)| &\leq \frac{\epsilon}{2}\int_0^L u_t^2 dx + \frac{C\epsilon}{2}\int_0^L \left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)^2 dx \\ &+ C\int_0^L (w_t^2 + w_{xt}^2 + w_{xx}^2) dx + \frac{C\epsilon}{2}[E_\epsilon(0) + \|k_1\|_\infty^2]\int_0^L w_{xx}^2 dx \\ &\leq (C\epsilon + C + C\epsilon[E_\epsilon(0) + \|K_1\|_\infty^2]) E_\epsilon(t) \\ &\leq \tilde{C}(1 + \epsilon[E_\epsilon(0) + \|k_1\|_\infty^2]) E_\epsilon(t) \end{aligned}$$

where $\tilde{C}$ is a positive constant independent of $0 < \epsilon < 1$. Thus,

$$\begin{aligned} |G_{\epsilon,\delta}(t) - E_\epsilon(t)| = \delta|F_\epsilon(t)| &\leq \delta\tilde{C}[1 + E_\epsilon(0) + \|k_1\|_\infty^2]E_\epsilon(t) \\ &\leq \delta\tilde{C}E_\epsilon(t) \end{aligned} \tag{4.13}$$

for some positive constant $\tilde{C}$ depending only on the initial data and $\|k_1\|_\infty^2$ (since $E_\epsilon(0)$ is bounded in ϵ).

Then, (4.13) together with (4.12) and our choice of δ implies the conclusion of Theorem 2.

5 Final remarks and comments

When $\alpha = 0$ the global well-posedness of (1.4)–(1.6) is still valid for each $\epsilon > 0$ but, in this case, the asymptotic limit as $\epsilon \to 0$ is of a different nature. In fact, when $\alpha = 0$ the limit system is of the form

$$\begin{cases} v_t = \dfrac{2}{1-\mu}\left[v_x + \dfrac{1}{2}z_x^2 + k_1 z\right]_x \\ z_{tt} + z_{xxxx} - z_{xxtt} = \dfrac{\partial}{\partial x} f(v,z) - g(v,z) - z_t + z_{xxt} \end{cases} \tag{5.1}$$

for $0 < x < L, t > 0$. System (5.1) has initial conditions

$$v(x,0) = u_0(x), z(x,0) = w_0(x), z_t(x,0) = w_1(x), \quad 0 < x < L \tag{5.2}$$

and boundary conditions

$$v(0,t) = v(L,t) = z(0,t) = z(L,t) = z_x(0,t) = z_x(L,t) = 0. \tag{5.3}$$

System (5.1)–(5.3) is the coupling between a parabolic equation and a fourth order hyperbolic equation, thus it has a similar structure to a system of thermoelasticity. The total energy associated with (4.1) is given by

$$E(t) = \frac{1}{2}\int_0^L \left\{ z_t^2 + z_{xx}^2 + z_{xt}^2 + \left(v_x + \frac{1}{2}z_x^2 + k_1 z \right)^2 \right\} dx$$

and satisfies

$$\frac{dE}{dt} = -\int_0^L (v_t^2 + z_t^2 + z_{xt}^2)dx.$$

According to the discussion of Theorem 4.1 we can pass to the limit as $\varepsilon \to 0$ to obtain the following decay property for the solution of (4.1)–(4.3)

$$E(t) \le CE(0) \exp\left(-\frac{\beta t}{1 + E(0) + \|k_1\|_\infty^2} \right)$$

for all $t > 0$.

We refer to [6] for further developments of this issue in the case of the classical von Kármán and Timoshenko equations.

The analysis developed in this paper can be adapted to a variety of situations, including different boundary conditions. The interested reader is referred to [6] and [7] for the discussion of these issues in the case of von Kármán and Timoshenko equations.

Acknowledgments. The first author was partially supported by a Grant of CNPq and PRONEX (MCT, Brasil). Part of this work was done while he was visiting the "Centro de Modelamiento Matemático (CMM)" of the "Universidad de Chile" as part of the Chair "Marcel Dassault". He expresses his gratitude for their kind hospitality and support. The second author was supported by grant PB96-0663 of the DGES (Spain) and the TMR project of the EU "Homogenization and multiple scales". This work was done while the second author was visiting the Universidad Autónoma de Madrid, for the academic year 01-02.

References

[1] J.M. Ball, *Initial-boundary value problems for and extensible beam*, J. Math. Anal. Appl., **41** (1973), 69–90.

[2] Ph. Ciarlet, *Mathematical Elasticity. Volume III. Theory of Shells*, Studies in Mathematics and its Applications, **29** (2000), North Holland.

[3] J.E. Lagnese and G. Leugering, *Uniform stabilization of a nonlinear beam by nonlinear boundary feedback*, J. Diff. Equations, **91** (1991), 355–388.

[4] J.L. Lions, *Quelques Méthodes de Résolution des problèmes aux Limites non Linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[5] K. Marguerre, *Zur theorie der gekrümmten platte grosser Formänderung, Jahrbuch der deutschen Luftfahrtforschung*, (1939), 413–418.

[6] A. Pazoto, G. Perla Menzala and E. Zuazua, *Stabilization of Timoshenko's equation as limit of the von Kármán system of beams and plates*, to appear.

[7] G. Perla Menzala and E. Zuazua, *Timoshenko's beam equation as limit of a nonlinear one-dimensional von Karman system*, Proceedings of the Royal Society of Edinburgh, **130A** (2000), 855–875.

[8] V.I. Sedenko, *On the uniqueness theorem for generalized solutions of initial-boundary problems for the Marguerre-Vlasov vibrations of shallow shells with clamped boundary conditions*, Appl. Math. Optim., **39** (1999), 309–326.

[9] I.I. Vorovich, *On the existence of solutions in nonlinear shell theory*, Dokl. Akal. Nauk SSSR, **117** (1957), 203–206.

G. Perla Menzala
National Laboratory of Scientific Computation, LNCC / MCT
25651-070 Petrópolis, RJ
Brasil

E-mail: perla@lncc.br

Enrique Zuazua
Departamento de Matemática Aplicada
Universidad Complutense de Madrid
28040, Madrid
Spain

E-mail: enrique.zuazua@uam.es