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A strong uniqueness theorem for planar vector fields*

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— Dedicated to Constantine Dafermos on his 60th birthday

Abstract. This work establishes a strong uniqueness property for a class of planar locally integrable vector fields. A result on pointwise convergence to the boundary value is also proved for bounded solutions.

Keywords: weak boundary values, locally integrable vector fields, uniqueness in the Cauchy problem.

Mathematical subject classification: Primary 35F15, 35B30, 42B30; Secondary 42A38, 30E25.

0. Introduction

Consider a complex, smooth vector field

$$L = \frac{\partial}{\partial y} + a(x, y)\frac{\partial}{\partial x}$$

defined in a neighborhood of the origin in $\mathbb{R}^2$. We are interested in the following uniqueness question: if a function $u(x, y)$ defined in a neighborhood of the origin satisfies

$$\begin{cases} Lu = 0 & \text{for } y > 0 \text{ and} \\ u(x, 0) = 0, \end{cases} \tag{0.1}$$

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can we conclude that $u(x, y)$ vanishes identically in a neighborhood of the origin? In 1960 P. Cohen [C] (see also [Z] and the references therein) constructed smooth functions $u(x, y)$ and $a(x, y)$ defined on the plane such that

(1) $Lu(x, y) = (u_y + au_x)(x, y) = 0$;

(2) $u(x, y) = a(x, y) = 0$ for all $y \leq 0$;

(3) $supp\, u = supp\, a = \{(x, y) : \quad y \geq 0\}$.

In particular, u satisfies (0.1) and does not vanish identically in any neighborhood of the origin which shows that some additional hypothesis must be made on L if one wants to obtain uniqueness. A quite satisfactory answer is known for the class of locally integrable vector fields ([T1]): if $u(x, y) = 0$ satisfies (0.1) and L is locally integrable, u must vanish on a small rectangle $(-\delta, \delta) \times (0, \delta)$.

In this article we investigate a stronger uniqueness property for locally integrable vector fields, replacing the condition that $u(x, 0)$ vanish identically by the weaker hypothesis that the integral of $\ln|u(x, 0)|$ be equal to $-\infty$, and consider one-sided solutions, i.e., $u(x, y)$ is only assumed to satisfy the equation on one side of the initial curve $\{y = 0\}$, conditions that are classically known to guarantee uniqueness for the Cauchy-Riemann operator ([F], [RR], see also [Du] and the references therein). After an appropriate local change of variables that preserves the initial curve $\{y = 0\}$, any elliptic vector field can be transformed into a multiple of the Cauchy-Riemann operator and this shows that elliptic vector fields share this strong uniqueness property. However, this condition is not enough to guarantee uniqueness for the vector field ∂_y so an additional hypothesis has to be made on L if it is to possess the strong uniqueness property under scrutiny. It turns out that a much weaker assumption than ellipticity is enough to ensure that L will share with the Cauchy-Riemann operator this strong uniqueness property for bounded solutions. All we need to assume is that the integral curve of $X = \mathrm{Re}\, L$ through the origin contains a sequence of points on which L is elliptic and the sequence converges to the origin (see Theorem 1.2 below for the precise statement). It is also shown that this geometric condition is necessary for the validity of the strong uniqueness property. The work [J] and the recent article [Co] contain results on this kind of uniqueness property. In [Co] the author established the strong uniqueness property for approximate solutions of a class of planar vector fields. A continuous function u is said to be an *approximate solution* for a vector field L if Lu is locally integrable and satisfies the inequality $|Lu| \leq M|u|$ for some constant M. In [J] the setup involves a generic CR manifold $\mathcal{M}$ in $\mathbb{C}^N$ and a submanifold E of $\mathcal{M}$ satisfying

$T\mathcal{M}+J(T\mathcal{M}) = TE+J(TE)$ over the points of E and where J is the complex structure map. It was proved that if u is a continuous CR function in a neighborhood of $p \in E$ and if for $z \in E$, $|u(z)| \leq h(|z-p|)$ for some continuous, increasing h on $[0,\infty)$ with $\int_0^1 \ln h(r)\,dr = -\infty$, then u vanishes on the Sussmann orbit through p. We emphasize that in the result of [J], the function u is a solution in a full neighborhood of E while in our result (Theorem 1.2), we consider a solution defined only on one side.
This paper is organized as follows. In sections 1 and 2 the proof of the main result Theorem 1.2 is presented. Section 3 is devoted to a theorem on pointwise convergence to weak boundary values.

1 A uniqueness criterion for locally integrable vector fields

We recall that a vector field with smooth complex coefficients

$$L = \frac{\partial}{\partial y} + a(x,y)\frac{\partial}{\partial x}$$

defined in a neighborhood of the origin in $\mathbb{R}^2$ is said to be locally integrable at the origin if there exists a smooth function $Z(x,y)$ defined in a neighborhood of the origin such that

(1) $LZ = 0$; and

(2) $dZ(0,0) \neq 0$.

A function $Z(x,y)$ satisfying these properties is called a first integral of L. Throughout this paper we assume that L is locally integrable at the origin. Let $u(x,t)$ be a bounded measurable function that satisfies $Lu = 0$ on $(-a,a)\times(-b,b)$ in the sense of distributions. For a general L^∞ function the restriction to a horizontal line $t=$ constant is not defined because lines have measure zero with respect to the Lebesgue two dimensional measure. On the other hand, Fubini's theorem implies that there exist a null set $E \subset (-b,b)$ such that if $t_0 \notin E$ the function $(-a,a) \ni x \mapsto u(x,t_0)$ is measurable. Since u is assumed to satisfy the equation $Lu = 0$ we may assert more: since the wave front set $WF(u)$ is contained in the characteristic set of L

$$\left\{(x,y;\xi,\eta) \in (-a,a)\times(-b,b)\times\mathbb{R}^2 : \quad \operatorname{Im} a(x,y)\xi = 0,\ \eta + \operatorname{Re} a(x,y)\xi = 0\right\}$$

it follows ([Hor,Corollary 8.2.7]) that *for all* $|t_0| < b$ there is a well defined restriction of u to the horizontal lines $t = t_0 \in (-b,b)$ – called the trace of u at $t = t_0$ – that will be denoted by $u(x,t_0) \in \mathcal{D}^*(-a,a)$. Furthermore, the distribution $u(x, t_0)$ depends continuously on t_0, i.e., if $\phi(x) \in C_c^\infty((-a, a))$ the function $(-b, b) \ni t \mapsto \langle u(x, t), \phi(x)\rangle$ is continuous, in fact, it is smooth. We are here using the same notation, namely $u(x, t_0)$, to denote two different notions: the distribution trace of u and the pointwise restriction; however this may not cause confusion as they coincide for almost every t (see for instance [HT,Lemma B.2]).

The most basic and general uniqueness theorem for L is a by-product of the Baouendi-Treves approximation formula ([BT], [T3],[T1]): if $u(x, 0) = 0$ then u must vanish in a neighborhood of the origin. This uniqueness result can be strengthened as follows for one-sided solutions (*cf.*, e.g., [T2], [HM,p.1313]): if $u(x, t)$ is only defined on $(-a, a) \times (0, b)$ where it satisfies $Lu = 0$ and we assume that $u(x, 0) = 0$ in the sense that

$$\lim_{t \searrow 0} \langle u(x, t), \phi(x)\rangle = 0, \quad \phi \in C_c^\infty((-a, a)), \tag{1.1}$$

then $u(x, t)$ must vanish identically for $|x| < \delta, 0 < t < \delta$ if $\delta > 0$ is sufficiently small. For a general distribution that satisfies $Lu = 0$ on $(-a, a) \times (0, b)$ the limit on the right hand side of (1.1) may not exist but here it does because u is bounded. Indeed

Lemma 1.1. *Let*

$$L = \frac{\partial}{\partial y} + a(x, y)\frac{\partial}{\partial x},$$

$a(x, t) \in C^\infty$ on $(-a, a) \times (-b, b)$, be a not necessarily locally integrable vector field. Let f be a bounded function on $(-a, a) \times (0, b)$ such that $Lf = 0$. Then, as $y \searrow 0$, $f(x, y)$ converges in $\mathcal{D}'((-a, a))$ to a bounded function $bf(x) \doteq \lim_{y \searrow 0} f(x, y) \in L^\infty(-a, a)$.

Remark. This lemma is a variation of Lemma 1.2 in [BH1].

We postpone the proof of the Lemma – to be found in the Appendix – and continue our discussion of uniqueness. According to our previous discussion, it is known that if $u(x, y)$ is defined and bounded for $y > 0$, satisfies the equation $Lu = 0$ and its boundary value $bu(x)$ is zero, then u must vanish identically for $|x| < \delta, 0 < t < \delta$, if δ is small. On the other hand, if L is the Cauchy Riemann operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

so $U(x+iy) = u(x, y)$ is bounded and holomorphic for $y > 0$ it is well known that it is enough to assume that

$$\int_{-a}^{a} \ln |bu(x)| = -\infty \tag{1.2}$$

to conclude that u must vanish. Note that if

i) $bu(x)$ vanishes on a set of positive measure, or

ii) $bu(x)$ has a zero of exponential order, i.e., $|bu(x)| \leq A \exp(-B|x-x_0|^{-1})$ for some $x_0 \in (-\delta, \delta)$,

then (1.2) holds. We wish to extend this finer type of uniqueness property to a class of locally integrable vector fields. Of course, the class cannot contain the vector field $L = \partial_y$ which obviously fails to have this type of uniqueness as any function $u(x)$ independent of y satisfies the homogeneous equation $\partial_y u = 0$. This strong divergence in behavior between the Cauchy-Riemann operator and ∂_y is explained by the fact that the former is elliptic at every point while the second is not elliptic at any point. Loosely speaking, our class must exhibit some degree of ellipticity in order to have a chance to share the uniqueness property we are interested in with the Cauchy-Riemann operator. A precise geometric property that characterizes vector fields L possessing this type of uniqueness is given by the theorem below. Let's write $L = X + iY$ with X and Y real and note that L is elliptic precisely at the points where the real vector fields X and Y are linearly independent.

Theorem 1.2. *Let*

$$L = \frac{\partial}{\partial y} + a(x, y)\frac{\partial}{\partial x} = X + iY,$$

$a(x, t) \in C^\infty$ on $(-a, a) \times (-b, b)$, be locally integrable. Assume that on the integral curve of X that passes through the origin there is a sequence of points $p_n = (x_n, y_n)$ such that

(1) *L is elliptic at p_n, i.e., $X(p_n)$ and $Y(p_n)$ are linearly independent;*

(2) *$y_n > 0$ and $p_n \to (0, 0)$.*

Then there exists $0 < \delta < a$ such that every function $u(x, y) \in L^\infty((-a, a) \times (0, b))$ that satisfies

(3) $Lu(x, y) = 0$ *for* $y > 0$;

(4) $\displaystyle\int_{-\delta}^{\delta} \ln |bu(x)|\, dx = -\infty$;

must vanish identically on $(-\delta, \delta) \times (0, \delta)$.

Conversely, if no sequence $p_n = (x_n, y_n)$ *on the integral curve of* X *that passes through the origin satisfies both* (1) *and* (2), *there exists a function* $u(x, y) \in C^\infty((-\delta, \delta) \times [0, \delta))$ *that satisfies*(3) *and* (4) *but does not vanish identically on the intersection of* $(-\delta, \delta) \times (0, \delta))$ *with any neighborhood of the origin.*

Proof. We may find new local coordinates in a neighborhood of the origin that preserve the x-axis and the upper half plane $\{y > 0\}$ in which a first integral $Z(x, y)$ of L has the form

$$Z(x, y) = x + i\varphi(x, y), \qquad \varphi(0, 0) = \varphi_x(0, 0) = 0,$$

with $\varphi(x, y)$ real. Since $LZ = 0$, modulo a nonvanishing factor, the expression of L in the new coordinates is

$$L = X + iY = \frac{\partial}{\partial y} - \frac{i\varphi_y}{1 + i\varphi_x}\frac{\partial}{\partial x},$$

$$X = \frac{\partial}{\partial y} - \frac{\varphi_y \varphi_x}{1 + \varphi_x^2}\frac{\partial}{\partial x}, \qquad Y = -\frac{\varphi_y}{1 + \varphi_x^2}\frac{\partial}{\partial x}.$$

We see that X and Y become linearly dependent precisely at the points where φ_y vanishes. Note that $y \mapsto \varphi(0, y)$ cannot vanish identically on any interval $[0, \epsilon]$, $0 < \epsilon < b$. Indeed, if it did we would conclude that $X = \partial_y$ on the segment $\{0\} \times [0, \epsilon)$ which would then be an integral curve of X on which Y vanishes, contradicting hypothesis (1). For $0 < \delta < b$ to be determined later (we will have to shrink δ several times) set

$$M(x) = \sup_{0 \le y \le \delta} \varphi(x, y), \qquad m(x) = \inf_{0 \le y \le \delta} \varphi(x, y).$$

Then $M(x)$ and $m(x)$ are Lipschitz continuous functions, $m(x) \le M(x)$ and $m(0) < M(0)$. Assume that $u(x, y) \in L^\infty((-a, a) \times (-0, b))$ satisfies hypotheses (3) and (4) and we wish to show that u vanishes near the origin. Now the Baouendi-Treves approximation formula ([BT], [HM,Cor2.2]) furnishes a sequence of holomorphic polynomials $P_k(\xi + i\eta)$ such that the functions $u_k(x, y) = P_k(Z(x, y))$ satisfy the following properties for some fixed $\delta > 0$ and $K > 0$:

(1) $|u_k(x, y)| \leq K$ on $(-\delta, \delta) \times (0, \delta)$;

(2) $u_k(x, y) \to u(x, y)$ a.e. on $(-\delta, \delta) \times (0, \delta)$;

(3) $u_k(x, 0) \to bu(x)$ a.e. on $(-\delta, \delta)$.

Let's denote by Q the rectangle $(-\delta, \delta) \times (0, \delta)$ and by Ω the region of the complex plane bounded by the Lipschitz curves $M(x)$ and $m(x)$, more precisely,

$$\Omega = \{\zeta = \xi + i\eta : \quad |\xi| < \delta, \ m(\xi) < \eta < M(\xi)\},$$

where we have shrunk $\delta > 0$ to ensure that $m(x) < M(x)$ for $|x| < \delta$. It follows from (1) that the sequence $P_k(\zeta)$ is uniformly bounded on Ω and by Montel's theorem we may assume, after passing to a subsequence, that $P_k(\zeta)$ converges uniformly over compact subsets of Ω to a bounded holomorphic function $U(\zeta)$. Since the functions $u_k(x, t) = P_k \circ Z(x, t)$, $k = 1, 2 \ldots$, satisfy $Lu_k = 0$, converge pointwise to $u^*(x, t) = U \circ Z(x, t)$ on $Z^{-1}(\Omega)$ and the u_k are uniformly bounded, we see that $u^*(x, t)$ also satisfies $Lu^* = 0$ on $Z^{-1}(\Omega)$. Furthermore, there is a set $E \subset Q$ with measure $|E| = 0$ such that $u_k(x, t) \to u(x, t)$ if $(x, t) \in Q \setminus E$ so we conclude that $u(x, t) = u^*(x, t)$ a.e. in $Q \cap Z^{-1}(\Omega)$. This shows that $u(x, t)$ can be extended to a solution defined on $Q \cup Z^{-1}(\Omega)$. Since $u^*(x, t)$ is smooth on $Z^{-1}(\Omega)$ we will as usual modify $u(x, y)$ in a null set to obtain $u(x, t) = u^*(x, t)$ everywhere on $Z^{-1}(\Omega) \cap Q$, which means that we are picking a representative in the class of $u \in L^\infty$ which restricted to $Z^{-1}(\Omega) \cap Q$ is continuous. Assume without loss of generality that $m(0) \leq 0 < M(0)$ and let's look at the boundary behavior of the holomorphic function U on Ω. Since $U(\zeta)$ is bounded and Ω has a Lipschitz boundary the nontangential limit at a boundary point $\zeta_0 = x_0 + i\varphi(x_0, 0)$, $-\delta < x_0 < \delta$,

$$\lim_{\Gamma(\zeta_0) \ni \zeta \to \zeta_0} U(\zeta) = bU(\zeta_0) \in \mathbb{C} \quad \text{exists for a.e. } \zeta_0 \in \partial\Omega$$

where $\Gamma(\zeta_0)$ is a nontangential region with vertex at ζ_0. At points ζ_0 where the limit does not exist we define $bU(\zeta_0) = 0$ so bU is now everywhere defined on $\partial\Omega$. We denote by $\widehat{U}(\zeta)$ the natural extension to $\overline{\Omega}$ of $U(\zeta)$, i.e.,

$$\widehat{U}(\zeta) = \begin{cases} U(\zeta), & \zeta \in \Omega, \\ bU(\zeta), & \zeta \in \partial\Omega, \end{cases} \tag{1.3}$$

which is measurable and bounded and set $\widehat{u}(x, y) = \widehat{U}(x + i\varphi(x, y))$ for $(x, y) \in Q$. The proof of the theorem will rely heavily on the following representation formula for $u(x, y)$.

Lemma 1.3. *Let $u(x, y)$, $U(\zeta)$ and $\widehat{U}(\zeta)$ be defined as above. After modification of $u(x, y)$ on a null set, the following identity holds*

$$u(x, y) = \widehat{U}(x + i\varphi(x, y)) \qquad (x, y) \in Q. \tag{1.4}$$

Proof. We wish to prove that $u(x, y) = \widehat{u}(x, y)$ as distributions on Q. Let (x_0, y_0) be an arbitrary point in Q. If $x_0 + i\varphi(x_0, y_0) \in \Omega$ then $u(x, y) = U \circ Z(x, y) = \widehat{U} \circ Z(x, y) = \widehat{u}(x, y)$ on a neighborhood of (x_0, y_0). This shows that $u(x, y) = \widehat{u}(x, y)$ everywhere on $Q \cap Z^{-1}(\Omega)$. If $x_0 + i\varphi(x_0, y_0) \in \partial\Omega$, then either $\varphi(x_0, y_0) = m(x_0)$ or $\varphi(x_0, y_0) = M(x_0)$. Let's assume first that $m(x_0) = \varphi(x_0, y_0) < M(x_0)$. We consider the maximal vertical interval $\{x_0\} \times I(x_0) \subset Q$ that contains (x_0, y_0) on which $\varphi(x, y) = m(x_0)$ and distinguish two cases.

Case 1. $I(x_0) = [y_0 - \eta, y_0 + \rho]$.

Here $\rho, \eta \geq 0$ and $[y_0 - \eta, y_0 + \rho] \subset (0, \delta)$. Thus, $\{x_0\} \times I(x_0)$ is contained in a rectangle $R = (x_0 - \mu, x_0 + \mu) \times (y_0 - \eta', y_0 + \rho') \subset Q$, $\eta' > \eta$, $\rho' > \rho$, such that

$$Z\Big(\big([x_0 - \mu, x_0 + \mu] \times \{y_0 - \eta'\}\big) \cup \big([x_0 - \mu, x_0 + \mu] \times \{y_0 + \rho'\}\big)\Big) \subset \Omega$$

and

$$\varphi(x, y) < M(x) \quad \text{on } \overline{R}.$$

In particular, $\widehat{u}(x, y) = u(x, y)$ on a neighborhood of the horizontal edges of the boundary of R. By the basic uniqueness result on the Cauchy problem for locally integrable vector fields mentioned at the beginning of this section, in order to conclude that $\widehat{u} = u$ as distributions on a neighborhood of $\{x_0\} \times I(x_0)$ it will be enough to show that $L\widehat{u} = 0$ on R. For small $\epsilon > 0$ consider the function

$$u_\epsilon(x, y) = U(x + i\varphi(x, y) + i\epsilon), \qquad (x, y) \in R.$$

While $U \circ Z$ is only defined on $Z^{-1}(\Omega)$, so it fails to be defined at points $(x, y) \in R$ such that $\varphi(x, y) = m(x)$, this is not the case for $u_\epsilon(x, y)$ because the strict inequality $\varphi(x, y) + \epsilon > m(x)$ always holds. If $(x, y) \in Q \cap Z^{-1}(\Omega)$ it is clear that $\lim_{\epsilon \to 0} u_\epsilon(x, y) = U \circ Z(x, y) = u(x, y) = \widehat{u}(x, y)$ by the continuity of U. Let's study the limit at a point $(x, y) \in R$, for which $\varphi(x, y) = m(x)$. Taking account of (1.3) we see that

$$\lim_{\epsilon \searrow 0} u_\epsilon(x, y) = \lim_{\epsilon \searrow 0} U(x + i\varphi(x, 0) + i\epsilon) = bU(x + i\varphi(x, 0)) = \widehat{u}(x, y)$$

unless x belongs to an exceptional set E_1 of measure $|E_1| = 0$. Since $Lu_\epsilon = 0$, $u_\epsilon(x, y) \to \widehat{u}(x, y)$ a.e. and the u_ϵ are uniformly bounded, we conclude that $L\widehat{u} = 0$ on R which implies that $\widehat{u} = u$ on a neighborhood of (x_0, y_0).

Case 2. $I(x_0) = (0, y_0 + \rho]$.

Here $\rho \geq 0$ and $(0, y_0 + \rho] \subset (0, \delta)$. Now $\{x_0\} \times I(x_0)$ is contained in a rectangle $R = (x_0 - \mu, x_0 + \mu) \times (0, y_0 + \rho') \subset Q$, $\rho' > \rho$, such that

$$Z\big([x_0 - \mu, x_0 + \mu] \times \{y_0 + \rho'\}\big) \subset \Omega$$

and

$$\varphi(x, y) < M(x) \quad \text{on } \overline{R}.$$

This time we conclude immediately that $\widehat{u}(x, y) = u(x, y)$ on a neighborhood of the upper horizontal edge of the boundary of R. But this is enough to repeat the argument of Case 1 and obtain via uniqueness in the Cauchy problem that $\widehat{u}(x, y) = u(x, y)$ on a neighborhood of (x_0, y_0).

Note that $I(x_0)$ cannot be equal to $(0, \delta)$ because this would imply that $m(x_0) = M(x_0)$. Hence, we have proved that if $m(x_0) = \varphi(x_0, y_0)$ then $\widehat{u} = u$ on a neighborhood of (x_0, y_0). Similarly, we could prove that the same holds if $\varphi(x_0, y_0) = M(x_0)$ by an analogous reasoning. This proves the representation formula (1.4). □

We continue with the proof of the theorem assuming that $u(x, y)$ has been modified in a null set so (1.4) holds everywhere. As a consequence $u(x, y)$ is constant on the fibers $F(x_0, y_0) = \{(x, y) \in Q : x = x_0, \varphi(x_0, y) = \varphi(x_0, y_0)\}$. In particular, if $\varphi(x, y)$ is constant on some vertical segment of the form $\{x_0\} \times (0, y_0] \subset Q$, the function $u(x, y)$ will also be constant on that segment and it follows that

$$\lim_{\epsilon \searrow 0} u(x_0, \epsilon) = u(x_0, y_0) = \widehat{u}(x_0, y_0) = \widehat{u}(x_0, 0) = \widehat{U}(x_0 + i\varphi(x_0, 0)) \quad (1.5)$$

We wish to see that (1.5) holds for almost all points $x_0 \in (-\delta, \delta)$ such that $\varphi(x_0, 0) = m(x_0)$. Let's assume then that $m(x_0) = \varphi(x_0, 0) < M(x_0)$. If $m(x_0) = \varphi(x_0, y)$ for all $0 < y \leq y_0$ for some $0 < y_0 < \delta$ we already saw the validity of (1.5) so we may assume that there is a sequence $y_n \searrow 0$ such that $\varphi(x_0, 0) < \varphi(x_0, y_n) < M(x_0)$ which implies that $x_0 + i\varphi(x_0, y_n) \in \Omega$. In this case we have

$$\begin{aligned} \lim_{y_n \searrow 0} u(x_0, y_n) &= \lim_{y_n \searrow 0} U(x_0 + i\varphi(x_0, y_n)) \\ &= bU(x_0 + i\varphi(x_0, 0) = \widehat{U}(x_0 + i\varphi(x_0, 0)) \end{aligned}$$

unless x_0 belongs to the exceptional set E_1 of measure $|E_1| = 0$ introduced in the proof of Lemma 1.3.

Shrinking δ once again we may assume that $\varphi(x,0) < M(x)$ for $|x| < \delta$. If $m(x_0) < \varphi(x_0,0) < M(x_0)$ and ρ is small enough, the extension of u (still denoted by u) to $Q \cup Z^{-1}(\Omega)$ is defined for $|x - x_0| < \rho$, $-\rho < y < \rho$, for some small ρ such that $(x_0 - \rho, x_0 + \rho) \subset (-\delta, \delta)$, and given by $u(x,y) = U(x + i\varphi(x,y))$. In particular, $u(x,y)$ is smooth in a neighborhood of $(x_0, 0)$ and (1.5) follows by continuity and the fact that $\widehat{U}(x_0 + i\varphi(x_0,0)) = U(x_0 + i\varphi(x_0,0))$ because $x_0 + i\varphi(x_0,0) \in \Omega$. Summing up, we have proved that

$$\lim_{\epsilon \searrow 0} u(x,\epsilon) = \widehat{U}(x + i\varphi(x,0)) \qquad \text{a.e. } |x| < \delta. \tag{1.6}$$

Let $\psi(x) \in C_c^\infty(-\delta,\delta)$. The dominated convergence theorem gives

$$\langle bu, \psi \rangle = \lim_{\epsilon \searrow 0} \int u(x,\epsilon)\psi(x)\,dx = \int \widehat{U}(x + i\varphi(x,0))\psi(x)\,dx,$$

showing that

$$bu(x) = \widehat{U}(x + i\varphi(x,0)) \quad \text{a.e. } x \in (-\delta,\delta). \tag{1.7}$$

Consider now the domain

$$\Omega^+ = \{\zeta = \xi + i\eta : \quad |\xi| < \delta,\ \varphi(\xi,0) < \eta < M(\xi)\},$$

and denote by $\widehat{U}^+$ the restriction of $\widehat{U}$ to Ω^+. Then (1.7) may be rephrased as

$$bu(x) = bU^+(x + i\varphi(x,0)) \quad \text{a.e. } x \in (-\delta,\delta).$$

Indeed, it is clear that

$$bU^+(\zeta) = bU(\zeta) \text{ if } \zeta \in \partial\Omega \cap \partial\Omega^+$$

and

$$bU^+(\zeta) = U(\zeta) \text{ if } \zeta \in \Omega \cap \partial\Omega^+.$$

We now invoke hypothesis (4) made on $bu(x)$. A change of variables shows that

$$\int_{-\delta}^{\delta} \ln |bU^+(x)|\,dx = -\infty$$

and then it is classical that U^+ must vanish identically, forcing U and therefore u to vanish on Q.

So far we have proved the first half of the theorem. The second part will be given in the next section.

2 End of the proof of Theorem 1.2

We continue to work in the local coordinates (x, y) that were used for the proof of the first part, so the first integral has the form $Z(x, y) = x + i\varphi(x, y)$ and

$$L = X + iY = \frac{\partial}{\partial y} - \frac{i\varphi_y}{1 + i\varphi_x}\frac{\partial}{\partial x},$$

$$X = \frac{\partial}{\partial y} - \frac{\varphi_y\varphi_x}{1+\varphi_x^2}\frac{\partial}{\partial x}, \qquad Y = -\frac{\varphi_y}{1+\varphi_x^2}\frac{\partial}{\partial x}.$$

It is easy to see that the hypothesis implies that $\varphi(0, y) = 0$ for all $0 \leq y \leq \delta$ for some $\delta > 0$. Consider the function defined on $(-\delta, \delta) \times [0, \delta)$ by

$$u(x, y) = \begin{cases} 0, & \text{for } x \leq 0 \\ \exp(Z^{-1}(x, y)), & \text{for } x > 0. \end{cases}$$

Since $Z(x, y) \neq 0$ for $x \neq 0$ it is clear that $u(x, y)$ is smooth for $x \neq 0$. Furthermore, as $x \searrow 0$ we easily see that

$$|u(x, y)| = \exp\left(-\frac{x}{x^2 + \varphi^2(x, y)}\right) \longrightarrow 0$$

because $\varphi(x, y) = O(x)$ for any $0 \leq y < \delta$. The same conclusion is valid for any derivative $D_x^j D_y^k u(x, y)$ in place of $u(x, y)$. This proves that $u \in C^\infty((-\delta, \delta) \times [0, \delta))$. Furthermore, $Lu = 0$ for $x \neq 0$ because u is constant for $x < 0$ whereas u is a holomorphic function of Z defined on a neighborhood of $Z((0, \delta) \times (0, \delta))$ for $x > 0$. Since $Lu(x, y)$ is continuous, it follows that $Lu = 0$ on $(-\delta, \delta) \times (0, \delta)$. Also, since u is continuous up to $y = 0$ it is apparent that $bu(x) = u(x, 0)$. In particular, $bu(x, 0) = 0$ for $x < 0$ and

$$\int_{-\epsilon}^{\epsilon} \ln |bu(x)|\, dx = -\infty$$

for any $\epsilon > 0$. Finally, $u(x, y) \neq 0$ for any $x > 0$. □

3 Pointwise convergence to the initial data

Always using the special coordinates (x, y) defined in a neighborhood of $[-a, a] \times [-b, b]$ where the first integral is given by $Z = x + i\varphi(x, y)$, consider a function $u(x, y)$ defined on $(-a, a) \times (0, b)$ that is measurable, bounded and satisfies the equation $Lu = 0$. Keeping the notation of Section 1, we set

$$M(x) = \sup_{0 \leq y \leq b} \varphi(x, y), \qquad m(x) = \inf_{0 \leq y \leq b} \varphi(x, y),$$

and define

$$F = \{x \in (-a, a) : \quad m(x) = M(x)\}.$$

Then F is closed and its complement may be written as a union of open intervals

$$(-a, a) \setminus F = \bigcup_{j=1}^{\infty} (\alpha_j, \beta_j).$$

For any $j = 1, 2, \ldots$ the set $(\alpha_j, \beta_j) \times (0, b)$ is mapped by Z into the domain

$$\Omega_j = \left\{\zeta = \xi + i\eta : \quad \alpha_j < \xi < \beta_j,\ m(\xi) < \eta < M(\xi)\right\}. \tag{3.1}$$

Since $m(x) < M(x)$ for $\alpha_j < x < \beta_j$ the proof of Theorem 1.2 shows that there is a holomorphic function U_j defined on Ω_j having an extension $\widehat{U}_j$ to $\overline{\Omega}$ such that $u(x, y) = \widehat{U}_j \circ Z(x, y)$ for $\alpha_j < x < \beta_j$. Furthermore, (1.6) holds, i.e.,

$$\lim_{\epsilon \searrow 0} u(x, \epsilon) = \widehat{U}_j(x + i\varphi(x, 0)) \qquad \text{a.e. } \alpha_j < x < \beta_j.$$

In particular, the limit $\lim_{\epsilon \searrow 0} u(x, \epsilon)$ exists for almost every $x \in (-a, a) \setminus F$. On the other hand, it is easy to see that the same limit exists for almost every $x \in F$. Since this is a local property, it will be enough to prove it in a neighborhood of a given point $x_0 \in F$. To simplify the notation let's assume that $x_0 = 0$. If $x \in F$, $\varphi(x, 0) = \varphi(x, y)$ for any $0 < y < b$, so $Z(x, y)$ is constant on $\{x\} \times (0, b)$, $x \in F$, and so is $u_k(x, y) = P_k(Z(x, y))$, where $P_k(\zeta)$ is the sequence of polynomials obtained from the Baouendi-Treves approximation scheme used at the beginning of the proof of Theorem 1.2. Since $u_k(x, y) \to u(x, y)$ a.e., for $|x| < \delta$, $0 < y < \delta$ for some $\delta > 0$, there is a set G of 2-dimensional measure zero such that $u_k(x, y) \to u(x, y)$ for $(x, y) \notin G$. By Fubini's theorem the 1-dimensional measure of $G_x = (\{x\} \times (0, \delta)) \cap G$ is zero for a.e. $|x| < \delta$, i.e., for $x \notin H \subset (-\delta, \delta)$, $|G_x| = 0$, with $|H| = 0$. If $x \in F \setminus H$, $u_k(x, y) \to u(x, y)$ for a.e. y and we see that $y \mapsto u(x, y)$ is constant almost everywhere. Thus, modifying $u(x, y)$ in a set of 2-dimensional measure zero, we obtain that $y \to u(x, y)$ is constant for $x \in F \setminus H$ and $\lim_{\epsilon \to 0} u(x, \epsilon)$ exist for $x \in F \setminus H$. Observe that this limit has to equal $bu(x)$ a.e. since we can find a sequence $y_j \mapsto 0$ such that the traces $u(., y_j)$ converge to bu weakly in $L^\infty(-a, a)$. Hence, we have proved the existence of the limit $\lim_{\epsilon \to 0} u(x, \epsilon)$ for a.e. $x \in (-\delta, \delta)$ and therefore for a.e. $x \in (-a, a)$. In particular, this gives an alternative proof of Lemma 1.1 when L is locally integrable.

In the convergence result just proved, the boundary point $(x_0, 0)$ is approached vertically to the initial curve $\{y = 0\}$ (in the special coordinates we are using).

Since in the classical case where L is the Cauchy-Riemann operator and u is holomorphic the limit is still valid on nontangential regions of approach, it is natural to try to replace the normal set of approach $\{(x_0, \epsilon)\}$ by a larger set. Keeping in mind that L is elliptic exactly at the points where $\varphi_y \neq 0$, we distinguish two types of points $x_0 \in (-a, a)$:

(I) there exists $\eta = \eta(x_0) > 0$ such that $\varphi_y(x_0, y) = 0$ for $0 \leq y \leq \eta$;

(II) there exists a sequence of points $y_n > 0$ such that $y_n \searrow 0$ and $\varphi_y(x_0, y_n) \neq 0$.

We will attach to every point x_0 a set of approach $\Gamma(x_0)$. In case (I) we keep the normal set of approach and simply define $\Gamma(x_0) = \{x_0\} \times (0, b)$. In case (II) we proceed as follows: since the hypothesis implies that $m(x_0) < M(x_0)$ it follows that $x_0 \in (\alpha_j, \beta_j)$ for some j and $u(x, y) = \widehat{U}_j \circ Z(x, y)$ for $\alpha_j < x < \beta_j$ with U_j holomorphic and bounded in the open set Ω_j given by (3.1). Since $U(\zeta) \to bu(x_0 + i\varphi(x_0, 0))$ nontangentially unless x_0 belongs to an exceptional set of measure zero, we define $\Gamma(x_0)$ as the interior of

$$Z^{-1}\big(\{\xi + i\eta : \quad |\xi - x_0| \leq |\eta - \varphi(x_0, 0)|\}\big) \cap \{y > 0\}$$

which is an open set that contains $\{x_0\} \times (0, b)$ and for almost every $x_0 \in (\alpha_j, \beta_j)$ satisfies

$$\lim_{\Gamma(x_0) \ni (x,y) \to (x_0, 0)} u(x, y) = bu(x_0),$$

as follow from the convergence of $U_j(\zeta) \to \zeta_0 = x_0 + i\varphi(x_0, 0)$ as $\zeta = \xi + i\eta \to x_0 + i\varphi(x_0, 0)$, when $|\eta - \varphi(x_0, 0)| \geq |\xi - x_0|$. Note that if L is elliptic at $(x_0, 0)$, i.e., $\varphi_y(x_0, 0) \neq 0$, then $\Gamma(x_0)$ contains a cone $y > \mu|x - x_0|$, with $\mu > 0$, so we recover the classical nontangential convergence valid for elliptic vector fields. On the other hand, if $\varphi_y(x_0, 0) = 0$ and x_0 is of type (II)) then $\Gamma(x_0)$ is still an open neighborhood of $\{x_0\} \times (0, b)$ which cannot contain any cone $y > \mu|x - x_0|$, $\mu > 0$, because its width at height y is $O(|y|^2)$ and it is contained in a cuspoidal region $y > \mu|x - x_0|^{1/2}$ with vertex x_0. We summarize these facts in a more invariant way as

Theorem 3.1. *Let*

$$L = \frac{\partial}{\partial y} + a(x, y)\frac{\partial}{\partial x} = X + iY,$$

$a(x, t) \in C^\infty$ on $(-a, a) \times (-b, b)$, be a locally integrable vector field. Denote by γ_x the integral curve of $X =$ Re *L that stems from $\{x\}$ and enters $\{y > 0\}$.*

Then to each $(x, 0)$ *we may associate a set of convergence* $\Gamma(x)$, *which is an open neighborhood of* γ_x *if* $(x, 0)$ *is of type (II) and reduces to* γ_x *if* $(x, 0)$ *is of type (I) so that for any* $u(x, y) \in L^\infty((-a, a) \times (0, b))$ *that satisfies* $Lu(x, y) = 0$ *for* $y > 0$ *we have*

$$\lim_{\Gamma(x_0) \ni (x,y) \to (x_0,0)} u(x, y) = bu(x_0), \qquad \text{a.e. } x_0 \in (-a, a).$$

In the example below there are points $(x_0, 0)$ of type (I) and type (II). The points of type (II) are all elliptic points and the solution $u(x, y)$ converges nontangentially to its boundary values $bu(x)$ at those points while the convergence occurs strictly along γ_x at points $(x, 0)$ of type (I). The example shows that in general we cannot hope to enlarge $\Gamma(x)$ to an open neighborhood of γ_x at points of type (I). We refer the reader to [BH2] for related results on pointwise convergence to the boundary value.

Example. Let $K \subset (-1, 1) \subset \mathbb{R}$ be a Cantor set with positive measure $|K| > 0$ and denote by (α_j, β_j), $j = 1, 2, \ldots$, its complementary intervals in $(-1, 1)$. Let $b(x) \in C^\infty(-1, 1)$ such that

$$\begin{cases} b(x) = 0 & \text{if } x \in K \text{ and} \\ b(x) > 0, & \text{if } x \in (-1, 1) \setminus K. \end{cases}$$

Define

$$Z(x, y) = x + iyb(x); \qquad L = \frac{\partial}{\partial y} - \frac{ib(x)}{1 + iyb'(x)} \frac{\partial}{\partial x},$$

where $b'(x)$ denotes the derivative of $b(x)$. It is readily checked that $LZ = 0$ and Z_x does not vanish so L is locally integrable. Note that L is nonelliptic exactly at the points $(x, y) \in (-1, 1) \times \mathbb{R}$ where $b(x) = 0$, i.e., at the points of $K \times \mathbb{R}$. Thus, the points of K are of type (I) and those of $(-1, 1) \setminus K$ are elliptic points of type (II).

Consider the characteristic function of the set K, $\chi(x) = 1$ if $x \in K$ and $\chi(x) = 0$ otherwise and set $u(x, y) = \chi(x)$. Clearly, $u \in L^\infty(\mathbb{R}^2)$ and since $u(x, y)$ is independent of y it follows that $\partial_y u = 0$. Furthermore, $\chi'(x)$ is a distribution supported in K and since $b(x)$ vanishes to infinite order on K it follows that $b\chi' = 0$. This shows that $Lu = 0$. Finally, we point out that if $x_0 \in K$, any neighborhood of $\{x_0\} \times (0, 1)$ in $\{y > 0\}$ contains a sequence (x_n, y_n) of points converging to $(x_0, 0)$ such that $u(x_n, y_n) = 0$ while $bu(x_0) = 1$.

4 Appendix

Proof of Lemma 1.1. Since f is bounded, we need only show that $f(x, y)$ converges in $\mathcal{D}'(-a, a)$ to a distribution $bf(x)$ as $y \searrow 0$ as the weak compactness of the unit ball of $L^\infty(-a, a)$ will then show that $bf \in L^\infty(-a, a)$. We will proceed as in [BH1] with minor modifications. Let $\phi \in C_0^\infty(-a, a)$. For $\epsilon \geq 0$ sufficiently small, set

$$L^\epsilon = \frac{\partial}{\partial y} + a(x, y+\epsilon)\frac{\partial}{\partial x}$$

We will choose ϕ_0^ϵ and $\phi_1^\epsilon \in C^\infty((-a, a) \times [0, b))$ such that if

$$\Phi^\epsilon(x, y) = \phi_0^\epsilon(x, y) + y\phi_1^\epsilon(x, y),$$

then

$$(1) \quad \Phi^\epsilon(x, 0) = \phi(x), \quad \text{and} \quad (2) \quad |(L^\epsilon)^*\Phi^\epsilon(x, y)| \leq Cy,$$

where $(L^\epsilon)^*$ denotes the formal transpose of L^ϵ and C depends only on the derivatives of ϕ up to order 2. In particular, C will be independent of ϵ. Define $\phi_0^\epsilon(x, y) = \phi(x)$ and write

$$L^\epsilon = \frac{\partial}{\partial y} + Q^\epsilon\left(x, y, \frac{\partial}{\partial x}\right),$$

and define

$$\phi_1^\epsilon(x, y) = -\frac{\partial}{\partial y}\phi_0^\epsilon(x, y) + (Q^\epsilon)^*\phi_0^\epsilon$$

One easily checks that (1) and (2) above hold with these choices of the ϕ_j^ϵ. We will next use the integration by parts formula of the form

$$\int u(x, T)w(x, T)\, dx - \int u(x, 0)w(x, 0)\, dx = \int_0^T \int_{\mathbb{R}^n} (wPu - uP^*w)\, dxdy$$

which is valid for P a vector field, u and w in $C^1(\mathbb{R} \times [0, T])$ and the $x-$support of w contained in a compact set in $\mathbb{R}$. Note that the x-support of $\Phi^\epsilon(x, y)$ is contained in the support of $\phi(x)$. Let $\psi \in C_0^\infty(B_1(0))$, where $B_1(0)$ denotes the ball of radius 1 centered at the origin in $\mathbb{R}^2$, be of the form $\psi(x, y) = \alpha(x)\beta(y)$. Assume $\int \alpha(x)\, dx = \int \beta(y)\, dy = 1$, and for $\delta > 0$, let $\psi_\delta(x, y) = \delta^{-2}\psi(x/\delta), y/\delta) = \alpha_\delta(x)\beta_\delta(y)$. For $\epsilon > 0$, set $f_\epsilon(x, y) = f(x, y+\epsilon)$. Observe that if $\delta < \epsilon$, then the convolution $f_\epsilon * \psi_\delta(x, y)$ is C^∞ in the region $y \geq 0$. In the integration by parts formula above set $u(x, y) = f_\epsilon * \psi_\delta(x, y)$, $w(x, y) = \Phi^\epsilon(x, y)$ and $P = L^\epsilon$. We get

$$\int_{-a}^{a} f_\epsilon * \psi_\delta(x, 0)\phi(x)\, dx = \int_{-a}^{a} f_\epsilon * \psi_\delta(x, T)\Phi^\epsilon(x, T)\, dx - \int_0^T \int_{-a}^{a} L^\epsilon\,(f_\epsilon * \psi_\delta)\, \Phi^\epsilon\, dxdy + \int_0^T \int_{-a}^{a} f_\epsilon * \psi_\delta (L^\epsilon)^* \Phi^\epsilon\, dxdy. \tag{a.1}$$

We have chosen $0 < T < b$ such that $x \mapsto f(x, T)$ is bounded and measurable. Fix $\epsilon > 0$. Let $\delta \to 0^+$. Note that $\{f_\epsilon * \psi_\delta(x, y)\}$ is uniformly bounded and converges almost everywhere to $f_\epsilon(x, y)$ on a neighborhood W of $\operatorname{supp} \phi \times [0, T]$. Hence,

$$L^\epsilon\,(f_\epsilon * \psi_\delta) \to L^\epsilon f_\epsilon$$

in $\mathcal{D}'(W)$ as $\delta \to 0^+$. Moreover, $L^\epsilon f_\epsilon(x, y) = Lf(x, y + \epsilon) \in L^\infty$. Hence, by Friederichs' Lemma,

$$L^\epsilon\,(f_\epsilon * \psi_\delta) \to L^\epsilon f_\epsilon$$

in $L^2(W)$ as $\delta \to 0^+$. Furthermore, using that the trace map $(0, b) \ni y \mapsto f(x, y) \in \mathcal{D}'((-a, a))$ is continuous, we also see that the first integral on the right hand side of (a.1) converges to

$$\int_{-a}^{a} f(x, T + \epsilon)\Phi^\epsilon(x, T)\, dx.$$

We thus get

$$\int_{-a}^{a} f(x, \epsilon)\phi(x)\, dx = \int_{-a}^{a} f(x, T + \epsilon)\Phi^\epsilon(x, T)\, dx - \int_0^T \int_{-a}^{a} L^\epsilon f_\epsilon(x, y)\Phi^\epsilon(x, y)\, dxdy + \int_0^T \int_{-a}^{a} f_\epsilon(x, y)(L^\epsilon)^* \Phi^\epsilon(x, y)\, dxdy$$

In the third integral on the right, we have

$$|f_\epsilon(x, y)(L^\epsilon)^* \Phi^\epsilon(x, y)| \le Cy,$$

where C depends only on the derivatives of ϕ upto order 2. By the dominated convergence theorem, as $\epsilon \to 0$, this third integral converges to

$$\int_0^T \int_{-a}^a f L^* \Phi^0 \, dx dy.$$

In the second integral on the right, note that since $Lf \in L^2(X \times (0, T))$, as $\epsilon \to 0$, the translates $L^\epsilon f_\epsilon = (Lf)_\epsilon \to Lf$ in L^2. We thus get

$$\langle bf, \phi \rangle = \int_{-a}^a f(x, T)\Phi(x, T)\, ds - \int_0^T \int_{-a}^a Lf \Phi \, dx dy + \int_0^T \int_{-a}^a f L^* \Phi \, dx dy,$$

where $\Phi = \Phi^0$. □

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