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Microscopic structure of the k-step exclusion process

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Abstract. We review the hydrodynamics and discuss the shock, rarefaction fan and contact discontinuity at a microscopic level for a one-dimensional totally asymmetric k-step exclusion process. In particular we define a microscopical object that identifies the shock in the decreasing case.

Keywords: k-step exclusion process, non-concave non-convex flux, microscopic shock, rarefaction fan, contact discontinuity, simulation.

Mathematical subject classification: Primary: 60K35; Secondary: 82C22.

1 Introduction and Notation

To study the so called long range exclusion process, Liggett (1980) introduced a Feller non conservative approximation of it. A conservative version of this dynamics, called k-step exclusion process, was investigated in Guiol (1999). We summarize some of its features: It is described in the following way in dimension 1.

Let $k \in \mathbb{N}^* := \{1, 2, \dots\}$, $\mathbf{X} := \{0, 1\}^{\mathbb{Z}}$ be the state space, and let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain on $\mathbb{Z}$ with transition matrix $p(.,.)$ and for which $\mathbb{P}^x(X_0 = x) = 1$. The hypothesis $\sup_{y\in\mathbb{Z}} \sum_{x\in\mathbb{Z}} p(x, y) < +\infty$ ensures that L_k is an infinitesimal pregenerator:

$$L_k f(\eta) = \sum_{\eta(x)=1,\eta(y)=0} q_k(x, y, \eta)\left[f(\eta^{x,y}) - f(\eta)\right], \tag{1}$$

where f is a cylinder function,

$$q_k(x, y, \eta) = \mathbb{E}^x\left[\prod_{i=1}^{\sigma_y - 1} \eta(X_i), \sigma_y \le \sigma_x, \sigma_y \le k\right]$$

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is the intensity to move from x to y on configuration η, $\sigma_y = \inf\{n \geq 1 : X_n = y\}$ is the first (non zero) arrival time to site y of the chain starting at site x and $\eta^{x,y}$ is configuration η where the states of sites x and y were exchanged.

In words if a particle at site x wants to jump it may go to the first empty site encountered before returning to site x following the chain X_n (starting at x) provided it takes less than k attempts; otherwise the movement is cancelled.

By Hille-Yosida's theorem, the closure of L_k generates a continuous Markov semi-group $S_k(t)$ on $C(\mathbf{X})$, the set of continuous functions on $\mathbf{X}$, which corresponds to the k-step exclusion process $(\eta_t)_{t\geq 0}$. A constructive way to define the process is to adapt the graphical construction due to Harris (1972); we present this construction in section 2. When $k = 1$, $(\eta_t)_{t\geq 0}$ reduces to the well-known simple exclusion process (see Liggett (1985) and Liggett (1999) for a complete study of the latter). The k-step exclusion shares some of the properties of simple exclusion: For instance, it is an *attractive process.*

Let $\mathcal{I}_k$ — resp. $\mathcal{S}$ — be the set of invariant measures for $(\eta_t)_{t\geq 0}$ — resp. of translation invariant measures on $\mathbf{X}$. If $p(.,.)$ is translation invariant and irreducible then (Guiol (1999)):

$$(\mathcal{I}_k \cap \mathcal{S})_e = \{\nu_\alpha : \alpha \in [0,1]\},$$

where the index e mean extremal and ν_α is the Bernoulli product measure with constant density α, *i.e.*, the measure with marginal

$$\nu_\alpha\{\eta \in \mathbf{X} : \eta(x) = 1\} = \alpha.$$

In Bahadoran *et al.* (2002), a constructive method, relying on the explicit construction of Riemann solutions without assuming convexity of the flux function, lead to Euler hydrodynamics of one-dimensional attractive processes with irreducible jumps and product invariant measures. The k-step exclusion process is a natural illustration, since its flux function, neither concave nor convex, gives rise to non-standard Riemann solutions for the hydrodynamics: They present stable increasing and decreasing shocks, rarefaction fans and contact discontinuities.

In this paper, we consider the one-dimensional totally asymmetric k-step exclusion process, for which we investigate analogues of these macroscopic structures at a microscopic level: Is there a microscopic object corresponding to a shock, a rarefaction fan or a contact discontinuity? (for the study of the microscopic structure of shock for simple exclusion, look at Liggett (1999)). We give some answers to this question. Our techniques rely on couplings and particular interpretations of the process: Namely, we introduce an auxiliary process, the stack process (see section 3.1); and since we are in the totally asymmetric case the process has a

Pushing interpretation 1. *When a particle at some site x is able to jump to an empty site $y > x$ (observe that $y - x \leq k$ and all the sites between x and y are occupied), on the original system we interpret the dynamics as follows. All particles between site x (included) and y (excluded) are pushed of one unit to the right, preserving the relative order of the particles.*

In section 2 we present the graphical construction for the one-dimensional k-step exclusion process and review its hydrodynamics in the totally asymmetric case. Sections 3 to 6 analyse the microscopic counterparts to hydrodynamics dictated by the flux function under Riemann initial condition. More precisely, in section 3, for the convex part of the convex envelope of the flux function, we describe a microscopic object identifying the shock when initially there is no particle at the right of the origin. For this case we perform some simulations confirming the idea that the rightmost particle identifies the shock. In section 4 for the the concave part of the convex envelope of the flux function we look at the shock when starting from the initial condition where all sites to the right of the origin are occupied. In section 5 we study the rarefaction fan, and in section 6 the contact discontinuity.

2 Preliminaries

In the first part of this section we present the graphical construction for the k-step exclusion process in dimension one associated to a finite range Markov chain. The graphical representation is essential to understand how to simulate the process. In the second part we describe the results of Bahadoran *et al.* (2002) for the totally asymmetric k-step exclusion process.

2.1 Graphical Construction of the k-step exclusion process

This construction is an adaptation of those given by Ferrari (1992) for the simple exclusion process and Durrett (1995) for spin systems (see also Seppäläinen (2000)). It is based on a graphical construction for the (finite) long range exclusion process (Ferrari, *private communication*). It relies on coupling.

Recall that $p(.,.)$ is the transition matrix of a Markov chain with finite range on $\mathbb{Z}$. Let M be a two-dimensional Poisson process on $\mathbb{R} \times \mathbb{R}^+$ with rate 1. For all $x \in \mathbb{Z}$ define a family of partitions $(I_i^x)_{1 \le i \le k}$ of the interval $[0, 1)$ such that

$$I_1^x = \bigcup_{y \in \mathbb{Z}} \left\{ I^{x,y} := \left[\sum_{z<y} p(x,z), \sum_{z \le y} p(x,z) \right) \right\};$$

and for any $1 < i \le k$ partition I_i^x is a refining of partition I_{i-1}^x in such a way that each element $I^{x,y_1,\dots,y_{i-1}}$ of I_{i-1}^x (with length $p(x,y_1)\dots p(y_{i-2},y_{i-1})$) is partitioned into intervals $(I^{x,y_1,\dots,y_i})_{y_i \in \mathbb{Z}}$ with length $p(x,y_1)\dots p(y_{i-1},y_i)$. Observe that in this way if $x \in I^{x,y_1,\dots,y_k}$ then $x \in I^{x,y_1,\dots,y_j}$ for all $1 \le j \le k$.

Now define a family of random times $(\tau_n^x)_{n \in \mathbb{N}^*, x \in \mathbb{Z}}$ such that for each $x \in \mathbb{Z}$

$$\tau_1^x := \inf\{t > 0 \ : \ M([x, x+1) \times [0,t]) > 0\},$$

and

$$\tau_n^x := \inf\{t > \tau_{n-1}^x \ : \ M([x, x+1) \times (\tau_{n-1}^x, t]) > 0\}.$$

At each time τ_n^x we draw from x a family of k arrows $(a_1^{x,n}, \dots, a_k^{x,n})$ with decreasing order of priority according to the following rule: if $s := \inf\{u > 0 : M([x, x+u) \times \{\tau_n^x\}) > 0\} \in I^{x,y_1,\dots,y_k}$ then the sequence of arrows with decreasing priority is $(a_1^{x,n} := x \to y_1, a_2^{x,n} := x \to y_2, \dots, a_k^{x,n} := x \to y_k)$.

We denote by P the distribution of the above configurations of arrows, we call α such a random graph; P is only determined by the Poisson process M. This construction enables to compute the time evolution of our process by the following argument due to Harris (1972): there exists a time $t_0 > 0$ such that P-*a.s.*, all the connected components from time 0 to t_0 in the random graph are finite. This result is a direct adaptation of Theorem 2.1. in Durrett (1995) (see also Seppäläinen (2000)).

Thus we can start from an initial configuration $\eta \in \mathbf{X}$ at time 0 and we construct $(\eta_t)_{0 \le t \le t_0}$ as a function of η and α. In each finite component of the random graph we label the sequences of (k) arrows in their order of appearance (in time). If the first sequence of arrows starts at a site, say x, where a particle stands, then we select the first arrow in that sequence that leads to a vacant site, say y, and the particle jumps to site y; if there is no such arrow the jump is cancelled and the particle stays at x (site x is considered vacant during the jump). Then we have a second sequence of arrows and we repeat this procedure, until the last sequence of arrows encountered in the component.

In this way we have constructed the process up to time t_0. We iterate the same argument to construct the process up to time $2t_0$ and so on ...

2.2 Hydrodynamics in the Riemann case

In this section, we particularize Theorem 2.1 of Bahadoran *et al.* (2002) for the totally asymmetric k-step exclusion process: The limiting density profile at time t is the (weak) entropy solution of equation

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t} + \dfrac{\partial G_k(u)}{\partial x} = 0 \\ \\ u(x,0) = u^0(x), \end{array} \right. \tag{2}$$

where G_k represents the flux of particles:

$$G_k(u) = \sum_{j=1}^{k} j u^j (1-u).$$

We claim that for any $k \geq 2$, G_k admits a unique inflection point $a := a(k)$ in $(0,1)$ and G_k is convex on $(0,a)$ and concave on $(a,1)$. Indeed:

Lemma 4.1 of Bahadoran *et al.* (2002) states that $G_k(u)$ has at most one inflection point in $(0,\infty)$. To see that this inflection point exists and is in $(0,1)$ observe that $G_k(u)$ is a polynomial in u, $G_k(0) = G_k(1) = 0$, $G'_k(0) = 1 > 0$ and $G''_k(0) = 2 > 0$. This means that G_k is strictly convex and increasing in some neighborhood of 0. Then the only possibility, starting from $u = 0$ with value 0, to reach the same value again at $u = 1$ is the existence of an inflection point in between. This also shows the second part of the claim.

Let $H_k(u) = G'_k(u)$ denote the characteristic speed and let

$$S_k[\lambda;\rho] := \frac{G_k(\lambda) - G_k(\rho)}{\lambda - \rho} \tag{3}$$

be the shock speed.

Let $h_{k,1}$ and $h_{k,2}$ be the inverses of H_k respectively restricted to $(-\infty, a)$ and to $(a, +\infty)$. For $u < a$ we consider the upper convex envelope $(G_k)^c$ of G_k on $(u, +\infty)$, and we denote by $u^* := u^*(k)$ the first point where $(G_k)^c$ coincides with G_k. In the same way when $u > a$, $u_* := u_*(k)$ denotes the first point where the lower convex envelope $(G_k)_c$ of G_k on $(-\infty, u)$ coincides with G_k.

For the k-step exclusion process, $k \geq 2$, the result of Bahadoran *et al.* (2002) reduces to

Theorem 1. *Let $v \in \mathbb{R}$, $\lambda, \rho \neq a$, and $\mu_{\lambda,\rho}$ be the product measure on $\mathbb{Z}$ with densities λ for $x \leq 0$ and ρ for $x > 0$, i.e., the product measure with marginals*

$$\mu_{\lambda,\rho}\{\eta \in \mathbf{X} : \eta(x) = 1\} = \left\{ \begin{array}{ll} \lambda, & x < 0; \\ \rho, & x \geq 0. \end{array} \right.$$

Then

$$\lim_{t\to\infty} \mu_{\lambda,\rho}\tau_{\lfloor vt\rfloor} S_k(t) = \nu_{u(v,1)}$$

at every continuity point of $u(.,1)$*, where* $\tau_{\lfloor vt\rfloor}$ *is the spatial shift by* $\lfloor vt\rfloor$*, the integer part of* vt*, and* $\nu_{u(v,1)}$ *denotes the Bernoulli product measure with density* $u(v,1)$ *defined by:*

Case 1. $\lambda < \rho < a$*: continuous solution, with a rarefaction fan*

$$u(x,1) = \begin{cases} \lambda, & x \le H_k(\lambda); \\ h_{k,1}(x), & H_k(\lambda) < x \le H_k(\rho); \\ \rho, & H_k(\rho) < x. \end{cases}$$

Case 2. $\rho < \lambda < \rho^*$ $(\rho < a)$*: entropy shock*

$$u(x,1) = \begin{cases} \lambda, & x \le S_k[\lambda;\rho]; \\ \rho, & x > S_k[\lambda;\rho]. \end{cases}$$

Case 3. $\rho < \rho^* < \lambda$ $(\rho < a)$*: contact discontinuity*

$$u(x,1) = \begin{cases} \lambda, & x \le H_k(\lambda); \\ h_{k,2}(x), & H_k(\lambda) < x \le H_k(\rho^*); \\ \rho, & H_k(\rho^*) < x. \end{cases}$$

Case 4. $a < \rho < \lambda$*: continuous solution, with a rarefaction fan*

$$u(x,1) = \begin{cases} \lambda, & x \le H_k(\lambda); \\ h_{k,2}(x), & H_k(\lambda) < x \le H_k(\rho); \\ \rho, & H_k(\rho) < x. \end{cases}$$

Case 5. $\rho > \lambda > \rho_*$ $(\rho > a)$*: entropy shock*

$$u(x,1) = \begin{cases} \lambda, & x \le S_k[\lambda;\rho]; \\ \rho, & x > S_k[\lambda;\rho]. \end{cases}$$

Case 6. $\rho > \rho_* > \lambda$ $(\rho > a)$*: contact discontinuity*

$$u(x,1) = \begin{cases} \lambda, & x \le H_k(\lambda); \\ h_{k,1}(x), & H_k(\lambda) < x \le H_k(\rho_*); \\ \rho, & H_k(\rho_*) < x. \end{cases}$$

In particular when $k = 2$, the flux function reads $G_2(u) = u + u^2 - 2u^3$ and has one inflection point $a = 1/6$. The characteristic speed is $H_2(u) = 1 + 2u - 6u^2$

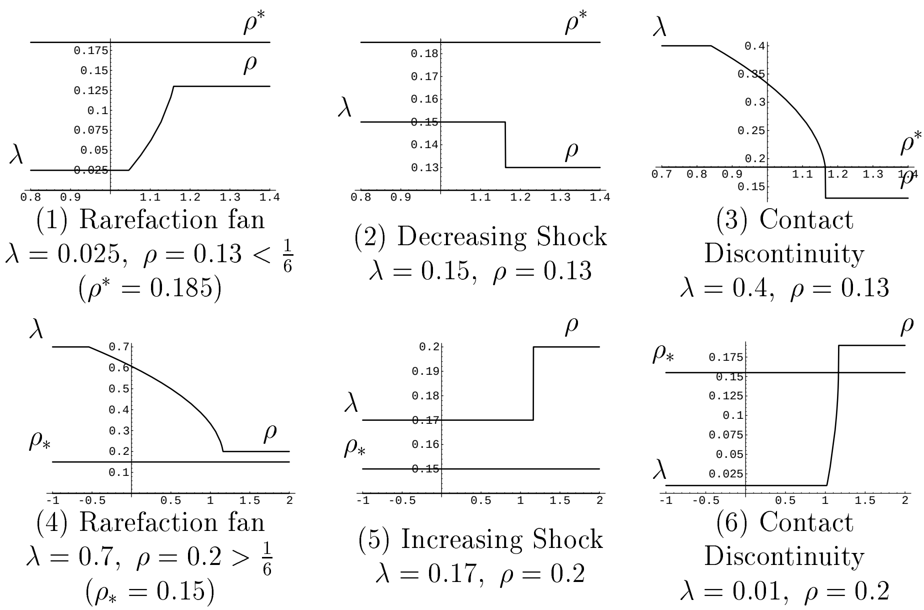

Figure 1: Hydrodynamic behavior of the 2-step exclusion with Riemann initial conditions, graph of the exact solution $u(x/t, 1)$ for different values of λ and ρ.

and the shock speed reads $S_2[\lambda; \rho] = 1 + (\lambda + \rho) - 2(\lambda^2 + \lambda\rho + \rho^2)$. Finally $h_{2,1}(x) = (1/6)(1 - \sqrt{7 - 6x})$, $h_{2,2}(x) = (1/6)(1 + \sqrt{7 - 6x})$ on $(-\infty, 7/6)$ and $u^* = u_* = (1 - 2u)/4$.

Figure 1 shows the six possible behaviors of the (self-similar) solution $u(v, 1)$ for the 2-step exclusion process. Cases 1 and 2 present respectively a rarefaction fan with increasing initial condition and a preserved decreasing shock. These situations as well as cases 3 and 6 cannot occur for simple exclusion. Observe also that $\rho \geq 1/2$ implies $\rho_* \leq 0$, which leads only to cases 4 and 5, and excludes case 6 (going back to a simple exclusion behavior).

3 Shock in the convex part of the flux

We consider here Case 2 of Theorem 1 with a right density 0, *i.e.*, a decreasing entropy shock: $u_0(x) = \lambda \mathbf{1}_{\{x \leq 0\}}$, so that $\rho = 0 < \lambda < \rho^*$. We take the Pushing interpretation 1 of the dynamics.

3.1 The Stack Process

We introduce an auxiliary process to k-step exclusion, called *the stack process*. Let $\eta \in \{0,1\}^{\mathbb{Z}}$ be a given configuration of the k-step exclusion process. To it we associate a *stack configuration* $\xi \in \mathbb{N}^{\mathbb{Z}}$ in the following way: We label the holes in η, the first hole to the left of (or on) the origin is labeled 0-th hole, others getting the natural order. Then $\xi(\ell)$, the height of the ℓ-th stack, denotes the number of particles in η between the ℓ-th and the $(\ell+1)$-st hole.

The k-step exclusion dynamics on $(\eta_t)_{t\geq 0}$ induces the following dynamics on the stack process $(\xi_t)_{t\geq 0}$. Associate with each stack, say the ℓ-th one, a Poisson process of rate $g(\ell) := k \wedge \xi(\ell)$. When its clock rings, with a probability $(1/g(\ell))\mathbf{1}_{\{g(\ell)>0\}}$ the topmost $j \in \{1,\ldots,g(\ell)\}$ particles on the ℓ-th stack are transferred to the bottom of the $(\ell+1)$-st stack.

To study Case 2 of Theorem 1, we consider a k-step exclusion with two types of particles: Particles η, of initial distribution $\mu_{\lambda,0}$, are *first class particles*, and particles η', of initial distribution $\mu_{0,\lambda}$, are *second class particles*. It means that particles η evolve as if they were alone in the system (they "do not see" particles η'): We summarize in the following the two possible classes of situations where a first class particle, (the leftmost) denoted by 1, ends its jump on a site occupied by a second class particle, denoted by 2. An empty site is denoted by 0, $*$ denotes 1 or 2, and $\clubsuit$ denotes 0, 1 or 2 (the total number of figured sites, in each case, is $k+1$).

$11\cdots12*\cdots*0\clubsuit\cdots\clubsuit \dashrightarrow 01\cdots11*\cdots*2\clubsuit\cdots\clubsuit$
$11\cdots12*\cdots* \dashrightarrow 21\cdots11*\cdots*$

The stack process $(\xi_t)_{t\geq 0}$ will be associated to $(\eta_t+\eta'_t)_{t\geq 0}$. Therefore its initial distribution is product geometric with average height $\lambda/(1-\lambda)$, and is invariant for the evolution of stacks. We will say that a stack is a *first class stack* if it is either empty or contains only first class particles; otherwise it is called a *second class stack*.

We observe that the stacks to the left of the origin are and remain at equilibrium with a product geometric distribution with parameter λ. Let N_t be their flux through the origin up to time t. Observe that this is the flux crossing a tagged hole in the k-step exclusion process. The following are straightforward consequences of results in Ferrari (1986), Ferrari (1996). For the tagged hole in a totally asymmetric k-step exclusion process, or equivalently tagged particle $\mathcal{Y}(t)$ in a totally asymmetric (in the opposite direction) discrete k-range Hammersley process $(\zeta_t)_{t\geq 0}$ (see Ferrari (1996) for an account on the discrete Hammersley process; its k-range version restricts the size of the jumps up to k sites), we have

1. $(\zeta_t, \mathcal{Y}(t))_{t\geq 0}$ and $(\zeta_t')_{t\geq 0} = (\tau_{\mathcal{Y}(t)}\zeta_t)_{t\geq 0}$ are Markov processes.
2. ν_ρ is invariant and translation invariant for the process (this is Guiol (1999)).

Thus ν_ρ' is invariant for $(\zeta_t')_{t\geq 0}$ which says that the tagged hole sees equilibrium. Now using the ergodicity of the flux through the origin we obtain,

$$\lim_{t\to+\infty} \frac{N_t}{t} = \sum_{j=1}^{k} j\lambda^j \text{ a.s.} \tag{4}$$

We denote by v_s the velocity of the shock (cf. (3)) $S_k[\lambda; 0] = G_k(\lambda)/\lambda$. Let X_t and S_t be the number of *first class and second class stacks respectively* up to $v_s t$ (*i.e.*, in the region $\{0, 1, 2, \ldots, \lfloor v_s t \rfloor\}$). Then

Theorem 2.

$$\lim_{t\to+\infty} \frac{X_t}{t} = v_s \text{ in probability.}$$

Proof. We denote by M_t the total number of particles in the stacks in $[0, \lfloor v_s t \rfloor]$. Since the number of particles in the stacks are independent geometric random variables

$$\lim_{t\to+\infty} \frac{M_t}{v_s t} = \frac{\lambda}{1-\lambda} \text{ a.s.} \tag{5}$$

Let Y_t be the position (in the η process) of the topmost particle on the right-most occupied stack in $[0, v_s t]$. $M_t + \lfloor v_s t \rfloor$ gives the number of holes and particles in the stacks in the region $[0, v_s t]$. Since the bottom of 0-th stack $\xi(0)$ is at a distance given by a geometric random variable from the origin at time zero, and the hole in the η process corresponding to the bottom of the 0-th stack moves by $-N_t$ in time t we have

$$Y_t = (M_t + \lfloor v_s t \rfloor) - N_t - G,$$

where G is a geometric random variable of parameter λ. Since

$$\frac{\lambda}{1-\lambda} v_s = \frac{G_k(\lambda)}{1-\lambda} = \sum_{j=1}^{k} j\lambda^j$$

we have by (4) and (5)

$$\lim_{t\to+\infty} \frac{M_t - N_t}{t} = 0 \text{ a.s.}$$

which implies

$$\lim_{t\to+\infty} \frac{Y_t}{t} = v_s \text{ a.s.} \tag{6}$$

Recall that S_t is the number of second class stacks in $[0, v_s t]$. The result would follow if we can prove that $S_t/t \to 0$ in probability. First we observe that the stacks in $[0, v_s t]$ to the right of the rightmost occupied stack (if there are any) are all first class stacks since they are all empty stacks. Since S_t is bounded by the number of second class particles in $[0, v_s t]$, it is bounded by Z_t, the number of second class particles (in the η process) on or to the left of Y_t. By (6) for all $\delta > 0$

$$\lim_{t\to+\infty} \mathbf{P}\left(\frac{Y_t}{t} > v_s + \frac{\delta}{3}\right) = 0.$$

Let $Z_t = Z_t^1(\delta) + Z_t^2(\delta)$ where $Z_t^1(\delta)$ ($Z_t^2(\delta)$) is the number of second class particles in $\mathbb{Z} \cap [(v_s - \delta/3)t, Y_t]$ ($\mathbb{Z} \cap (-\infty, (v_s - \delta/3)t)$). This in turn implies

$$\lim_{t\to+\infty} \mathbf{P}\left(\frac{Z_t^1(\delta)}{t} > \frac{2\delta}{3}\right) = 0.$$

From the hydrodynamic limit we know that the empirical density of second class particles to the left of $(v_s - \delta/3)t$ (in the η process) goes to zero in probability as $t \to \infty$. This gives us

$$\lim_{t\to+\infty} \mathbf{P}\left(\frac{Z_t^2(\delta)}{t} > \frac{\delta}{3}\right) = 0$$

thus

$$\lim_{t\to+\infty} \mathbf{P}\left(\frac{Z_t}{t} > \delta\right) = 0$$

proving that $S_t/t \to 0$ in probability. □

3.2 A lower bound for the speed of the rightmost particle

Distribute particles according to $\mu_{\lambda,0}$. Let W_t be the position of the rightmost particle, that is $W_t = \sup\{x \in \mathbf{Z} : \eta_t(x) = 1\}$.

Corollary 3.

$$\lim_{t\to+\infty} \mathbf{P}\left(\frac{W_t}{t} \geq v_s\right) = 1.$$

Proof. Suppose there exists $\varepsilon > 0$ and $\delta > 0$ such that for any $T > 0$ there exists $t > T$ such that

$$\mathbf{P}\left(v_s - \frac{W_t}{t} > \delta\right) > \varepsilon.$$

Let Y_t be as before, the position of the topmost particle in the rightmost occupied stack before $v_s t$. From (6) there exists T_1 such that if $t > T_1$

$$\mathbf{P}\left(|Y_t - v_s t| \leq \frac{\delta}{2}t\right) > 1 - \frac{\varepsilon}{2}$$

Therefore for $t > T \vee T_1$

$$\mathbf{P}\left(Y_t - W_t > \frac{\delta}{2}t\right) \geq \mathbf{P}\left(Y_t - W_t > \frac{\delta}{2}t;\, |Y_t - v_s t| \leq \frac{\delta}{2}t\right) > \frac{\varepsilon}{2}.$$

Let us denote by ξ_t^Y and ξ_t^W respectively the rightmost occupied stack in $[0, v_s t]$ and the stack containing the particle located at W_t in the η process, which means that in this proof we call a stack what we may find between two holes (*e.g.* we consider either empty or occupied stacks). We want to show that with positive probability a positive fraction of the occupied stacks in $[0, v_s t]$ lies between ξ_t^Y and ξ_t^W. Since ξ_t^W is at or behind ξ_t^Y this would imply that a positive fraction of stacks in $[0, v_s t]$ are second class stacks with positive probability leading to a contradiction of Theorem 2. This is equivalent to saying that the number R_t of nonempty blocks of particles between Y_t and W_t is greater than βt for some constant $\beta > 0$ with positive probability. Let D_j^t, $0 \leq j \leq \lfloor v_s t \rfloor$, be the number of nonempty blocks of particles in the interval $(j, j + \delta t/2)$. A nonempty block of particles is counted as being in $(j, j + \delta t/2)$ if and only if the holes at the left and right ends of the block are in $[j, \lfloor j + \delta t/2 \rfloor]$. Then:

$$\begin{aligned}\mathbf{P}\left(Y_t - W_t > \frac{\delta}{2}t,\ |Y_t - v_s t| \leq \frac{\delta}{2}t,\ \min_{0\leq j\leq\lfloor v_s t\rfloor} D_j^t > \beta t\right)&\\ \leq \mathbf{P}\left(Y_t - W_t > \frac{\delta}{2}t,\ |Y_t - v_s t| \leq \frac{\delta}{2}t,\ R_t > \beta t\right).& \qquad (7)\end{aligned}$$

Divide $[0, v_s t + \delta t/2]$ into C intervals of size $\delta t/4$. Let B_i^t, $1 \leq i \leq C$ be the number of nonempty blocks of particles in the i-th interval. Since for all $0 \leq j \leq \lfloor v_s t \rfloor$, D_j^t contains a B_i^t for some $1 \leq i \leq C$,we have for any $\beta > 0$:

$$\mathbf{P}\left(\min_{0\leq j\leq \lfloor v_s t\rfloor} \frac{D_j^t}{\delta t/2} > \beta\right) \geq \mathbf{P}\left(\min_{1\leq i\leq C} \frac{B_i^t}{\delta t/4} > 2\beta\right)$$

By law of large numbers for independent geometric random variables (cf. (5)), and because there is a hole before every block of particles, there exists $M = M(\varepsilon/(4C))$ such that for $t > M$,

$$\mathbf{P}\left(\frac{B_i^t}{\delta t/4} > \frac{\lambda}{2}\left(\frac{\lambda}{1-\lambda}+1\right)^{-1}\right) > 1 - \frac{\varepsilon}{4C}$$

which implies

$$\mathbf{P}\left(\min_{0\leq j\leq \lfloor v_s t\rfloor} \frac{D_j^t}{\delta t/2} > \frac{\lambda(1-\lambda)}{4}\right) \geq \mathbf{P}\left(\min_{1\leq i\leq C} \frac{B_i^t}{\delta t/4} > \frac{\lambda(1-\lambda)}{2}\right) > 1 - \frac{\varepsilon}{4}$$

Therefore there exists $t > T \vee M \vee T_1$ such that

$$\mathbf{P}\left(Y_t - W_t > \frac{\delta}{2}t, |Y_t - v_s t| \leq \frac{\delta}{2}t, \min_{0\leq j\leq \lfloor v_s t\rfloor} D_j^t > \frac{\lambda(1-\lambda)}{4}\frac{\delta t}{2}\right) > \frac{\varepsilon}{4}.$$

This proves, by (7), that for any $T > 0$ there exists $t > T$ such that

$$\mathbf{P}\left(R_t > \frac{\lambda(1-\lambda)}{4}\frac{\delta}{2}t\right) > \frac{\varepsilon}{4}$$

Thus the number S_t of second class stacks in $[0, v_s t]$ is greater than $\beta t = \lambda(1-\lambda)\delta t/8$. Since $S_t + X_t = v_s t$, this contradicts Theorem 2. $\square$

3.3 Simulations on the convex part of the flux function

To conclude this section we present some computer simulations for the 2-step exclusion process with Riemann initial profile with $\rho = 0 < \lambda < \rho^* = 1/4$. The simulation was programmed in Ox (version 2.20 for AIX, see Doornik (1999)).

As we have shown in the last paragraph, v_s is a lower bound for the speed of the rightmost particle W_t. However the intuition indicates that it should be, indeed, the correct limit. Loosely speaking if, by contradiction, we suppose that the rightmost particle is at a (long) distance in front of the shock then its velocity should be 1 (the speed of a non interacting particle). Thus as $v_s > 1$ (for the set of λ we consider) the distance between the shock and this particle cannot be maintained.

The following simulations confirm that impression. We simulate the process and get some estimates for the mean velocity of the rightmost particle $\overline{W}_t/t$ and its variance

$$\sigma^2(t) := \mathbf{E}(W_t - \mathbf{E}W_t)^2. \tag{8}$$

The simulation

The original process (infinite system) is approximated by a finite system with border conditions on a moving frame (following the shock). As we treat the case with $\rho = 0$ we will not need a border condition to the right of the frame. Our main worry is to keep "enough space" between the rightmost particle and the left border (where we put a reservoir of particles) so that the shock does not reach the left border which ensures a good approximation of local equilibrium.

The totally asymmetric 2-step exclusion process is constructed on a set of sites, we call the *frame*, $\Psi_t := \{L_t, L_t + 1, \dots, R_t\}$ where $L_0 < 0 < R_0$. Denote by $|\Psi_t|$ the number of sites in Ψ_t. On and to the left of site 0 particles are initially distributed according to a Bernoulli product measure with parameter λ. Sites to right of 0 are initially empty.

The algorithm is as follows: At each step, say at time t_j, we randomize an exponential random variable T with parameter $|\Psi_{t_j}|+2$. We then set $t_{j+1} = t_j + T$ and we draw uniformly a site x in $\{L_{t_j} - 2, L_{t_j} - 1\} \cup \Psi_{t_j}$. If $x \in \Psi_{t_j}$ and is occupied by a particle then this particle goes to the first empty site it encounters to its right in $\{x + 1, x + 2\}$ (2-step exclusion rule). If both sites are occupied the movement is cancelled. If $x = L_{t_j} - 2$ or $L_{t_j} - 1$ respectively, a uniform random variable U is drawn on $(0, 1)$; if $U \leq \lambda^2$ or $U \leq \lambda$ respectively, then a particle of the reservoir tries to enter into the system according to the 2-step exclusion rule; otherwise nothing happens.

To (re)adjust the frame Ψ_t we proceed as follows. We take "pictures" of the system at each time interval, say I, suitably fixed at the beginning of the program. After each picture and before resuming the simulation, we check if the left side and the right side of the frame are at least at a $2I$ distance from the rightmost particle in the frame and setup the system. The first check guarantees there is enough space between the reservoir and the rightmost particle and the second check prevents the rightmost particle of going out of the frame before the next picture.

For some fixed values of λ, we simulated the dynamics of 1,000 independent instances of the 2-step exclusion process and observed the mean position of the rightmost particle and its dispersion for t varying from 0 to 15,000.

We choose to present some pictures for $\lambda = 0.2$ at different times: $t = 0, 100, 200, 300, 400$ and 500 with 250 independent instances for sake of visualization.

For each time the first picture in Figure 2 presents the mean density of the process: The curve in the middle is surrounded by a one standard deviation interval. Initially we observe equilibrium at density 0.2 to the left of the origin and no particles to the right. The second picture represents the distribution of the position of the rightmost particle at time t.

Figure 3 shows the shock speed $v_s(\lambda) = 1 + \lambda - 2\lambda^2$ (the solid line) and the rescaled simulated mean position (*i.e.*, $\overline{W}(t)/t$) for different values of λ for $t = 7,500$ and $t = 15,000$. It gives a glance of the adequation between the real curve of v_s and the mean of the simulated velocity of the rightmost particle.

The lower bound given in section 3.2 and the results of the simulation indicate that the rightmost particle is a good candidate for a microscopic indicator of the position of the shock for $\rho = 0$.

For the simple exclusion process (*i.e.*, $k = 1$), in the stable shock case (*i.e.*, increasing shock $\lambda < \rho$), Ferrari & Fontes (1994) proved that the diffusion coefficient $\lim_t \Sigma^2(t)/t$ is constant, where $\Sigma^2(t)$ was the variance, at time t, of the position of a second class particle originally at 0; for $\lambda = 0$ this second class particle corresponds with the leftmost particle. Observe that in the case we consider in this section (*i.e.*, decreasing stable shock for $k = 2$, with $\lambda < 1/4$ and $\rho = 0$) the second class particle does not correspond to the rightmost particle because it may jump back (even with the pushing interpretation).

The simulations for $k = 2$ and $\rho = 0$, make us suspect that $\sigma^2(t)$ (cf. (8)) may also grow linearly, as shown in Figure 4. Here, $\sigma^2(t)$ is estimated by the empirical variance

$$\hat{\sigma}^2(t) = \frac{1}{K} \sum_{i=1}^{K} \left(W_{t,i} - \overline{W}_t\right)^2 ,$$

where K is the number of independent realizations of the process and

$$\overline{W}_t = K^{-1} \sum_{i=1}^{K} W_{t,i} .$$

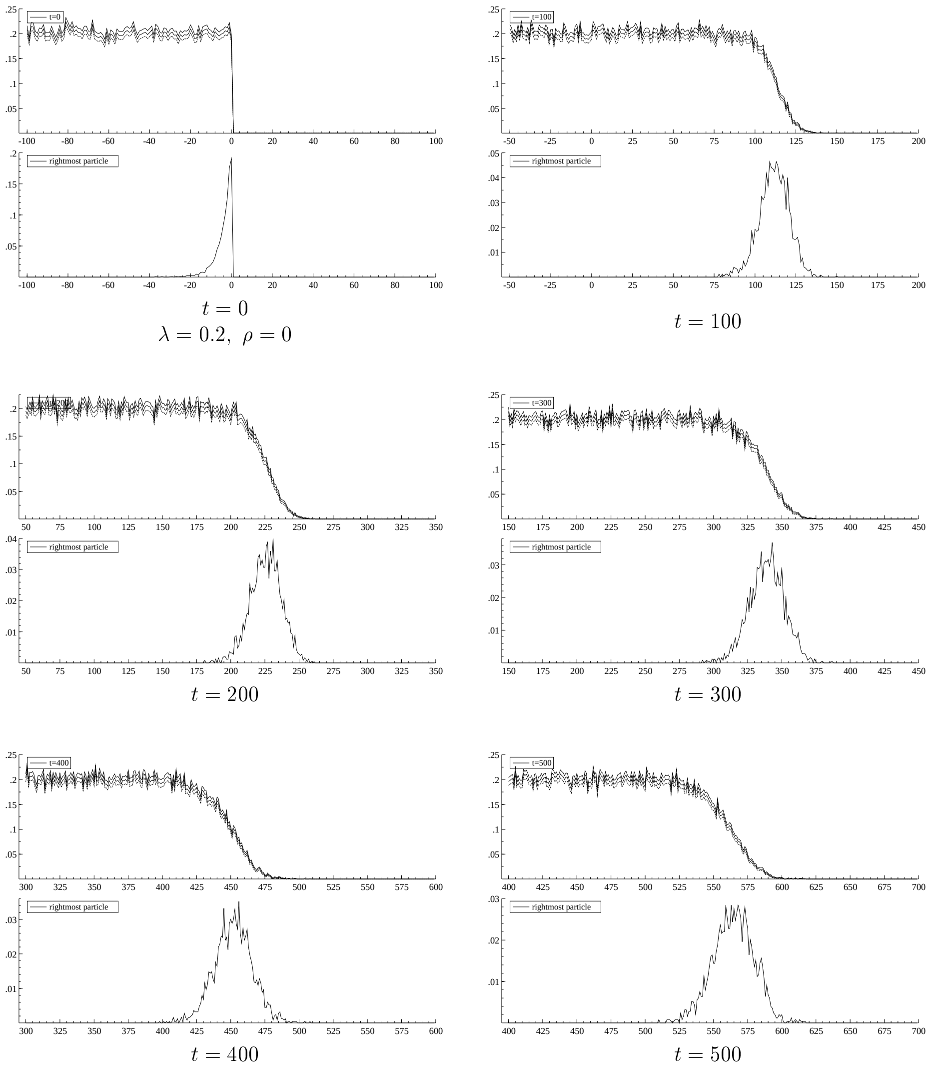

Figure 2: Evolution of the mean density and the mean position of the rightmost particles for initial profile $\lambda = 0.2$ and $\rho = 0$ from time 0 to 500.

4 Shock in the concave part of the flux

We treat here Case 5 of Theorem 1 with a right density $\rho = 1$, *i.e.*, $u_0(x) = \lambda \mathbf{1}_{\{x \leq 0\}} + \mathbf{1}_{\{x > 0\}}$, with $\lambda > \rho_* = 1_*$.

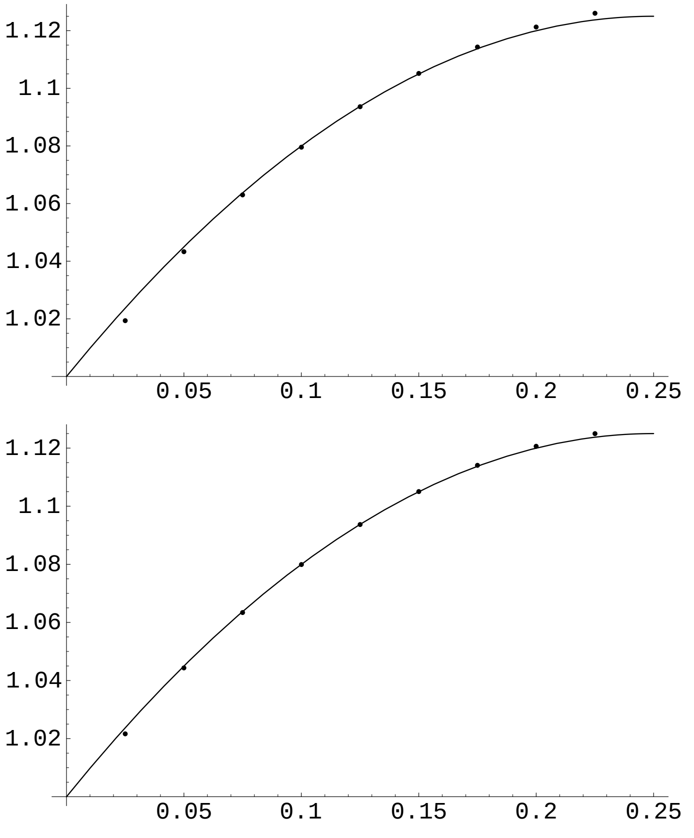

Figure 3: The velocity curve of the shock $v_s(\lambda) = S_2[\lambda, 0]$ and the simulated velocity of the rightmost particle W_t/t for $t = 7,500$ (above) an $t = 15,000$ (below) in the convex part of the flux: $\lambda \in (0, 1/4)$.

Theorem 4. *When $u_0(x) = \lambda \mathbf{1}_{\{x \leq 0\}} + \mathbf{1}_{\{x>0\}}$, the rightmost hole identifies the shock.*

Proof. The motion of the rightmost hole does not depend on the distribution of particles to the right of it; therefore it moves exactly as a tagged hole under the equilibrium distribution ν_λ. According to (4) the tagged hole speed is $-\sum_{j=1}^{k} j\lambda^j$ which is exactly the shock speed $S_k[\lambda; 1]$. □

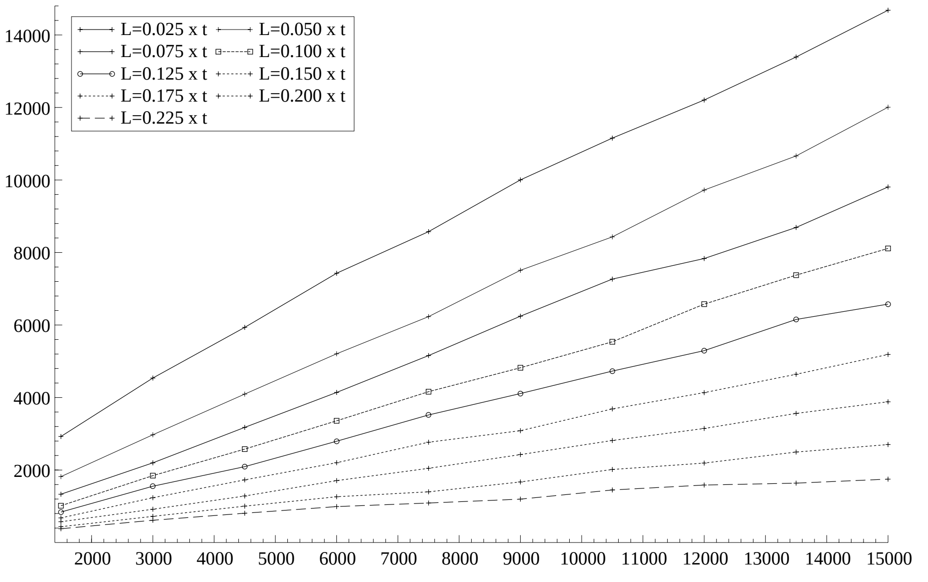

Figure 4: $\hat{\sigma}^2(t)$ for some values of λ.

5 The rarefaction fan

We consider here Cases 1 and 4 of Theorem 1: $\lambda < \rho < a$, and $a < \rho < \lambda$. We follow section 2 of Ferrari & Kipnis (1995). Under the initial distribution $\mu_{\lambda,\rho}$, we put a second class particle at the origin, denote by X_t its position at time t, and by $\widetilde{\mathbf{P}}$ the expectation of this process. We prove the

Theorem 5. *When $\lambda < \rho < a$ or $a < \rho < \lambda$, X_t/t converges in distribution to a probability measure concentrated on $[H_k(\lambda), H_k(\rho)]$, absolutely continuous w.r.t. the Lebesgue measure.*

$$\lim_{t\to\infty} \widetilde{\mathbf{P}}\left(\frac{X_t}{t} > r\right) = \frac{u(r,1)-\rho}{\lambda-\rho}$$

$$= \begin{cases} 1 & \text{if } r \leq H_k(\lambda) \\ \dfrac{h_{k,i}(r)-\rho}{\lambda-\rho} & \text{if } H_k(\lambda) < r \leq H_k(\rho) \\ 0 & \text{otherwise;} \end{cases}$$

where $i = 1$ in Case 1 and $i = 2$ in Case 4.

Proof. For a given initial configuration η, $J_{r,t}(\eta)$ is the flux of particles that have jumped through the space-time line with velocity r at time t. We denote by X_t^x the position at time t of the particle starting originally at site x:

$$J_{r,t}(\eta) = \sum_{x \le 0} \eta(x)\mathbf{1}_{\{X_t^x > \lfloor r \rfloor\}} - \sum_{x>0} \eta(x)\mathbf{1}_{\{X_t^x \le \lfloor r \rfloor\}} \tag{9}$$

that we can also write, since in the pushing interpretation particles cannot jump over each other,

$$\begin{aligned} J_{r,t}(\eta) &= \sum_{x \le 0} \eta(x)\mathbf{1}_{\{X_t^x > \lfloor r \rfloor\}} - \sum_{x>0} \eta(x)(1 - \mathbf{1}_{\{X_t^x \le \lfloor r \rfloor\}}) \\ &= \sum_{x} \eta(x)\mathbf{1}_{\{X_t^x > \lfloor r \rfloor\}} - \sum_{x>0} \eta(x) \\ &= \sum_{x > \lfloor r \rfloor} \eta_t(x) - \sum_{x>0} \eta(x) \end{aligned} \tag{10}$$

We prove the result for Case 4.

We consider two different couplings of two versions of k-step exclusion processes, starting respectively from η^0 of distribution $\mu_{\lambda,\rho}$, and η^1 of distribution $\tau_{-1}\mu_{\lambda,\rho}$. We denote by $\overline{E}$ the expectation of a coupled process w.r.t. this initial distribution $\overline{\mu}$. The first coupling is the basic coupling, under which we assume that $\eta^0(x) = \eta^1(x)$ for all $x \neq 0$, and with probability $\lambda - \rho$, on 0 there is a particle for η^0 and none for η^1. This possible discrepancy between the two marginals has a second class particle behavior, so that by (9)

$$\int \overline{E}(J_{rt,t}(\eta^0) - J_{rt,t}(\eta^1))d\overline{\mu}(\eta^0, \eta^1) = (\lambda - \rho)\widetilde{\mathbf{P}}(X_t > \lfloor rt \rfloor) \tag{11}$$

The second coupling is a "particle to particle coupling", under which $\eta^1 = \tau_{-1}\eta^0$. It means that we label particles of both configurations, in such a way that particles number ℓ for both configurations occupy sites distant by one (initially, particle number 0 is the first one to the left -including it- of the origin); to keep this situation during the evolution, when the clock rings for the ℓ-th particle of the first configuration, particles labeled ℓ for both configurations try to move. This is possible thanks to the pushing interpretation 1 under which particles keep their relative order. This way, by (10),

$$\int \overline{E}(J_{rt,t}(\eta^0) - J_{rt,t}(\eta^1))d\overline{\mu}(\eta^0, \eta^1) = \int \overline{E}(\eta_t^0(\lfloor rt \rfloor + 1) - \eta_0^0(0))d\overline{\mu}(\eta^0, \eta^1) \tag{12}$$

Putting together (11) and (12), and applying Theorem 1, Case 4 yields the result.

The proof is similar for Case 1, replacing $\tau_{-1}\mu_{\lambda,\rho}$ by $\tau_1\mu_{\lambda,\rho}$. □

6 Contact discontinuity

We consider here Cases 3 and 6 of Theorem 1: $\rho < \rho^* < \lambda$ ($\rho < a$), and $\rho > \rho_* > \lambda$ ($\rho > a$). We proceed as in the preceding section, to get the limiting behavior of a second class particle initially on site 0:

Theorem 6. *When $\rho < \rho^* < \lambda$ ($\rho < a$) or $\rho > \rho_* > \lambda$ ($\rho > a$), X_t/t converges in distribution to a probability measure which is a convex combination of a measure absolutely continuous w.r.t. the Lebesgue measure, and of a Dirac mass. For r a continuity point of $u(., 1)$,*

$$\lim_{t\to\infty} \widetilde{\mathbf{P}}\left(\frac{X_t}{t} > r\right) = \begin{cases} 1 & \textit{if } r \le H_k(\lambda) \\ \dfrac{h_{k,i}(r) - \rho}{\lambda - \rho} & \textit{if } H_k(\lambda) < r < H_k(\rho_i) \\ 0 & \textit{otherwise} \end{cases}$$

and

$$\lim_{t\to\infty} \widetilde{\mathbf{P}}\left(\frac{X_t}{t} = H_k(\rho_i)\right) = \frac{\rho_i - \rho}{\lambda - \rho}$$

where in Case 6, $i = 1$, $\rho_1 = \rho_*$, and in Case 3, $i = 2$, $\rho_2 = \rho^*$.

Proof. For every continuity point r of $u(., 1)$, the proof is the same as in Theorem 5. If we look at Case 3 for $r = H_k(\rho^*)$, the result follows from

$$\lim_{r\to H_k(\rho^*)^+} \widetilde{\mathbf{P}}\left(\frac{X_t}{t} > r\right) = 0,$$

$$\lim_{r\to H_k(\rho^*)^-} \widetilde{\mathbf{P}}\left(H_k(\lambda) < \frac{X_t}{t} \le r\right) = 1 - \frac{h_{k,2}(H_k(\rho^*)) - \rho}{\lambda - \rho} = 1 - \frac{\rho^* - \rho}{\lambda - \rho}.$$

□

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