BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 34(1), 43-57

Sommerfeld condition for a Liouville equation and concentration of trajectories

Benoît Perthame and Luis Vega

— Dedicated to IMPA on the occasion of its 50^{th} *anniversary*

Abstract. We analyse the concentration of trajectories in a Liouville equation set in the full space with a potential which is not constant at infinity. Our motivation comes from geometrical optics where it appears as the high freqency limit of Helmholtz equation. We conjecture that the mass and energy concentrate on local maxima of the refraction index and prove a result in this direction. To do so, we establish a priori estimates in appropriate weighted spaces and various forms of a Sommerfeld radiation condition for solutions of such a stationary Liouville equation.

Keywords: Sommerfeld conditions, Liouville equation, geometrical optics, Helmholtz equation.

Mathematical subject classification: 35Q99, 35F05, 35A05.

1 Introduction

We consider the stationary Liouville equation on the density $f_\alpha(x,\xi)$ with x, $\xi \in \mathbb{R}^d$,

$$\alpha f_\alpha + \xi \cdot \nabla_x f_\alpha + \frac{1}{2}\nabla_x V(x) \cdot \nabla_\xi f_\alpha = \Sigma(x,\xi). \tag{1.1}$$

Much of our results refer to the mass density $\int_{\mathbb{R}^d} f_\alpha d\xi$ and to the energy density $\int_{\mathbb{R}^d} \frac{|\xi|^2}{2} f_\alpha d\xi$. Our purpose is to study the behavior of solutions as the absorption parameter α vanishes in the case where the potential V (index at infinity in geometrical optics) is not constant at infinity. More precisely, we establish uniform estimates in appropriate weighted spaces of Morrey type, and we show a radiation condition that we express in various ways.

Received 3 September 2002.

The solution is also directly related to the system of ordinary differential equations

$$\begin{aligned} \dot{X}(t, y) &= \zeta(t, y), \\ \dot{\zeta}(t, y) &= \tfrac{1}{2}\nabla_x V\Big(X(t, y)\Big). \end{aligned}$$

Our results strongly depend on its Hamiltonian structure which implies $|\zeta(t, y)|^2 = V\big(X(t, y)\big)$, and on the the large time behaviour of the solutions which concentrate on critical points of V. However, our motivation for studying this problem comes from geometrical optics. More precisely this equation can be derived as the high frequency limit of a Helmholtz equation for a smoothly varying media as proved in Benamou *et al* [3] and Castella *et al* [4]. At this level it may be useful to notice that another high frequency limit exists for random media which writes (see Ryzhik *et al* [17], Erdös and Yau [8], Poupaud and Vasseur [15] at least for the evolution case)

$$\alpha f_\alpha + \xi \cdot \nabla_x f_\alpha + \int_{\mathbb{R}^d} K(\xi, \xi')[f(\xi) - f(\xi')]d\xi' = 0,$$

but the existence of a limit and the related uniqueness theory seems to rely on different methods.

Several mathematical features are in common between (1.1) and Helmholtz equation. Specially *a priori* estimates, uniform in α, are not obvious and cannot be obtained in usual Lebesgue spaces. In Perthame and Vega [13] such estimates in Morrey-Campanato spaces were derived for Helmholtz equation with the right space scale which makes them 'uniform' in frequency. Therefore the method also yields estimates for equation (1.1) which are uniform in $\alpha > 0$ (see Theorem 2.1). The only point here is to translate in terms of the phase space variables the manipulations made in the single variable x for Helmholtz equation.

More deep is to understand the uniqueness condition at infinity, so-called Sommerfeld radiation condition. It expresses that no rays (or no mass, or no energy) are incoming. Roughly it says that, in the limit $\alpha = 0^+$, we have $f(x, \xi) = 0$ for $x \cdot \xi \leq 0$ and $|x|$ large. The question is to give a precise meaning to this statement and in particular to take into account possible variations of V at infinity as we do here. We establish that it can be written as

$$\frac{1}{R}\int_{\{|x|\leq R\}}\int_{\xi\in\mathbb{R}^d} |\xi - \frac{x}{|x|}V^{1/2}|^2 f(x, \xi)dxd\xi \to 0 \quad \text{as} \quad R \to \infty. \tag{1.2}$$

This is a little surprising because several authors have proved an alternative condition which involves the phase in a natural way. It is given by

$$\begin{cases} \frac{1}{R}\int_{\{|x|\leq R\}}\int_{\xi\in\mathbb{R}^d} |\xi-\nabla\phi(x)|^2 f(x,\xi)dxd\xi \to 0 \quad \text{as} \quad R\to\infty, \\ |\nabla\phi| = V^{1/2} \quad \text{in } \mathbb{R}^d, \end{cases} \tag{1.3}$$

see Agmon *et al* [1], Saito [18] or for different applications Zhang [19], Eidus [6], [7]. And, following the classical theory of Hamilton-Jacobi equation, $\nabla\phi \not\equiv \frac{x}{|x|}V^{1/2}$. In fact the compatibility between the two conditions is explained by a concentration of trajectories (characteristics), and thus of f, on critical points of V where the two quantities coincide. This fact was discovered in [14] for Helmholtz equation and we extend it here to (1.1) in Theorem 2.1, equation (2.10). Notice however that a similar statement can be given directly for trajectories (differential equation) for large times rather than in the limit $\alpha\to 0$. This was done by Herbst [9]. Here, we develop the same theory with PDE methods and we state various asymptotic forms relating the limits as $\alpha\to 0^+$ and as $R\to\infty$ in expressions like the above Sommerfeld radiation condition.

The outcome of this paper is as follows. We first state our precise assumptions and results. Then, in the last two sections, we prove these results. The last section is devoted to a uniqueness proof based on our Sommerfeld condition.

2 Main results

For the sake of simplicity we restrict ourselves to the case where the potential V is positively homegeneous of degree 0 as considered in Herbst [9], although the extension to assumptions in the spirit of [13], [14] is possible. Hence, we assume throughout this paper that

$$\alpha > 0, \tag{2.1}$$

$$V = V(\frac{x}{|x|}) \in C^2(S^{d-1}), \quad V > 0, \tag{2.2}$$

$$\begin{cases} \Sigma(x,\xi) = \sigma(x)\delta\Big(\xi = \frac{x}{|x|}V(x)^{1/2}\Big), \\ \sigma(x)\geq 0, \quad \sigma\not\equiv 0, \quad \sigma\in C^1_{comp}(\mathbb{R}^d). \end{cases} \tag{2.3}$$

We also use the following notations

$$\xi^t(x,\xi) = \xi - \frac{x}{|x|^2} x \cdot \xi, \tag{2.4}$$

$$\nabla_x V = \frac{\nabla_\omega V}{|x|}, \qquad \omega = \frac{x}{|x|}. \tag{2.5}$$

We begin with the problem set with $\alpha > 0$ and establish uniform a priori bounds that are used later to study the limit $\alpha \to 0$.

Theorem 2.1. **(A priori bounds).** *There is a unique nonnegative, locally bounded, measure f_α, solution to (1.1) and it satisfies, with right hand sides independent of α,*

$$\alpha \int_{\mathbb{R}^{2d}} f_\alpha(x,\xi)dxd\xi = \int_{\mathbb{R}^d} \sigma(x)dx := M, \tag{2.6}$$

$$f_\alpha \textit{ is supported by } \{|\xi|^2 = V(x)\}, \tag{2.7}$$

$$\int_{\mathbb{R}^{2d}} \frac{|\xi^t|^2}{|x|} f_\alpha(x,\xi)dxd\xi \le \|V\|_{L^\infty}^{1/2} M, \tag{2.8}$$

$$\frac{1}{R}\int_{\{|x|\le R\}}\int_{\xi\in\mathbb{R}^d} |\xi|^2 f_\alpha(x,\xi)dxd\xi \le \|V\|_{L^\infty}^{1/2} \int_{\mathbb{R}^d} \sigma(x)dx, \tag{2.9}$$

$$\int_{\mathbb{R}^{2d}} \frac{|\nabla_\omega V|^2}{|x|} f_\alpha(x,\xi)dxd\xi \le C(V, D^2V)\, M \qquad \textit{(see remark 2)}. \tag{2.10}$$

Remark 2.1. To see why estimate (2.10) is relevant we point out that, in the limit $\alpha \to 0^+$ (see also Theorem 2.3), there holds

$$\int_{\mathbb{R}^{2d}} \frac{1}{1+|x|} f(x,\xi)dxd\xi = \infty.$$

This can be seen from estimate (2.14) below, which implies that for R large enough

$$\int_{\{R\le|x|\le 2R\}}\int_{\mathbb{R}^d} \frac{1}{|x|} f(x,\xi)dxd\xi \ge \frac{M}{4\|V^{1/2}\|_{L^\infty}}.$$

In the case $V = Constant$, say $V = 1$, this can also be computed directly from the representation formula (3.1) which gives

$$\int_{\mathbb{R}^{2d}} \frac{1}{|x|} f(x,\xi) dx d\xi = \int_0^\infty \int_{\mathbb{R}^d} \frac{\sigma(y)}{1+|y|+t} dy\, dt = \infty.$$

We obtain here uniform (in α) estimates which use the norm

$$\sup_{R>0} \frac{1}{R} \int_{\{|x|\leq R\}} \int_{\xi\in\mathbb{R}^d} |\dots| dx d\xi$$

and specially that preservs the right space homogeneity. These are typical of Helmholtz equations, see Agmon and Hörmander [2], [13], [14] and have been used recently in the context of dispersive equations, see Kenig, Ponce and Vega [11], and in the context of kinetic equations, see Lions and Perthame [12]. This space homogeneity is also the reason why they allow, in the context of Helmholtz equations, uniformity in the frequency [3]. The extra decay provided in estimate (2.10) is fundamental to establish the Sommerfeld condition in its simple form (1.2) i.e. without refering to the phase as in (1.3).

The limit measure $f(x,\xi)$ obtained for $\alpha \to 0^+$ satisfies the Liouville equation

$$\xi \cdot \nabla_x f + \frac{1}{2} \nabla_x V(x) \cdot \nabla_\xi f = \Sigma(x,\xi). \tag{2.11}$$

In order to establish uniqueness (see section 5) for the above equation a condition of Sommerfeld radiation type is needed. Indeed, even with $V \equiv 0$ there are infinitely many solutions given for instance by $f = F(x - \frac{\xi}{|\xi|^2} x \cdot \xi)$ for any smooth function F. It is given in our next result.

Theorem 2.2. (Sommerfeld radiation condition). *Uniformly in α we have as $R \to \infty$,*

$$\frac{1}{R} \int_{\{|x|\leq R\}} \int_{\xi\in\mathbb{R}^d} |\xi - \frac{x}{|x|} V^{1/2}|^2 f_\alpha(x,\xi) dx d\xi \to 0.$$

And also, as $\alpha \to 0$,

$$\alpha \int_{\mathbb{R}^{2d}} |\xi - \frac{x}{|x|} V^{1/2}|^2\ f_\alpha(x,\xi) dx d\xi \to 0.$$

Notice that (2.10) indicates that f_α concentrates along the critical points of V. Our next result says that in fact the mass concentrates rather on high values of V.

Theorem 2.3. (Asymptotics $\alpha \to 0$). *As $\alpha \downarrow 0^+$ we have $f_\alpha \uparrow f$ and therefore f satisfies the a priori bounds in theorem 2.1; moreover, additionally to the statements in Theorem 2.2,*

$$\alpha \int_{\mathbb{R}^{2d}} |\xi^t|^2\, f_\alpha(x,\xi)dxd\xi \to 0, \quad \alpha \int_{\mathbb{R}^{2d}} |\nabla_\omega V|^2\, f_\alpha(x,\xi)dxd\xi \to 0, \tag{2.12}$$

$$\begin{aligned}\alpha \int_{\mathbb{R}^{2d}} V(x)^{1/2} f_\alpha(x,\xi)dxd\xi \to \int_{\mathbb{R}^d} V(x)^{1/2}\,\sigma(x)dx \\ + \int_{\mathbb{R}^{2d}} \frac{|\xi^t|^2}{|x|} f(x,\xi)dxd\xi,\end{aligned} \tag{2.13}$$

$$\frac{1}{R}\int_{\{R\le|x|\le 2R\}} \int_{\mathbb{R}^d} V(x)^{1/2} f(x,\xi)dxd\xi \to \int_{\mathbb{R}^{2d}} \sigma(x)dx := M, \tag{2.14}$$

$$\begin{aligned}\frac{1}{R}\int_{\{R\le|x|\le 2R\}} \int_{\mathbb{R}^d} V(x) f(x,\xi)dxd\xi \to \int_{\mathbb{R}^d} V(x)^{1/2}\,\sigma(x)dx \\ + \int_{\mathbb{R}^{2d}} \frac{|\xi^t|^2}{|x|} f(x,\xi)dxd\xi.\end{aligned} \tag{2.15}$$

The limits (2.13) compared to (2.6), and (2.15) compared to (2.14), express that the mass $f(x,\xi)dxd\xi$ not only concentrates on extrema of V (see (2.10)), but for large values of x it goes rather to larger values of V compared to the source. Indeed, except very special situations when the source is only supported by extrema of V, we always have $\int_{\mathbb{R}^{2d}} \frac{|\xi^t|^2}{|x|} f(x,\xi)dxd\xi > 0$.

3 Proof of Theorem 2.1 and Theorem 2.2

Proof of Theorem 2.1. The existence of a unique measure solution f_α is a classical matter. It is given through the characteristics

$$\begin{aligned}\dot X(t,y) &= \zeta(t,y), & X(0,y) &= y,\\ \dot\zeta(t,y) &= \tfrac12 \nabla_x V\Big(X(t,y)\Big), & \zeta(0,y) &= \tfrac{y}{|y|} V^{1/2}(y).\end{aligned}$$

The formula is now

$$f_\alpha(x,\xi) = \int_0^\infty \int_{\mathbb{R}^d} e^{-\alpha t}\delta\Big(x - X(t,y)\Big)\delta\Big(\xi - \zeta(t,y)\Big)\sigma(y)dydt. \tag{3.1}$$

To prove (2.6), we just integrate in x and ξ equation (1.1) or the representation formula (3.1). To prove (2.7), we multiply (2.6) by $\Big(|\xi|^2 - V(x)\Big)^2$. Using (2.3) and since

$$\xi \cdot \nabla_x \Big(|\xi|^2 - V(x)\Big)^2 + \frac{1}{2}\nabla_x V(x) \cdot \nabla_\xi \Big(|\xi|^2 - V(x)\Big)^2 = 0,$$

we find after integrating by parts

$$\alpha \int_{\mathbb{R}^{2d}} \Big(|\xi|^2 - V(x)\Big)^2 f_\alpha(x,\xi)dxd\xi = 0.$$

To prove (2.8), we multiply equation (1.1) by $\xi \cdot \frac{x}{|x|}$. Since $\xi \cdot \nabla_x(\xi \cdot \frac{x}{|x|}) = \frac{|\xi^t|^2}{|x|}$, we obtain using the signs of V and Σ

$$\begin{aligned}\int_{\mathbb{R}^{2d}} \frac{|\xi^t|^2}{|x|} f_\alpha(x,\xi)dxd\xi &= \alpha \int_{\mathbb{R}^{2d}} \xi \cdot \frac{x}{|x|} f_\alpha(x,\xi)dxd\xi \\ &\quad - \int_{\mathbb{R}^{2d}} V^{1/2}(x)\Sigma(x,\xi)dxd\xi \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\le \alpha \textstyle\int_{\mathbb{R}^{2d}} |\xi| f_\alpha(x,\xi)dxd\xi \\ &= \alpha \textstyle\int_{\mathbb{R}^{2d}} V^{1/2}(x) f_\alpha(x,\xi)dxd\xi, \end{aligned}$$

and we conclude thanks to (2.6).

Next, we prove (2.9). Following [13], we use the multiplier $\xi \cdot \nabla_x \Psi_R(x)$ with

$$\nabla_x \Psi_R(x) = \begin{cases} \dfrac{x}{R} & for \quad |x| \le R, \\ \dfrac{x}{|x|} & for \quad |x| \ge R. \end{cases}$$

We obtain

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} \left[\frac{|\xi|^2}{R}\, \mathbf{1}_{\{|x|\le R\}} + \frac{|\xi^t|^2}{|x|}\, \mathbf{1}_{\{|x|\ge R\}}\right] f_\alpha(x,\xi)dxd\xi \\ &= \alpha \int_{\mathbb{R}^{2d}} \xi \cdot \nabla_x \Psi_R(x) f_\alpha(x,\xi)dxd\xi - \int_{\mathbb{R}^{2d}} \xi \cdot \nabla_x \Psi_R(x)\Sigma(x,\xi)dxd\xi, \end{aligned} \tag{3.3}$$

and we obtain the result noticing that $\xi \cdot \nabla_x \Psi_R(x)\Sigma(x,\xi) \ge 0$.

We now turn to the proof of (2.10). We use the multiplier $\xi^t \cdot \nabla_\omega V(x)$ and obtain,

$$\int_{\mathbb{R}^{2d}} \left[\frac{D^2_\omega V.(\xi^t, \xi^t)}{|x|} - \frac{x \cdot \xi}{|x|^2} \xi^t \cdot \nabla_\omega V(x) + \frac{1}{2} \frac{|\nabla_\omega V(x)|^2}{|x|} \right] f_\alpha(x, \xi) dx d\xi$$
$$= \alpha \int_{\mathbb{R}^{2d}} \xi^t \cdot \nabla_\omega V(x) f_\alpha(x, \xi) dx d\xi - \int_{\mathbb{R}^{2d}} \xi^t \cdot \nabla_\omega V(x) \Sigma(x, \xi) dx d\xi.$$

This identity uses the relations

$$D^2_{x\omega} V.(\xi, \xi) = \xi_i \xi_j D^2_{\omega_i \omega_k} \; V \left(\frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3} \right),$$

and $0 = D_{\omega_j} V + x_i D^2_{\omega_i \omega_k} V \left(\frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3} \right)$, for all j, which lead to

$$D^2_{x\omega} V.(\xi, \xi) = \frac{D^2_\omega V.(\xi^t, \xi^t)}{|x|} - \frac{x \cdot \xi}{|x|^2} \xi^t \cdot \nabla_\omega V(x).$$

We therefore conclude

$$\int_{\mathbb{R}^{2d}} \frac{|\nabla_\omega V(x)|^2}{|x|} f_\alpha(x, \xi) dx d\xi \le (1 + \alpha) \|V\|^{1/2}_{L^\infty} \|\nabla V\|_{L^\infty} M$$
$$+ \int_{\mathbb{R}^{2d}} \left[\|D^2_\omega V\|_{L^\infty} \frac{|\xi^t|^2}{|x|} + \|V\|^{1/2}_{L^\infty} \frac{|\xi^t| \, |\nabla_\omega V|}{|x|} \right] f_\alpha(x, \xi) dx d\xi.$$

And we conclude by a Cauchy-Schwarz inequality using the previous estimates.

Proof of Theorem 2.2. We define

$$\rho_R(x) = inf\big(1, \frac{|x|}{R}\big), \qquad \nabla \Psi_R = \frac{x}{|x|} \rho_R,$$

and use the multiplier $-2\xi \cdot \nabla_x \Psi_R + 2V^{1/2} \rho_R$. Using that $|\xi|^2 = V$, we obtain

$$\begin{aligned}
&\alpha \int_{\mathbb{R}^{2d}} \frac{\rho_R(x)}{V^{1/2}} |\xi - \frac{x}{|x|} V^{1/2}|^2 f_\alpha(x, \xi) dx d\xi \\
&\quad + \frac{1}{R} \int_{\mathbb{R}^{2d}} \left[|\xi - \frac{x}{|x|} V^{1/2}|^2 \, \mathbf{1}_{\{|x| \le R\}} + 2 \frac{|\xi^t|^2}{|x|} \, \mathbf{1}_{\{|x| \ge R\}} \right] f_\alpha(x, \xi) dx d\xi \\
&\qquad = -2 \int_{\mathbb{R}^{2d}} \xi \cdot \nabla_x V^{1/2} \rho_R \; f_\alpha(x, \xi) dx d\xi \\
&\qquad \le \int_{\mathbb{R}^{2d}} \left(\frac{|\xi^t|^2}{|x|} + \frac{|\nabla_\omega V^{1/2}|^2}{|x|} \right) \rho_R \; f_\alpha(x, \xi) dx d\xi.
\end{aligned} \tag{3.4}$$

Notice that

$$\int_{\mathbb{R}^{2d}} \left(\frac{|\xi^t|^2}{|x|} + \frac{|\nabla_\omega V^{1/2}|^2}{|x|} \right) \rho_R \, f_\alpha(x,\xi) dx d\xi \leq$$

$$\leq \int_{\mathbb{R}^{2d}} \left(\frac{|\xi^t|^2}{|x|} + \frac{|\nabla_\omega V^{1/2}|^2}{|x|} \right) f(x,\xi) dx d\xi$$

is uniformly bounded using estimates (2.8) and (2.10) and following the argument of the proof of Theorem 2.3 that f_α is increasing to f as $\alpha \downarrow 0$. Therefore we may pass to the limit as $R \to \infty$ in (3.4) and this gives the first statement of Sommerfeld condition.

For the second statement, we use the idendity

$$\alpha \int_{\mathbb{R}^{2d}} \frac{1}{V^{1/2}} \, |\xi - \frac{x}{|x|} V^{1/2}|^2 \; f_\alpha(x,\xi) dx d\xi =$$

$$= 2\,\alpha \int_{\mathbb{R}^{2d}} \left[V^{1/2} - \frac{x \cdot \xi}{|x|} \right] f_\alpha(x,\xi) dx d\xi.$$

Then, we use the idendity (3.2) and the result (2.13) of Theorem 2.3 which is proved independently, and this concludes the proof of Theorem 2.2.

4 Proof of Theorem 2.3

The monotonicity of f_α, and thus the existence of a limit in locally bounded measures, follows from the maximum principle or (3.1).

We now explain how the limits can be computed. We begin with the first limit in (2.12). We compute

$$[\xi \cdot \nabla_x + \frac{1}{2} \nabla_x V \cdot \nabla_\xi] \, |\xi^t|^2 = 2 \Big[- |\xi^t|^2 \frac{x \cdot \xi}{|x|} + \frac{1}{2} \xi \cdot \nabla_x V \Big]. \tag{4.1}$$

And thus, using that $|\xi^t(x,\xi)|^2 \Sigma(x,\xi) = 0$, we obtain

$$\alpha \int_{\mathbb{R}^{2d}} |\xi^t|^2 \; f_\alpha(x,\xi) dx d\xi = 2 \int_{\mathbb{R}^{2d}} \Big[-|\xi^t|^2 \frac{x \cdot \xi}{|x|} + \frac{1}{2} \xi \cdot \nabla_x V \Big] \, f_\alpha(x,\xi) dx d\xi$$

$$\to 2 \int_{\mathbb{R}^{2d}} \Big[-|\xi^t|^2 \frac{x \cdot \xi}{|x|} + \frac{1}{2} \xi \cdot \nabla_x V \Big] \, f(x,\xi) dx d\xi,$$

thanks to the integrability proved in Theorem 2.1.

We can compare the above identity to a direct computation based on equation (2.11), after integration by parts against the test function $|\xi^t|^2\varphi_R(x)$, with the truncation function

$$\varphi_R(x) = \varphi(\frac{|x|}{R}), \quad \varphi(r) = \begin{cases} 1 & for\ 0 \le r \le 1, \\ 2-r & for\ 1 \le r \le 2, \\ 0 & for\ r \ge 2. \end{cases} \tag{4.2}$$

We find, using again (4.1) and $|\xi^t(x,\xi)|^2\Sigma(x,\xi) = 0$,

$$-2\int_{\mathbb{R}^{2d}} \left[-|\xi^t|^2 \frac{x\cdot\xi}{|x|} + \frac{1}{2}\xi\cdot\nabla_x V \right] \varphi_R(x)\ f(x,\xi)dxd\xi =$$

$$= \int_{\mathbb{R}^{2d}} |\xi^t|^2 \xi\cdot\nabla_x\varphi_R\ f(x,\xi)dxd\xi,$$

and, since

$$\xi\cdot\nabla_x\varphi_R = -\xi\cdot\frac{x}{R|x|}\,\mathbf{1}_{\{R\le|x|\le 2R\}},$$

passing to the limit we obtain

$$\int_{\mathbb{R}^{2d}} \left[-|\xi^t|^2 \frac{x\cdot\xi}{|x|} + \frac{1}{2}\xi\cdot\nabla_x V \right] f(x,\xi)dxd\xi = 0,$$

which concludes the first limit of (2.12). The second one follows the same lines and we skip the proof.

The derivation of (2.13) uses the same type of arguments. As a first step, we compute

$$\alpha\int_{\mathbb{R}^{2d}} V^{1/2}\ f_\alpha(x,\xi)dxd\xi =$$

$$= \int_{\mathbb{R}^{2d}} \xi\cdot\nabla_x V^{1/2}\ f_\alpha(x,\xi)dxd\xi + \int_{\mathbb{R}^{2d}} V^{1/2}\ \Sigma(x,\xi)dxd\xi$$

$$\to \int_{\mathbb{R}^{2d}} \xi\cdot\nabla_x V^{1/2}\ f(x,\xi)dxd\xi + \int_{\mathbb{R}^{d}} V^{1/2}\ \sigma(x)dx,$$

and, working directly with the limit and the truncation function φ_R, we deduce using Sommerfeld condition

$$\lim_{R\to\infty} \frac{1}{R}\int_{R\le|x|\le 2R}\int_{\mathbb{R}^d} V^{1/2}\,\xi\cdot\frac{x}{|x|}\ f(x,\xi)dxd\xi$$

$$= \lim_{R\to\infty} \frac{1}{R} \int_{R\leq|x|\leq 2R} \int_{\mathbb{R}^d} V(x)\ f(x,\xi)dxd\xi$$

$$= \int_{\mathbb{R}^{2d}} \xi \cdot \nabla_x V^{1/2}\ f(x,\xi)dxd\xi + \int_{\mathbb{R}^d} V^{1/2}\,\sigma(x)dx.$$

On the other hand, we also have, as $\alpha \to 0^+$

$$\alpha \int_{\mathbb{R}^{2d}} \xi \cdot \frac{x}{|x|}\ f_\alpha(x,\xi)dxd\xi =$$

$$= \int_{\mathbb{R}^{2d}} \frac{|\xi^t|^2}{|x|}\ f_\alpha(x,\xi)dxd\xi + \int_{\mathbb{R}^{2d}} \xi \cdot \frac{x}{|x|}\ \Sigma(x,\xi)dxd\xi$$

$$\to \int_{\mathbb{R}^{2d}} \frac{|\xi^t|^2}{|x|}\ f(x,\xi)dxd\xi + \int_{\mathbb{R}^d} V^{1/2}\,\sigma(x)dx,$$

and, working directly with the limit and the truncation function φ_R, we deduce using Sommerfeld condition

$$\lim\ \frac{1}{R} \int_{R\leq|x|\leq 2R} \int_{\mathbb{R}^d} \Big(\xi \cdot \frac{x}{|x|}\Big)^2\ f(x,\xi)dxd\xi$$

$$= \lim\ \frac{1}{R} \int_{R\leq|x|\leq 2R} \int_{\mathbb{R}^d} V(x)\ f(x,\xi)dxd\xi$$

$$= \int_{\mathbb{R}^{2d}} \frac{|\xi^t|^2}{|x|}\ f(x,\xi)dxd\xi + \int_{\mathbb{R}^d} V^{1/2}\,\sigma(x)dx.$$

As a conclusion of these different limits we deduce a family of equalities

$$\int_{\mathbb{R}^{2d}} \frac{|\xi^t|^2}{|x|}\ f(x,\xi)dxd\xi = \int_{\mathbb{R}^{2d}} \xi \cdot \nabla_x V^{1/2}\ f(x,\xi)dxd\xi, \tag{4.3}$$

$$\lim\ \alpha \int_{\mathbb{R}^{2d}} V^{1/2}\ f_\alpha(x,\xi)dxd\xi = \lim\ \alpha \int_{\mathbb{R}^{2d}} \xi \cdot \frac{x}{|x|}\ f_\alpha(x,\xi)dxd\xi$$

$$= \lim\ \frac{1}{R} \int_{R\leq|x|\leq 2R} \int_{\mathbb{R}^d} V(x)\ f(x,\xi)dxd\xi. \tag{4.4}$$

From this the limits in (2.13), (2.15) follow.

As for (2.14), it follows by comparing (2.6) with the result obtained from working directly with the limit and the truncation function φ_R, and using Sommerfeld condition,

$$\lim \frac{1}{R}\int_{R\le|x|\le 2R}\int_{\mathbb{R}^d}\xi\cdot\frac{x}{|x|}\ f(x,\xi)dxd\xi=\int_{\mathbb{R}^d}\sigma(x)dx$$
$$=\lim \frac{1}{R}\int_{R\le|x|\le 2R}\int_{\mathbb{R}^d}V^{1/2}\ f(x,\xi)dxd\xi.$$

5 Uniqueness

For the sake of completeness, in this section we prove uniqueness under the Sommerfeld condition. Of course the method mimicks the case of Helmholtz equation and we indicate the arguments without too many details.

Theorem 5.1. *We make the assumptions (2.2), (2.3), then there is a unique measure f which satisfies*

(i) $\int_{\mathbb{R}^d}(1+|\xi|^2)|f(\cdot,\xi)|d\xi\in M^1_{loc}(\mathbb{R}^d)$,

(ii) *in distributional sense the equation holds*

$$\xi\cdot\nabla_x f+\frac{1}{2}\nabla_x V\cdot\nabla_\xi f=\Sigma(x,\xi),$$

(iii) *the Sommerfeld condition holds*

$$\frac{1}{R}\int_{R\le|x|\le 2R}\int_{\mathbb{R}^d}|\xi-\frac{x}{|x|}V^{1/2}|^2\ |f(x,\xi)|dxd\xi\to 0\quad \textit{as } R\to\infty.$$

Several variants of this result are possible, especially the integral in (iii) could be taken on spheres, and liminf is enough.

Also a counterexample which shows the necessity of the Sommerfeld condition is simple. For $V=1$, $\Sigma=0$, we choose

$$f(x,\xi)=F\big(x-x\cdot\xi\frac{\xi}{|\xi|^2}\big)G(\xi).$$

Notice that f satisfies condition (i) because

$$\frac{1}{R}\int_{|x|\le R}\int_{\mathbb{R}^d}(1+|\xi|^2)f(x,\xi)dxd\xi\le\|F\|_{L^1(\mathbb{R}^d)}\ \|(1+|\xi|^2)G\|_{L^1(\mathbb{R}^d)}.$$

Also, Liouville equation (ii) is always fulfilled, whatever is $F\in C^1$ and G such that $\|(1+|\xi|^2)G\|_{L^1(\mathbb{R}^d)}$. But a simple computation shows that the Sommerfeld condition (iii) fails (the limit is positive) except if F or G vanish.

Proof of Theorem 5.1. We recall that existence follows from our main results (section 2). Therefore we prove uniqueness and consider the difference of two possible solutions. We still call f this difference which satisfies the statements (i), (ii) with $\Sigma = 0$ and (iii) .

Using DiPerna and Lions [5] arguments, we can apply, thanks to the regularity of V, the chain rule to Liouville equation (ii) and thus

$$\xi \cdot \nabla_x |f| + \frac{1}{2} \nabla_x V \cdot \nabla_\xi |f| = 0. \tag{5.1}$$

Consider again the truncation function (4.2). Then after integration by parts we deduce from the above equation (this requires smoothing in x and truncation in ξ)

$$\frac{1}{R} \int_{\mathbb{R}^{2d}} \xi \cdot \frac{x}{|x|} \varphi'(\frac{|x|}{R}) \, |f(x,\xi)| dx d\xi = 0.$$

Therefore we also have

$$\begin{aligned}
&\tfrac{1}{R} \int_{R \le |x| \le 2R} \int_{\mathbb{R}^d} \tfrac{|\xi|^2 + V}{V^{1/2}} \ |f(x,\xi)| dx d\xi \\
&= \tfrac{1}{R} \int_{R \le |x| \le 2R} \int_{\mathbb{R}^d} \tfrac{1}{V^{1/2}} |\xi - \tfrac{x}{|x|} V^{1/2}|^2 \ |f(x,\xi)| dx d\xi \\
&= o(1).
\end{aligned}$$

We now come back to equation (5.1), and now we use the multiplier taken from [12]: $\frac{\xi \cdot x}{(1+|x|)^{1/2}} \varphi_R$. We obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} \xi \cdot \nabla \frac{\xi \cdot x}{(1+|x|^2)^{1/2}} \, \varphi_R \ |f(x,\xi)| dx d\xi = \\
&- \int_{\mathbb{R}^{2d}} \frac{|\xi|^2}{(1+|x|^2)^{3/2}} \, \xi \cdot \nabla \varphi_R \ |f(x,\xi)| dx d\xi
\end{aligned}$$

which we can rewrite also

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} \tfrac{|\xi|^2}{(1+|x|^2)^{3/2}} \, \varphi_R \ |f(x,\xi)| dx d\xi \\
&\le \int_{\mathbb{R}^{2d}} \tfrac{|\xi|^2 (1+|x|^2) - (\xi \cdot x)^2}{(1+|x|^2)^{3/2}} \, \varphi_R \ |f(x,\xi)| dx d\xi \\
&= \tfrac{1}{R} \int_{\mathbb{R}^{2d}} \xi \cdot \tfrac{x}{(1+|x|^2)^{1/2}} \xi \cdot \tfrac{x}{|x|} \varphi'(\tfrac{|x|}{R}) \, |f(x,\xi)| dx d\xi \\
&= o(1).
\end{aligned}$$

We now let $R \to \infty$ and obtain in the limit

$$\int_{\mathbb{R}^{2d}} \frac{|\xi|^2}{(1+|x|^2)^{3/2}} |f(x,\xi)| dx d\xi = 0.$$

This concludes the proof of Theorem 5.1.

Acknowledgment. This work was partially supported by HYKE European programme HPRN-CT-2002-00282 (http://www.hyke.org)

References

[1] S. Agmon, J. Cruz-Sampedro and I. Herbst, *Generalized Fourier transform for Schrödinger operators with potentials of order zero*, J. of Funct. Anal., **167** (1999), 345–369.

[2] S. Agmon and L. Hörmander, *Asymptotic properties of solutions of differential equations with simple characteristics*, J. Anal. Math. **30** (1976), 1–37.

[3] J.D. Benamou, F. Castella, T. Katsaounis and B. Perthame, *High frequency limit of the Helmholtz equations,* Rev. IberoAmer. (2002).

[4] F. Castella, B. Perthame and O. Runborg, *High frequency limit of the Helmholtz equation. Source on a general manifold*, Comm. P.D.E. **27** n. 3-4 (2002), 607–651.

[5] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **98**(3) (1989), 511–547.

[6] D. M. Eidus, *The principle of limiting absorption*, Math. Sb., **57** (1962), 13–44. Amer. Math. Soc. Transl. (2) **47** (1965), 157–191.

[7] D. M. Eidus, *The limiting absorption and amplitud principles for the diffraction problem with two unbounded media*, Comm. Math. Phys. **107** (1986), 29–38.

[8] L. Erdös and H. T. Yau, *Linear Boltzmann equation as the weak coupling limit of a random Schrödinger equation*, Comm. Pure Appl. Math. **53** (2000), 667–735.

[9] I. Herbst, *Spectral and scattering theory for Schrödinger operators with potentials independent of* $|x|$, Amer. J. Math. **113**(3) (1991), 509–565.

[10] T. Ikebe and Y. Saito, *Limiting absorption method and aboslut continuity for the Schrödinger operator*, J. Math. Kyoto Univ. **12-3** (1972), 512–542.

[11] C. Kenig, G. Ponce and L. Vega, *Small solutions to nonlinear Schrödinger equations,* Ann. Inst. H. Poincare Anal. Non Lineaire **10** (1993), 255–288.

[12] P.L. Lions and B. Perthame, *Lemmes de moments, de moyenne et de dispersion,* C. R. Acad. Sci. Paris, Série I **314** (1992), 801–806.

[13] B. Perthame and L. Vega, *Morrey-Campanato estimates for Helmholtz Equation*, J. Funct. Anal. **164**(2) (1999), 340–355.

[14] B. Perthame and L. Vega, *Energy decay and Sommerfeld condition for Helmholtz equation with variable index at infinity*, preprint (2002).

[15] F. Poupaud and A. Vasseur *Classical and quantum transport in random media*, preprint (2001).

[16] M. Reed and B. Simon, *Analysis of Operators. Methods of Modern Mathematical Physics IV*, Acad. Press, San Diego, 1978.

[17] L. Ryzhik, G. Papanicolaou and J. Keller, *Energy transport for elastic and other waves in a random medium* Wave motion **24** (1996), 327–370.

[18] Y. Saito, *Schrödinger operators with a nonspherical radiation condition*, Pacif. J. of Math. **126**(2) (1987), 331–359.

[19] Bo Zhang, *Radiation condition and limiting amplitude principle for acoustic propagators with two unbounded media*, Proc. of the Royal Soc. of Edinburgh, **128 A** (1998), 173–192.

Benoît Perthame
Ecole Normale Supérieure, DMA, UMR8553
45, rue d'Ulm 75230 Paris
FRANCE
E-mail: benoit.perthame@ens.fr

Luis Vega
Universidad del Pais Vasco, Apdo. 644
48080 Bilbao
SPAIN
E-mail: mtpvegol@lg.ehu.es