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A Littlewood-Paley type inequality

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Abstract. In this note we prove the following theorem:
Let u be a harmonic function in the unit ball $B \subset \mathbf{R}^n$ and $p \in \left[\frac{n-2}{n-1}, 1\right]$. Then there is a constant $C = C(p, n)$ such that

$$\sup_{0 \le r < 1} \int_S |u(r\zeta)|^p d\sigma(\zeta) \le C \left(|u(0)|^p + \int_B |\nabla u(x)|^p (1 - |x|)^{p-1} dV(x) \right).$$

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1 Introduction

Throughout this note n is an integer greater than or equal to 3, $B(a, r) = \{x \in \mathbf{R}^n \mid |x - a| < r\}$ denotes the open ball centered at a of radius r, where $|x|$ denotes the norm of $x \in \mathbf{R}^n$ and B is the open unit ball in the n-dimensional Euclidean space $\mathbf{R}^n$. $S = \partial B = \{x \in \mathbf{R}^n \mid |x| = 1\}$ is the Euclidean boundary of B. Further, $dV(x)$ denotes the Lebesgue volume measure on B, $d\sigma$ the normalized surface measure on S.

Let U be the unit disc in the complex plane and $dm(z) = r dr \frac{d\theta}{\pi}$ the normalized Lebesgue area measure on U. Let $\mathcal{H}(U)$ be the space of all harmonic functions on U and $\mathcal{H}^p(U)$ the Hardy harmonic space i.e., the set of harmonic functions on U such that

$$||u||_{\mathcal{H}^p(U)} = \sup_{0<r<1} \left(\int_{\partial U} |u(re^{it})|^p dt \right)^{1/p} < +\infty.$$

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It is well known that when $p \geq 1$ for a given $u^* \in \mathcal{L}^p(\partial U)$, the harmonic extension of u^* on U, denoted by u, is

$$u(z) = \frac{1}{2\pi}\int_{\partial U} \frac{1-|z|^2}{|e^{it}-z|^2} u^*(e^{it})dt, \qquad \text{for} \qquad z \in U \tag{1}$$

Also it is well known that

$$\lim_{r\to 1-0} u(re^{it}) = u^*(e^{it}), \qquad \text{a.e. on } \partial U$$

and $u \in \mathcal{H}^p(U)$.

The following theorem has been recently proved in [7].

Theorem A. *Suppose $p \geq 1$ and $0 < s < 1$. Then there is a constant $C > 0$ such that for any harmonic extension u of $u^* \in \mathcal{L}^p(\partial U)$ the following estimate holds:*

$$||u^* - u(0)||^p_{\mathcal{L}^p(\partial U)} \leq C \int_U |\nabla u|^p (1-|z|)^{p-ps-1} dm(z).$$

It is interesting that the proof given there holds also in the case $p \in (0, 1]$, $s = 0$. Hence, when $p = 1$ we have

$$||u - u(0)||^p_{\mathcal{L}^p(\partial U)} \leq C \int_U |\nabla u|^p (1-|z|)^{p-1} dm(z), \tag{2}$$

for any harmonic extension u of $u^* \in \mathcal{L}^1(\partial U)$. The proof is based on the fact that the integral means of subharmonic functions are nondecreasing.

Inequality (2) can be viewed as a Littlewood-Paley type inequality. The inequality of Littlewood and Paley is the one contained in the following theorem, see [4], [5] and [8].

Theorem B. *If u^* is a function in $L^p(\partial U)$ and if u is the harmonic function defined via Poisson integral of u^*, then*

$$\int_U |\nabla u(z)|^p (1-|z|^2)^{p-1} dm(z) \leq C \int_{\partial U} |u^*|^p d\sigma \quad \textit{for} \quad p \geq 2$$

and

$$\int_U |\nabla u(z)|^p (1-|z|^2)^{p-1} dm(z) \geq C \int_{\partial U} |u^*|^p d\sigma \quad \textit{for} \quad p \in (1, 2]$$

where C is a constant indepedent of u and p.

Theorem A motivated us to investigate analogous estimate when $p \in (0, 1]$. We consider similar estimate in the case of harmonic functions on the unit ball B. Let $\mathcal{H}(B)$ be the space of all harmonic functions on B and $\mathcal{H}^p(B)$ the Hardy harmonic space on B. In this paper we prove the following theorem.

Theorem 1. *Suppose* $p \in [\frac{n-2}{n-1}, 1]$ *and* $u \in \mathcal{H}(B)$. *Then there is a constant* $C = C(p, n)$ *such that*

$$\sup_{0 \le r < 1} \int_S |u(r\zeta)|^p d\sigma(\zeta) \le C \left(|u(0)|^p + \int_B |\nabla u(x)|^p (1 - |x|)^{p-1} dV(x) \right).$$

In particular, if $\int_B |\nabla u(x)|^p (1 - |x|)^{p-1} dV(x) < \infty$, *then* $u \in \mathcal{H}^p(B)$.

2 Auxiliary results and the proof of the main result

In order to prove the main result we need three auxiliary results. Throughout the paper C denotes a positive constant that may change from one step to the next.

The first one is well known Fefferman-Stein lemma that was proved in [1], see also [3].

Lemma 1. *Let* $0 < p < \infty$. *Then for every multy-index* β,

$$|D^\beta u(a)|^p \le \frac{C}{r^n} \int_{B(a,r)} |D^\beta u|^p dV \quad \text{whenever} \quad B(a, r) \subset B,$$

for all $u \in \mathcal{H}(B)$ *and some constant C depending only on* β, p *and* n.

Lemma 2. *Suppose* $0 < p < \infty$ *and* $\alpha \in \mathbf{R}$. *Then there is a constant* $C = C(p, \alpha, n)$ *such that*

$$\begin{aligned} M_\infty^p(u, 7/8) &= \max_{x \in B(0,7/8)} |u(x)|^p \\ &\le C \left(|u(0)|^p + \int_B |\nabla u(x)|^p (1 - |x|)^{p+\alpha} dV(x) \right), \end{aligned}$$

for all $u \in \mathcal{H}(B)$.

Proof. Since $u(x_0) - u(0) = \int_0^1 u'(tx_0)dt = \int_0^1 \langle \nabla u(tx_0), x_0 \rangle dt$, by elementary inequalities we obtain

$$|u(x_0)|^p \le c_p \left(|u(0)|^p + |x_0|^p \max_{|x| \le 7/8} |\nabla u(x)|^p \right), \tag{3}$$

for each $x_0 \in \overline{B(0, 7/8)}$, where $c_p = 1$ for $0 < p < 1$ and $c_p = 2^{p-1}$ for $p \ge 1$.

On the other hand by Lemma 1 and some simple calculations we obtain

$$|\nabla u(x)|^p \le C \int_{B(x,1/16)} |\nabla u(y)|^p dV(y)$$

for each $x \in \overline{B(0, 7/8)}$ and consequently

$$\max_{|x| \le 7/8} |\nabla u(x)|^p \le \max\{C\, 16^{p+\alpha}, C\} \int_{B(0,15/16)} |\nabla u(y)|^p (1 - |y|)^{p+\alpha} dV(y). \tag{4}$$

From (3) and (4) the result follows. □

For $x \in B \setminus B(0, 5/9)$, $x = r\zeta$, $\zeta \in S$, and a continuous function f let define the following "maximal" function:

$$f^{max}(x) = \sup \left\{ \ |f(t\zeta)| \ \middle| \ |x| - \frac{5(1 - |x|)}{4} < t < |x| + \frac{3(1 - |x|)}{4} \ \right\}.$$

Lemma 3. *Let $u \in \mathcal{H}(B)$. Then there is a constant $C = C(p, n)$ such that*

$$\int_{11/19}^1 M_p^p((\nabla u)^{max}, r)(1 - r)^{p-1} r^{n-1} dr \le C \int_0^1 M_p^p(\nabla u, r)(1 - r)^{p-1} r^{n-1} dr.$$

Proof. Let $x = r\zeta \in B \setminus B(0, 11/19)$, $\zeta \in S$. By Lemma 1 it follows that

$$(\,(\nabla u)^{max}(x)\,)^p \le \frac{C}{(1 - r)^n} \int_{B\left((r - \frac{1-r}{4})\zeta,\, \frac{9}{8}(1-r)\right)} |\nabla u|^p dV. \tag{5}$$

Replacing x in (5) by Ux, where U is an arbitrary orthogonal transformation of B, then using the change $y \to Uy$ and integrating with respect to the Haar measure on the orthogonal group $\mathcal{O}(n)$ we obtain

$$\int_{\mathcal{O}(n)} ((\nabla u)^{max}(Ux))^p dU \le \frac{C}{(1 - r)^n} \int_{\mathcal{O}(n)} \int_{B\left((r - \frac{1-r}{4})\zeta,\, \frac{9}{8}(1-r)\right)} |\nabla u(Uy)|^p dV(y) dU.$$

By Fubini's theorem and since $\int_{\mathcal{O}(n)} |g(Ux)|^p dU = \int_S |g(|x|\zeta)|^p d\sigma(\zeta)$ we obtain

$$M_p^p((\nabla u)^{max}, |x|) \leq \frac{C}{(1-r)^n} \int\limits_{B\left((r-\frac{1-r}{4})\zeta, \frac{9}{8}(1-r)\right)} M_p^p(\nabla u, |y|) dV(y). \tag{6}$$

Multiplying (6) by $(1-r)^{p-1}$, then integrating over $B \setminus B(0, 11/19)$, using the fact that

$$\frac{1}{8}(1-|x|) \leq 1-|y| \leq \frac{19}{8}(1-|x|) \quad \text{for} \quad y \in B\left(\left(r - \frac{1-r}{4}\right)\zeta, \frac{9}{8}(1-r)\right)$$

and using Fubini's theorem, we obtain

$$\begin{aligned} &\int\limits_{B\setminus B(0,11/19)} M_p^p((\nabla u)^{max}, |x|)(1-r)^{p-1} dV(x) \leq \\ \leq &C \int\limits_{B\setminus B(0,11/19)} \int\limits_{B\left((r-\frac{1-r}{4})\zeta, \frac{9}{8}(1-r)\right)} (1-|y|)^{p-1-n} M_p^p(\nabla u, |y|) dV(y) dV(x) \\ \leq &C \int\limits_B (1-|y|)^{p-1-n} M_p^p(u, |y|) \int\limits_{D(y)} dV(x) dV(y) \end{aligned} \tag{7}$$

where

$$D(y) \subset \left\{ x \,\middle|\, |x - \frac{1-|x|}{4|x|}x - y| < \frac{9}{8}(1-|x|) \right\} \subset \left\{ x \,\middle|\, |x-y| < \frac{11}{8}(1-|x|) \right\}.$$

From (7), since $V(D(y)) \leq V(B)11^n(1-|y|)^n$ and using the polar coordinates the result follows.

Proof of Theorem 1. Let $x \in B$, $x = r\zeta$, $\zeta \in S$. Clearly

$$u(x) - u(0) = \int_0^1 u'(tx)dt = \int_0^1 \langle \nabla u(tx), x \rangle dt. \tag{8}$$

Denote $t_k = 1 - 2^{-k}$, $k \in \mathbf{N} \cup \{0\}$. From (8) and using elementary inequalities we obtain

$$\begin{aligned}|u(x)|^p &\le |u(0)|^p + \left|\int_0^1 \langle \nabla u(tx), x\rangle dt\right|^p \\ &\le |u(0)|^p + \sum_{k=1}^{\infty}\left(\int_{t_{k-1}}^{t_k} |\langle \nabla u(tx), x\rangle| dt\right)^p \\ &\le |u(0)|^p + \sum_{k=1}^{\infty}\frac{1}{2^{pk}} \sup_{t_{k-1}<t<t_k} |\nabla u(tx)|^p.\end{aligned} \tag{9}$$

Integrating (9) over S using the fact that

$$\sup_{t_k<t<t_{k+1}} |\nabla u(tr\zeta)|^p \le (\nabla u)^{max}(\rho x),$$

for $\rho \in (t_{k-1}, t_k)$, applying Lemma 2 and then Lemma 3 to the function $f(x) = \nabla u(rx)$ we obtain:

$$\begin{aligned}M_p^p(u, r) \le& |u(0)|^p + C\sum_{k=0}^{\infty}\frac{1}{2^{p(k+1)}}\int_S \sup_{t_k<t<t_{k+1}} |\nabla u(tr\zeta)|^p d\sigma(\zeta) \\ \le& |u(0)|^p + C\max_{|x|\le 7/8}|u(x)| \\ &+ C\sum_{k=3}^{\infty}\frac{1}{2^{p(k+1)}}\int_S \min_{t_{k-1}<\rho<t_k}((\nabla u)^{max}(\rho r\zeta))^p d\sigma(\zeta) \\ \le& |u(0)|^p + C\max_{|x|\le 7/8}|u(x)| \\ &+ C\int_{3/4}^1 (1-\rho)^{p-1}\int_S ((\nabla u)^{max}(\rho r\zeta))^p \rho^{n-1} d\sigma(\zeta) d\rho \\ \le& C\left(|u(0)|^p + \int_0^1 (1-t)^{p-1} M_p^p(\nabla u, rt) t^{n-1} dt\right) \\ \le& C\left(|u(0)|^p + \int_0^1 (1-t)^{p-1} M_p^p(\nabla u, t) t^{n-1} dt\right),\end{aligned}$$

where in the last inequality we use the fact that for $p \ge \frac{n-2}{n-1}$, the function $|\nabla u|^p$ is subharmonic [6, Chap. 7.3], and consequently $M_p^p(\nabla u, s)$ is nondecreasing in s. From this the result follows.

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