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Complexification of foliations and complex secondary classes

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Abstract. Some properties of complex secondary classes are discussed. It is shown that the Godbillon-Vey class and the Bott class are related via complexification.

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Introduction

Secondary characteristic classes are one of the main tools in studying foliations. In the holomorphic category, there are some results on complex secondary characteristic classes, e.g. [5, 9, 1], but their properties are not yet fully understood. In this paper, the space of complex secondary characteristic classes, denoted by $H^*(\mathrm{WU}_q)$, is studied and the following results are shown:

Theorem A. *There exists a spectral sequence*

$$E_2^{p,s} \cong H^s(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}) \otimes H^p(\mathrm{BGL}(q;\mathbf{C})) \Rightarrow H^{p+s}(\mathrm{WU}_q).$$

In fact, $d_r = 0$ for $r > 2q^2 + 4q + 1$.

Theorem B. *Complexification of foliations induces a natural isomorphism between $H^{2q+1}(\mathrm{WU}_q)$ and $H^{2q+1}(\mathrm{W}_q)$ if q is even, between*

$$H^{2q+1}(\mathrm{W}_q \otimes \overline{\mathrm{W}_q})/H^{2q+1}(\mathrm{WU}_q)$$

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and $H^{2q+1}(\mathrm{W}_q)$ *if* q *is odd.*

See the first section for definitions concerning secondary characteristic classes. Complexification is introduced in the second section with an example.

The meaning of Theorem A is as follows, namely, the differential induces a natural mapping from the classifying space $B\Gamma_q^{\mathbf{C}}$ for transversely holomorphic foliations of complex codimension q to $\mathrm{BGL}(q;\mathbf{C})$. Its homotopy fiber $B\overline{\Gamma_q^{\mathbf{C}}}$ is the classifying space for transversely holomorphic foliations of complex codimension q with trivialized complex normal bundle. Thus the cohomology of $B\Gamma_q^{\mathbf{C}}$ might be calculated, if $H^*(B\overline{\Gamma_q^{\mathbf{C}}})$ were known, by using the Serre spectral sequence whose E_2-term is $H^*(B\overline{\Gamma_q^{\mathbf{C}}})\otimes H^*(\mathrm{BGL}(q;\mathbf{C}))$, because $\mathrm{BGL}(q;\mathbf{C})$ is simply connected. Theorem A asserts that this is still valid for secondary classes. See Section 1 for more details.

Theorem A will be shown by constructing a certain differential graded algebra $\mathcal{WU}_q$ which is isomorphic to $(\mathrm{W}_q\otimes\overline{\mathrm{W}_q})\otimes\mathbf{C}[b_1,b_2,\cdots,b_q]$ as graded algebras and whose cohomology $H^*(\mathcal{WU}_q)$ is isomorphic to $H^*(\mathrm{WU}_q)$. This is done in the first section. Theorem B will be shown in the second section.

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1 Definitions and Proof of Theorem A

First recall secondary characteristic classes in order to fix notations. The coefficient of cohomology is chosen as $\mathbf{C}$ unless otherwise stated.

Complex secondary classes are defined in terms of the following differential graded algebras (DGA's for short).

Definition 1.1 WU_q and $\mathrm{W}_q\otimes\overline{\mathrm{W}_q}$ are DGA's defined as follows. First let $\mathbf{C}[v_1,\cdots,v_q]$ be the polynomial ring with generators $v_1,\cdots,v_q$. The degree of v_i, denoted by $\deg v_i$, is set to be $2i$. Let I_q be the ideal generated by monomials of degree greater than $2q$, and set $\mathbf{C}_q[v_1,\cdots,v_q]=\mathbf{C}[v_1,\cdots,v_q]/I_q$. $\mathbf{C}_q[\bar{v}_1,\cdots,\bar{v}_q]$ is defined by replacing v_i with $\bar{v}_i$. We set

$$\begin{aligned}\mathrm{WU}_q &= \textstyle\bigwedge[\widetilde{u}_1,\cdots,\widetilde{u}_q]\otimes\mathbf{C}_q[v_1,\cdots,v_q]\otimes\mathbf{C}_q[\bar{v}_1,\cdots,\bar{v}_q],\\ \mathrm{W}_q\otimes\overline{\mathrm{W}_q} &= \textstyle\bigwedge[u_1,\cdots,u_q]\wedge\bigwedge[\bar{u}_1,\cdots,\bar{u}_q]\otimes\mathbf{C}_q[v_1,\cdots,v_q]\\ &\qquad\otimes\mathbf{C}_q[\bar{v}_1,\cdots,\bar{v}_q].\end{aligned}$$

The differential is defined by requiring $d\widetilde{u}_i=v_i-\bar{v}_i$, $du_i=v_i$, $d\bar{u}_i=\bar{v}_i$ and $dv_i=d\bar{v}_i=0$. We set $\deg\widetilde{u}_i=\deg u_i=\deg\bar{u}_i=2i-1$. Elements of DGA's are expressed as usual by using multi-indices, for example, $u_I = u_{i_1} \cdots u_{i_r}$ and $v_J = v_1^{j_1} \cdots v_q^{j_q}$ for $I = \{i_1, \cdots, i_r\}$ and $J = (j_1, \cdots, j_q)$.

The cohomologies of these DGA's are regarded as the spaces of complex secondary classes for foliations as follows. First let $B\Gamma_q^{\mathbf{C}}$ be the classifying space for transversely holomorphic foliations of complex codimension q, then the differential induces a natural mapping from $B\Gamma_q^{\mathbf{C}}$ to $\mathrm{BGL}(q;\mathbf{C})$. Its homotopy fiber, denoted by $B\overline{\Gamma_q^{\mathbf{C}}}$, is the classifying space for transversely holomorphic foliations of complex codimension q with trivialized complex normal bundle. It is known that elements of $H^*(\mathrm{WU}_q)$ (resp. $H^*(\mathrm{W}_q \otimes \overline{\mathrm{W}_q})$) determine characteristic classes of transversely holomorphic foliations of complex codimension q (resp. transversely holomorphic foliations of complex codimension q with trivialized complex normal bundle). Indeed, there are homomorphisms

$$\chi^{\mathbf{C}} : H^*(\mathrm{WU}_q) \to H^*(B\Gamma_q^{\mathbf{C}}) \quad \text{and} \quad \widetilde{\chi}^{\mathbf{C}} : H^*(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}) \to H^*(B\overline{\Gamma_q^{\mathbf{C}}})$$

called the universal characteristic mappings [5, 10]. In particular, the elements of $H^*(\mathrm{WU}_q)$ and $H^*(\mathrm{W}_q \otimes \overline{\mathrm{W}_q})$ which involve $\widetilde{u}_i$, u_i or $\bar{u}_i$ are called complex secondary classes. Let ι be the natural mapping from $B\overline{\Gamma_q^{\mathbf{C}}}$, to $B\Gamma_q^{\mathbf{C}}$ and ι^* be the induced mapping on the cohomology. If we define a homomorphism of DGA's, say ι', from WU_q to $\mathrm{W}_q \otimes \overline{\mathrm{W}_q}$ by the formulae $\iota'(\widetilde{u}_i) = u_i - \bar{u}_i$, $\iota'(v_i) = v_i$ and $\iota'(\bar{v}_i) = \bar{v}_i$, then the induced mapping $\iota'_* : H^*(\mathrm{WU}_q) \to H^*(\mathrm{W}_q \otimes \overline{\mathrm{W}_q})$ and the mapping ι^* together with the universal characteristic mappings as above form the following commutative diagram:

$$\begin{array}{ccc}
H^*(\mathrm{WU}_q) & \xrightarrow{\iota'_*} & H^*(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}) \\
{\scriptstyle \chi^{\mathbf{C}}}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\chi}^{\mathbf{C}}} \\
H^*(B\Gamma_q^{\mathbf{C}}) & \xrightarrow[\iota^*]{} & H^*(B\overline{\Gamma_q^{\mathbf{C}}}).
\end{array}$$

By abuse of notation we denote the mapping ι'_* again by ι^*.

Real secondary classes are defined in terms of the following DGA's. Coefficients are chosen to be in $\mathbf{C}$ for simplicity.

Definition 1.1' WO_q and W_q are DGA's defined as follows. Let $\mathbf{C}_q[c_1, \cdots, c_q]$ be the truncated polynomial ring obtained by replacing v_i with c_i. The degree of c_i is set to be $2i$. We now set

$$\begin{aligned}
\mathrm{WO}_q &= \textstyle\bigwedge[h_1, h_3, \cdots, h_{[q]}] \otimes \mathbf{C}_q[c_1, \cdots, c_q], \\
\mathrm{W}_q &= \textstyle\bigwedge[h_1, h_2, \cdots, h_q] \otimes \mathbf{C}_q[c_1, \cdots, c_q],
\end{aligned}$$

where $[q]$ denotes the greatest odd integer which is not greater than q. The differential is defined by requiring $dh_i = c_i$ and $dc_i = 0$. We set $\deg h_i = 2i - 1$.

Let $B\Gamma_q$ be the classifying space for real foliations of real codimension q. Then there is again a natural mapping from $B\Gamma_q$ to $\mathrm{BGL}(q; \mathbf{R})$. The homotopy fiber is denoted by $B\overline{\Gamma_q}$ and it is the classifying space for real foliations of real codimension q with trivialized normal bundle. As in the complex case, there is the following commutative diagram:

$$\begin{array}{ccc} H^*(\mathrm{WO}_q) & \longrightarrow & H^*(\mathrm{W}_q) \\ \big\downarrow & & \big\downarrow \\ H^*(B\Gamma_q) & \longrightarrow & H^*(B\overline{\Gamma_q}), \end{array}$$

where the vertical mappings are the universal characteristic mappings and the mapping in the bottom line is the mapping induced by the natural mapping $B\overline{\Gamma_q} \to B\Gamma_q$. Finally, the mapping in the top line is induced by the obvious inclusion from WO_q to W_q.

Elements of $H^*(\mathrm{WO}_q)$ (resp. $H^*(\mathrm{W}_q)$) can be considered as characteristic classes for real foliations of real codimension q (resp. real foliations of real codimension q with trivialized normal bundle). The elements of $H^*(\mathrm{WO}_q)$ and $H^*(\mathrm{W}_q)$ which involve h_i are called real secondary classes.

Remark 1.2. If $\mathbf{R}$ is chosen as the coefficients in Definition 1.1', it suffices to consider $H^*(B\Gamma_q; \mathbf{R})$ and $H^*(B\overline{\Gamma_q}; \mathbf{R})$. This is indeed the usual formulation.

The following classes are significant:

Definition 1.3.

1) The Godbillon-Vey class GV_q is the element of $H^{2q+1}(\mathrm{W}_q)$ or $H^{2q+1}(\mathrm{WO}_q)$ defined by the cocycle $h_1 c_1^q$.

2) The Bott class Bott_q is the element of $H^{2q+1}(\mathrm{W}_q \otimes \overline{\mathrm{W}_q})$ defined by the cocycle $u_1 v_1^q$.

3) The imaginary part of the Bott class ξ_q is the element of $H^{2q+1}(\mathrm{WU}_q)$ defined by the cocycle $\sqrt{-1}\widetilde{u}_1(v_1^q + v_1^{q-1}\bar{v}_1 + \cdots + \bar{v}_1^q)$.

It is known that $\iota^*\xi_q = -2\,\mathrm{Im}\ \mathrm{Bott}_q = \sqrt{-1}(u_1 v_1^q - \bar{u}_1 \bar{v}_1^q)$ in $H^{2q+1}(\mathrm{W}_q \otimes \overline{\mathrm{W}_q})$.

One of the ways to realize elements of $H^*(\mathrm{WU}_q)$ as elements of the de Rham cohomology [10] leads us to the following

Definition 1.4. We set

$$\mathcal{WU}_q = \bigwedge[k_1, k_2, \cdots, k_q] \wedge \bigwedge[\bar{k}_1, \bar{k}_2, \cdots, \bar{k}_q] \otimes \mathbf{C}_q[v_1, v_2, \cdots, v_q] \\ \otimes \mathbf{C}_q[\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_q] \otimes \mathbf{C}[b_1, b_2, \cdots, b_q],$$

where $\deg k_i = \deg \bar{k}_i = 2i - 1$ and $\deg v_j = \deg \bar{v}_j = \deg b_j = 2j$. $\mathcal{WU}_q$ is equipped with the differential determined by requiring $dk_i = v_i - b_i$, $d\bar{k}_i = \bar{v}_i - b_i$ and $dv_j = d\bar{v}_j = db_j = 0$. Note that the element $k_i - \bar{k}_i$ can be naturally identified with $\widetilde{u}_i$, and under this identification, WU_q is naturally a sub-DGA of $\mathcal{WU}_q$. The inclusion is formally denoted by α.

By following the usual construction, using connections, of the universal mapping $\chi^{\mathrm{C}} : H^*(\mathrm{WU}_q) \to H^*(B\Gamma_q^{\mathrm{C}})$, the universal mapping $\widehat{\chi}^{\mathrm{C}} : H^*(\mathcal{WU}_q) \to H^*(B\Gamma_q^{\mathrm{C}})$ can be constructed, and they satisfy $\chi^{\mathrm{C}} = \widehat{\chi}^{\mathrm{C}} \circ \alpha_*$. Moreover, they are essentially the same:

Lemma 1.5. *Let $\alpha\colon \mathrm{WU}_q \to \mathcal{WU}_q$ be the inclusion as in Definition 1.4, then the induced mapping $\alpha_*\colon H^*(\mathrm{WU}_q) \to H^*(\mathcal{WU}_q)$ is an isomorphism.*

Proof. The proof presented here is suggested by the referee. The original proof was more computational.

First define a sub-DGA $\mathcal{B}$ of $\mathcal{WU}_q$ by setting

$$\mathcal{B} = \bigwedge[\bar{k}_1, \cdots, \bar{k}_q] \otimes \mathbf{C}[b'_1, \cdots, b'_q],$$

where $b'_i = b_i - \bar{v}_i$. Note that $\mathcal{WU}_q = \mathcal{B} \otimes \mathrm{WU}_q$ as DGA's, where WU_q is considered as a sub-DGA via α. We now introduce a filtration on $\mathcal{WU}_q$ by setting

$$F_p = \left\langle c \cdot c' \in \mathcal{WU}_q \mid c' \in \mathcal{B} \text{ and } \deg c' \geq p \right\rangle,$$

where the right hand side means the subspace generated by the elements inside the bracket. Note that F_p is closed under d. It is straightforward to see that $E_1 \cong \mathcal{B} \otimes H^*(\mathrm{WU}_q)$, $E_2 \cong H^*(\mathcal{B}) \otimes H^*(\mathrm{WU}_q)$ and that $d_r = 0$ if $r \geq 2$ in the resulting spectral sequence, where $d_r : E_r \to E_r$ denotes the differential induced on E_r. Since it is well-known that $H^*(\mathcal{B}) \cong \mathbf{C}$, this completes the proof. $\square$

Remark 1.6. One can show by direct calculations that the inverse mapping of α_* is induced by the mapping α' determined by

$$\alpha'(k_i) = \frac{1}{2}\widetilde{u}_i, \quad \alpha'(\bar{k}_i) = -\frac{1}{2}\widetilde{u}_i, \quad \alpha'(v_j) = v_j,$$
$$\alpha'(\bar{v}_j) = \bar{v}_j \quad \text{and} \quad \alpha'(b_j) = \frac{1}{2}(v_j + \bar{v}_j).$$

The original proof was based on this fact.

Proof of Theorem A. Let F_p be the subspace of $\mathcal{WU}_q$ defined by

$$F_p = \left\langle k_I \bar{k}_M v_J \bar{v}_K b_L \,\middle|\, 2\,|L| = 2(l_1 + 2l_2 + \cdots + ql_q) \geq p \right\rangle,$$

then $\mathcal{WU}_q = F_0 \supset F_1 = F_2 \supset F_3 = F_4 \supset \cdots$ and F_p is closed under d. The E_1-terms of the associated spectral sequence satisfy $E_1^{p,s} \cong H^{p+s}(F_p/F_{p+1})$, where

$$F_p/F_{p+1} \cong \begin{cases} \left\langle k_I \bar{k}_M v_J \bar{v}_K b_L \,\middle|\, 2\,|L| = p \right\rangle & \text{if } p \text{ is even} \\ 0 & \text{if } p \text{ is odd.} \end{cases}$$

Let B_p be the subspace of $\mathbf{C}[b_1, \cdots, b_q]$ which consists of the elements of degree $2p$, then $F_p/F_{p+1} \cong \mathrm{W}_q \otimes \overline{\mathrm{W}_q} \otimes B_p$ as DGA's if p is even. Thus $E_1 \cong H^*(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}) \otimes H^*(\mathrm{BGL}(q; \mathbf{C}))$. Noticing that $d_1 = 0$, we see that $E_2^{p,s} \cong E_1^{p,s}$.

As $H^*(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}) = \{0\}$ if $* > 2q^2 + 4q$ [7], $d_r = 0$ for $r > 2q^2 + 4q + 1$ and this spectral sequence converges to $H^*(\mathcal{WU}_q) \cong H^*(\mathrm{WU}_q)$. □

Remark 1.7. There is another spectral sequence which converges to $H^*(\mathrm{WU}_q)$ faster [3]. The meaning of the filtration is however not clear.

Let us now determine the space $H^*(\mathrm{WU}_q)$ in lower degrees. There are three important mappings. First, let $\tau = d_r : E_r^{0,r-1} \to E_r^{r,0}$ be the transgression map. Second, the projection from $\mathcal{WU}_q$ to $F_0/F_1 \cong \mathrm{W}_q \otimes \overline{\mathrm{W}_q}$ induces at the cohomology level the natural mapping ι^* via the identification of Lemma 1.5. Finally, we denote by π^* the mapping from $H^*(\mathrm{BGL}(q; \mathbf{C})) = \mathbf{C}[b_1, \cdots, b_q]$ to $H^*(\mathrm{WU}_q) \cong H^*(\mathcal{WU}_q)$ induced by the inclusion. This corresponds naturally to the mapping induced by the projection from $B\Gamma_q^{\mathrm{C}}$ to $\mathrm{BGL}(q; \mathbf{C})$.

Lemma 1.8. *The cohomology $H^*(\mathrm{WU}_q)$ in lower degrees is determined as follows:*

1) $H^n(\mathrm{BGL}(q;\mathbf{C})) \cong H^n(\mathrm{WU}_q)$ *for* $n \leq 2q$.

2) *The following sequence is exact:*

$$0 \to H^{2q+1}(\mathrm{WU}_q) \xrightarrow{\iota^*} H^{2q+1}(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}) \xrightarrow{\tau} H^{2q+2}(\mathrm{BGL}(q;\mathbf{C})) \to 0.$$

3) $H^{2q+2}(\mathrm{WU}_q) = \{0\}$.

Proof. First, $H^n(\mathrm{W}_q) = \{0\}$ for $0 < n < 2q+1$ [7]. On the other hand, one can easily see that $H^1(\mathrm{WU}_q) = \{0\}$. It follows from general properties of spectral sequences that there is the following exact sequence in positive degrees up to $n = 2q+2$:

$$\begin{array}{ccccccc} \cdots & \to & H^{n-1}(\mathrm{BGL}(q;\mathbf{C})) & \xrightarrow{\pi^*} & H^{n-1}(\mathrm{WU}_q) & \xrightarrow{\iota^*} & H^{n-1}(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}) \\ & \xrightarrow{\tau} & H^n(\mathrm{BGL}(q;\mathbf{C})) & \to & \cdots & & \\ & & & & \cdots & \to & H^{2q+2}(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}). \end{array}$$

Since $H^*(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}) \cong H^*(\mathrm{W}_q) \otimes H^*(\overline{\mathrm{W}_q})$, we see that $H^n(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}) = \{0\}$ if $1 \leq n \leq 2q$. Moreover, one can see from the form of the Vey basis [7] that $H^{2q+2}(\mathrm{W}_q) = \{0\}$. Indeed, if $h_i h_I c_J$ represents a member of the Vey basis which is of degree $2q+2$, then the number of the entries of I is odd. We may now assume that $i' > i$ for any $i' \in I$, then it is shown in [7] that $h_i c_J$ is also a member of the Vey basis. In particular $h_i c_J$ is of degree at least $2q+1$. On the other hand, h_I is of degree at least 3, which is absurd. Hence

$$H^{2q+2}(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}) = \{0\}.$$

The claims now follow from the facts that $H^{2n+1}(\mathrm{BGL}(q;\mathbf{C})) = \{0\}$ and that $\pi^* = 0$ in degrees greater than $2q$ (this corresponds to the Bott vanishing theorem [5]). $\square$

Let

$$\mathcal{R} = \left\{ \frac{c + \bar{c}}{2} \,\middle|\, c \in H^{2q+1}(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}) \right\}$$

and

$$\mathcal{I} = \left\{ \frac{c - \bar{c}}{2\sqrt{-1}} \,\middle|\, c \in H^{2q+1}(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}) \right\}.$$

Recalling that the Vey basis of $H^{2q+1}(\mathrm{W}_q)$ is of the form $h_i c_J$ with $i + |J| = q + 1$ [7], let $\widetilde{\gamma}$ be the linear mapping from $H^{2q+1}(\mathrm{W}_q \otimes \overline{\mathrm{W}_q})$ to $H^{2q+1}(\mathrm{W}_q)$ satisfying the conditions $\widetilde{\gamma}(u_i v_J) = \sqrt{-1} h_i c_J$ and $\widetilde{\gamma}(\bar{u}_i \bar{v}_J) = -\sqrt{-1} h_i c_J$, and set $\gamma = \widetilde{\gamma} \circ \iota^*$. Let μ be the linear mapping from $H^{2q+1}(\mathrm{W}_q \otimes \overline{\mathrm{W}_q})$ to $H^{2q+1}(\mathrm{W}_q)$ determined by the condition $\mu(u_i v_J) = \mu(\bar{u}_i \bar{v}_J) = h_i c_J$.

As $\tau(u_i v_J) = \tau(\bar{u}_i \bar{v}_J) = -b_i b_J$, ι^* is an isomorphism from $H^{2q+1}(\mathrm{WU}_q)$ to $\mathcal{I}$. The exact sequence in 2) of Lemma 1.8 then can be read as follows:

Corollary 1.9.

1) *γ is an isomorphism from $H^{2q+1}(\mathrm{WU}_q) \cong \mathcal{I}$ to $H^{2q+1}(\mathrm{W}_q)$ such that $\gamma(-\frac{1}{2}\xi_q) = \mathrm{GV}_q$.*

2) *μ is an isomorphism from $\mathcal{R}$ to $H^{2q+1}(\mathrm{W}_q)$ such that $\mu(\mathrm{Re\ Bott}_q) = \mathrm{GV}_q$ and $\mu(\mathrm{Im\ Bott}_q) = 0$.*

Remark 1.10. The inverse mapping γ^{-1} is in general complicated. For example, when $q = 2$,

$$\gamma^{-1}(h_1 c_2) = \frac{1}{4\sqrt{-1}} \left(\widetilde{u}_1 (v_2 + \bar{v}_2) + \widetilde{u}_2 (v_1 + \bar{v}_1) \right).$$

2 Relation with Complexifications

Definition 2.1. Let $(N, \mathcal{G})$ be a transversely real analytic foliation of real codimension q. A transversely holomorphic foliation $(M, \mathcal{F})$ of complex codimension q is said to be a complexification of $(N, \mathcal{G})$ if there is an embedding $i : (N, \mathcal{G}) \to (M, \mathcal{F})$ such that $\mathcal{G}$ is transversely totally real with respect to $\mathcal{F}$.

Note that the complexifications discussed here are different from the ones considered by Haefliger and Sundararaman [8].

The mappings γ and μ are related to complexification as follows.

Proposition 2.2. *Let $(N, \mathcal{G})$ be a real foliation of codimension q whose normal bundle is trivial. Let $i : (N, \mathcal{G}) \to (M, \mathcal{F})$ be a complexification. Assume that the complex normal bundle of $\mathcal{F}$ is trivial when q is odd. Then $i^* : H^{2q+1}(M) \to H^{2q+1}(N)$ induces $(-1)^{\frac{q+1}{2}}\mu$ if q is odd, $(-1)^{\frac{q+2}{2}}\gamma$ if q is even.*

Proof. Let $Q(\mathcal{F})$ be the complex normal bundle of $\mathcal{F}$, namely, $Q(\mathcal{F})$ is the complex vector bundle locally spanned by $\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_q}$ modulo $T\mathcal{F} \otimes \mathbf{C}$, where $(z_1, \cdots, z_q)$ is a local holomorphic coordinate system in the transversal direction and $T\mathcal{F}$ is the set of leaf tangent vectors (see [2] for details). Similarly let $Q(\mathcal{G})$ be the normal bundle of $\mathcal{G}$ which is the real vector bundle locally spanned by the vectors $\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_q}$ modulo $T\mathcal{G}$, where $(y_1, \cdots, y_q)$ is a local coordinate system in the transversal direction.

Suppose for a while that $Q(\mathcal{F})$ is trivial and let s be a trivialization. Let ∇_s be the flat connection with respect to s. As $\mathcal{G}$ is totally real with respect to $\mathcal{F}$, we may assume that i^*s is the complexification of a trivialization, say, $s_{\mathbf{R}}$ of $Q(\mathcal{G})$ and $i^*\nabla_s$ is the complexification of the flat connection for $s_{\mathbf{R}}$. Similarly, if ∇_B be a complex Bott connection on $Q(\mathcal{F})$ then $i^*\nabla_B$ is the complexification of a Bott connection on $Q(\mathcal{G})$. Hence by suitably choosing Hermitian and Riemannian metrics on $Q(\mathcal{F})$ and $Q(\mathcal{G}) \otimes \mathbf{C}$, we may assume that $i^*u_i = (-\sqrt{-1})^i h_i$. It follows that $i^*\widetilde{u}_i = 2\left(-\sqrt{-1}\right)^i h_i$ if i is odd and that $i^*\widetilde{u}_i = 0$ if i is even. Let

$$u_i v_J \in H^{2q+1}(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}), \quad \text{then} \quad i^*(u_i v_J) = (-\sqrt{-1})^{i+|J|} h_i c_J,$$

where $|J| = j_1 + 2j_2 + \cdots + qj_q$. Assume now that q is odd, then

$$i^*(u_i v_J + \bar{u}_i \bar{v}_J) = 2(-1)^{\frac{q+1}{2}} h_i c_J \quad \text{and} \quad i^*(u_i v_J - \bar{u}_i \bar{v}_J) = 0.$$

Here we used the fact that $2i - 1 + 2|J| = 2q + 1$. On the other hand, if q is even, then $i^*(u_i v_J - \bar{u}_i \bar{v}_J) = 2(-1)^{\frac{q}{2}}(-\sqrt{-1}) h_i c_J$ and $i^*(u_i v_J + \bar{u}_i \bar{v}_J) = 0$. Noticing that $u_i v_J - \bar{u}_i \bar{v}_J$ is in the image of $\iota^* : H^*(\mathrm{WU}_q) \to H^*(\mathrm{W}_q \otimes \overline{\mathrm{W}_q})$ and such classes can be constructed without using trivializations but only using connections, we see that the triviality of $Q(\mathcal{F})$ is unnecessary if q is even. This completes the proof. □

Theorem B now follows from Corollary 1.9 and Proposition 2.2.

Remark 2.3. Let κ be the mapping from $\mathrm{W}_q \otimes \overline{\mathrm{W}_q}$ to W_q defined by the formulae

$$\kappa(u_i) = (-\sqrt{-1})^i h_i, \quad \kappa(\bar{u}_i) = (\sqrt{-1})^i h_i,$$
$$\kappa(v_i) = (-\sqrt{-1})^i c_i \quad \text{and} \quad \kappa(\bar{v}_i) = (\sqrt{-1})^i c_i,$$

then the induced mapping $\kappa_*\colon H^*(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}) \to H^*(\mathrm{W}_q)$ coincides with the complexification. Note that the above proof in fact shows that κ induces a mapping $\kappa_* : H^*(\mathrm{WU}_q) \to H^*(\mathrm{WO}_q)$. Notice also that

$$H^{2q+1}(\mathrm{WO}_q) \cong H^{2q+1}(\mathrm{W}_q)$$

if q is even.

The following fact is a simple consequence of the Bott vanishing theorem and the form of the Vey basis. An element of $H^*(\mathrm{W}_q \otimes \overline{\mathrm{W}_q})$ is said to be a product class if it is the product of elements of $H^*(\mathrm{W}_q)$ and $H^*(\overline{\mathrm{W}_q})$ of positive degree.

Proposition 2.4. $\kappa_* : H^*(\mathrm{W}_q \otimes \overline{\mathrm{W}_q}) \to H^*(\mathrm{W}_q)$ *annihilates product classes.*

Proof. Let $u_i u_I v_J$ and $\bar{u}_{i'} \bar{u}_{I'} \bar{v}_{J'}$ be elements of $H^*(\mathrm{W}_q)$ and $H^*(\overline{\mathrm{W}_q})$. We may assume that $i + |J| > q$ and $i \leq k$, where k is the minimum integer such that $j_k \neq 0$ [7]. Hence $2\,|J| > q$. Similarly $2\,\left|J'\right| > q$ and thus $2(|J| + \left|J'\right|) > 2q$. Therefore $c_J c_{J'} = 0$. □

Beginning with a transversely holomorphic foliation, one can first forget its transverse structure and then complexify it. Associated with this procedure, there is a composition of the mappings

$$H^*(\mathrm{WU}_{2q}) \stackrel{\kappa_*}{\to} H^*(\mathrm{WO}_{2q}) \stackrel{[\lambda]}{\to} H^*(\mathrm{WU}_q),$$

where $[\lambda]$ is the mapping obtained by forgetting transverse holomorphic structures [1]. Similarly, one can first consider a real foliation and complexify it, then forget its transverse structure. The resulting sequence is

$$H^*(\mathrm{WO}_{2q}) \stackrel{[\lambda]}{\to} H^*(\mathrm{WU}_q) \stackrel{\kappa_*}{\to} H^*(\mathrm{WO}_q).$$

We know little about $[\lambda] \circ \kappa_*$, while we have the following

Proposition 2.5. *The composition* $\kappa_* \circ [\lambda]$ *is equal to zero when restricted to the secondary characteristic classes.*

Proof. Let $c \in H^*(\mathrm{WO}_{2q})$ be a secondary class. By [7], we may assume that $c = h_i h_I c_J$ with $2i - 1 + 2\,|J| > 4q$, where I might be empty. The image of c under $\kappa_* \circ [\lambda]$ will be a linear combination of classes of the form $h_{i'} h_{I'} c_{J'}$ with $2i' - 1 + 2\left|J'\right| = 2i - 1 + 2\,|J|$ but now $i' \leq q$. This implies that $\left|J'\right| > q$ and hence $c_{J'} = 0$ by the Bott vanishing theorem. □

Example 2.6. Let Γ be a lattice in $\mathrm{SL}(q+1; \mathbf{C})$ such that $M = \Gamma \backslash \mathrm{SL}(q+1; \mathbf{C}) / \mathrm{SU}(q)$ is a closed manifold, where $\mathrm{SU}(q) = \{1\} \oplus \mathrm{SU}(q) \subset \mathrm{SL}(q+1; \mathbf{C})$. Assume moreover that $N = \Gamma_{\mathbf{R}} \backslash \mathrm{SL}(q+1; \mathbf{R}) / \mathrm{SO}(q)$ is also a closed manifold, where $\Gamma_{\mathbf{R}} = \Gamma \cap \mathrm{SL}(q+1;\mathbf{R})$ and $\mathrm{SO}(q) = \{1\} \oplus \mathrm{SO}(q) \subset \mathrm{SL}(q+1;\mathbf{R})$. It is well-known that such a Γ exists [4]. Let H be the subgroup of $\mathrm{SL}(q+1;\mathbf{C})$ defined by

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & D \end{pmatrix} \in \mathrm{SL}(q+1;\mathbf{C}) \,\middle|\, a \in \mathbf{C},\ b \in \mathbf{C}^q,\ D \in M(q;\mathbf{C}) \right\}$$

and set $H_{\mathbf{R}} = H \cap \mathrm{SL}(q+1;\mathbf{R})$.

Let $\mathcal{F}$ be the foliation of M induced by the cosets of H, namely, the foliation whose leaves are of the form $gH\,\mathrm{SU}(q)$, where $g \in \mathrm{SL}(q+1;\mathbf{C})$. Similarly, we denote by $\mathcal{F}_{\mathbf{R}}$ the foliation of N induced by $H_{\mathbf{R}}$. It is classically known that the Bott class of $\mathcal{F}$ and the Godbillon-Vey class of $\mathcal{F}_{\mathbf{R}}$ are non-trivial. They are represented as follows, namely, first let ω_{ij}, $0 \le i, j \le q$ be the natural dual basis of $M(q+1;\mathbf{C})$, where rows and columns are counted from zero. Note that ω_{ij} are naturally decomposed into the real and the imaginary parts: $\omega_{ij} = \eta_{ij} + \sqrt{-1}\nu_{ij}$. The Bott class of $\mathcal{F}$ is represented by the $(2q+1)$-form

$$\omega = \left(-\frac{q+1}{2\pi\sqrt{-1}} \right)^{q+1} \omega_{00} \wedge (d\omega_{00})^q$$

while the Godbillon-Vey class of $\mathcal{F}_{\mathbf{R}}$ is given by the $(2q+1)$-form

$$\omega_{\mathbf{R}} = \left(-\frac{q+1}{2\pi} \right)^{q+1} \eta_{00} \wedge (d\eta_{00})^q.$$

It follows that

$$(\sqrt{-1})^{q+1} i^*(\mathrm{Bott}_q(\mathcal{F})) = \mathrm{GV}(\mathcal{F}_{\mathbf{R}}),$$

where $i \colon N \to M$ is the natural inclusion. Noticing that

$$\mathrm{Im}\ \mathrm{Bott}_q(\mathcal{F}) = -\frac{1}{2}\xi_q(\mathcal{F}),$$

one can see that i^* coincides with either μ or γ according to the codimension q.

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