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The first Cohomology of Affine $\mathbb{Z}^p$-actions on Tori and applications to rigidity*

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Abstract. Let φ a *minimal* affine $\mathbb{Z}^p$-action on the torus T^q, $p \geq 2$ and $q \geq 1$. The cohomology of φ (see definition below) depends on both the algebraic properties of the induced action on $H^1(T^q, \mathbb{Z})$ and the arithmetical properties of the translation cocycle. We give a Diophantine condition that characterizes those affine actions whose first cohomology group is finite dimensional. In this case it is necessarily isomorphic to $\mathbb{R}^p$. Thus the $\mathbb{R}^p$-action F_φ obtained by suspension of φ is *parameter rigid*, i.e., any other $\mathbb{R}^p$-action with the same orbit foliation is smoothly conjugate to a reparametrization of F_φ by an automorphism of $\mathbb{R}^p$.

Keywords: minimal action, cocycle rigidity, diophantine condition, parameter rigidity, cohomology of ergodic actions.

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1 Introduction

Let Affine(T^q) denote the group of affine transformations of the torus T^q. By an affine $\mathbb{Z}^p$-action on T^q we mean a homomorphism φ of $\mathbb{Z}^p$ into Affine(T^q). The action φ induces a $\mathbb{Z}^p$-action by automorphisms of the ring $C^\infty(T^q)$ of the smooth functions on T^q given by $\ell \cdot f = f \circ \varphi(\ell)$ for $\ell \in \mathbb{Z}^p$ and $f \in C^\infty(T^q)$ defining a $\mathbb{Z}^p$-module structure $C^\infty_\varphi(T^q)$ on $C^\infty(T^q)$. The *cohomology of the action* φ is by definition the cohomology $H^*(\mathbb{Z}^p, C^\infty_\varphi(T^q))$of the $\mathbb{Z}^p$- module $C^\infty_\varphi(T^q)$, see for example [2].

The investigation of the cohomology of ergodic actions of higher rank abelian groups(e.g., $\mathbb{Z}^k$ and $\mathbb{R}^k$ for $k \geq 2$) has attracted considerable interest in recent years due to its connection with cocycle rigidity phenomena, i.e. every smooth

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cocycle is cohomologous to a constant one. Some cocycle rigidity phenomena appear in connection with hyperbolicity. For example A. Katok and R. Spatzier [10], [11] and [12] showed that real cocycles of certain Anosov actions of $\mathbb{Z}^p$ or $\mathbb{R}^p$ (standard Anosov) are cohomologous to constant cocycle. Similar results are obtained by A. Katok and K. Schmidt [9] for mixing expansive actions of $\mathbb{Z}^p$ on the automorphism group of an abelian group.

Another source of cocycle rigidity are certain $\mathbb{Z}^p$-actions on the group of translations of the torusT^q. This actions are determined by linear foliations with minimal leaves transverse to the fibers of the canonical projection of T^{p+q} to T^p, see [1], [22], [15], [20], [6], [4].

In this paper we give a *Diophantine condition* for affine $\mathbb{Z}^p$-actions on tori. Our main result states that the first cohomology group of an affine action is finite dimensional if and only if the action satisfies the Diophantine condition, see Section 1.2 for the precise statement of the result. This extends the work of J.L. Arraut and N.M. dos Santos done for actions of translations.

A consequence of our result is that, if the first cohomology group of the action is finite dimensional, then it is necessarily isomorphic to $\mathbb{R}^p$.

1.1 Geometrical consequences

Our main result, characterizing those affine actions with finite dimensional first cohomology group, has the following geometrical consequence.

Associated to each $\mathbb{Z}^p$- action on T^q we have its suspension foliated bundle $T^q \to (M, \mathcal{F}) \to^{\tau} T^q$ and an $\mathbb{R}^p$-action by automorphisms of the projection τ, called the *canonical action* of τ. The leafwise cohomology $H^*(\mathcal{F})$ is isomorphic to the cohomology of φ [22].

We say that a locally-free $\mathbb{R}^p$-action *is parameter rigid* if any other $\mathbb{R}^p$-action with the same orbit foliation is smooth conjugate to it, up to a reparametrization by an automorphism of $\mathbb{R}^p$. S. Matsumoto and Y. Mitsumatsu [19] proved that a locally-free action of $\mathbb{R}^p$ is parameter rigid if and only if the first cohomology group is isomorphic to $\mathbb{R}^p$. Thus the locally free $\mathbb{R}^p$-action induced by an affine $\mathbb{Z}^p$-action on $\mathbb{T}^q$ is parameter rigid if and only if the action satisfies the Diophantine condition, as defined below (Section 1.2).

This extends what was done by J.L. Arraut, N.M. dos Santos [1] and J. Moser [20] for action of translations. They gave a definition of a Diophantine $\mathbb{Z}^p$-action by translations on T^q and proved that the cohomology of such an action is isomorphic to the cohomology of T^q with real coefficients.

1.2 Statements of Results

An affine $\mathbb{Z}^p$-action φ on the torus T^q induces a $\mathbb{Z}^p$-action by automorphisms of $H_1(T^q, \mathbb{Z}) \cong \mathbb{Z}^q$ and $H^1(T^q, \mathbb{Z})$, denoted by φ_* and φ^*, respectively. The set $\sigma(\varphi_*)$ of all eigenvalues of all $\varphi_*(\ell)$ is referred to as the *spectrum* of φ_*. If φ is minimal then $\sigma(\varphi_*) = \{1\}$. The cohomology of φ depends on both the algebraic properties of φ^* and the arithmetical properties of the translation cocycle. The isotropy group of φ^* at $k \in \mathbb{Z}^q$ will be denoted by $I(k)$. Each isotropy group I of φ^* is a direct summand, i.e., there is a subgroup K of $\mathbb{Z}^p$ such that $\mathbb{Z}^p = I \oplus K$.

An affine $\mathbb{Z}^p$-action φ on T^q satisfies the *irrationality condition* if for $k \in \mathbb{Z}^q$ such that $\dim I(k) \geq p - 1$ there is $\ell \in I(k)$ so that

$$\langle k, \alpha(\ell)\rangle \notin \mathbb{Z}$$

where $\tilde{\varphi}(\ell) = \varphi_*(\ell) + \alpha(\ell)$ is a lifting of $\varphi(\ell)$ to the covering $\mathbb{R}^q$ and $\langle\, ,\, \rangle$ is the usual inner product on $\mathbb{R}^q$. Let Γ be the set of fixed points of φ^* i.e.

$$\Gamma = \{k \in \mathbb{Z}^q \mid \varphi^*(\ell)k = k \text{ for all } \ell \in \mathbb{Z}^p\}.$$

We note that if $\sigma(\varphi_*) = \{1\}$ then $\Gamma \neq 0$. An affine $\mathbb{Z}^p$-action φ on T^q satisfies a *Diophantine condition on* Γ if there are constants $\beta > 0$ and $C > 0$ so that for each $k \in \Gamma - \{0\}$ there is $j,\ 1 \leq j \leq p$ satisfying

$$\|\langle k, \alpha(e_j)\rangle\| \geq C|k|^{-\beta}$$

where $\{e_1, \ldots, e_p\}$ is the canonical basis of $\mathbb{Z}^p$, $|x| = \sup_j |x_j|$ and $\|x\| = \inf\{|x - l| \mid l \in \mathbb{Z}^n\}$.

Our main result is

Theorem 1. *Let φ be an affine $\mathbb{Z}^p$-action on T^q with $\sigma(\varphi_*) = \{1\}$, where $p \geq 2$ and $q \geq 1$. Suppose that φ satisfies the irrationality condition. Then the following statements are equivalent*

1. *φ satisfies a Diophantine condition on Γ.*
2. $H^1(\mathbb{Z}^p, C^\infty_\varphi(T^q)) \cong \mathbb{R}^p$.
3. *$H^1(\mathbb{Z}^p, C^\infty_\varphi(T^q))$ is Hausdorff.*

The main feature of an action by translations is that the induced action on cohomology is trivial and this greatly simplifies the calculation of the cohomology. This is not true for a general affine action and in fact the induced action on cohomology can be rather complicated. For this reason Theorem 1 is a nontrivial generalization of the corresponding result in [1].

The basic conjecture is that a minimal smooth $\mathbb{Z}^p$-action on T^q whose first cohomology is isomorphic to $\mathbb{R}^p$ should be smoothly conjugate to its corresponding affine $\mathbb{Z}^p$-action. This conjecture is supported by [16] and [26].

1.3 Examples

We now give the simplest example of affine $\mathbb{Z}^2$-action which does not act by translations and the first cohomology group is isomorphic to $\mathbb{R}^2$.

Let φ be an affine $\mathbb{Z}^2$-action on T^2 generated on the covering $\mathbb{R}^2$ by

$$\tilde{\varphi}_1(x, y) = (x + \alpha, x + y)$$

and

$$\tilde{\varphi}_2(x, y) = (x, y + \beta).$$

The action φ satisfies the irrationality condition and Diophantine condition on Γ if and only if α is a Diophantine number and β is irrational number. The second cohomology depends on β. It is isomorphic to $\mathbb{R}$ if β is Diophantine and it is non-Hausdorff if β is Liouville [15]. Contrasting with this a $\mathbb{Z}^p$-action by translations on T^q is Diophantine if only if the first cohomology is finite dimensional and thus its cohomology is necessarily isomorphic to the real cohomology of T^p [1].

An other interesting example is that of an affine $\mathbb{Z}^2$-action on torus T^5 such that the induced action on cohomology has infinitely many isotropy groups and the first cohomology group of the action is finite dimensional. The action is generated on the covering $\mathbb{R}^5$ by

$$\tilde{\varphi}_1(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_4, x_2 + x_5, x_3 + \beta, x_4 + \alpha, x_5)$$

and

$$\tilde{\varphi}_2(x_1, x_2, x_3, x_4, x_5) = (x_1 + \gamma, x_2 + x_4, x_3 + x_5, x_4, x_5 + \alpha).$$

If α is a Diophantine number and α, β and γ are linearly independent over the rational numbers then the action satisfies the hypothesis of the Theorem 1.

2 On affine minimal $\mathbb{Z}^p$-actions on T^q

Let φ be a $\mathbb{Z}^p$-action on T^q. Since φ_* acts by automorphisms of $\mathbb{Z}^q$ it induces a $\mathbb{Z}^p$- action φ_0 by automorphisms of T^q referred to as the *linear part of the action* φ. The action φ can be written uniquely as $\varphi(\ell) = \tau(\ell)\varphi_0(\ell)$, $\ell \in \mathbb{Z}^p$ where $\tau : \mathbb{Z}^p \longrightarrow T^q$ is a cocycle over φ_0 i.e. $\tau(\ell + \ell') = \tau(\ell)\varphi_0(\ell)\tau(\ell')$ for all $\ell, \ell' \in \mathbb{Z}^p$. We call τ is the *translation cocycle* of φ. A lifting $\alpha : \mathbb{Z}^p \longrightarrow \mathbb{R}^q$, on the covering $\mathbb{R}^q$ $\pi \circ \alpha = \tau$ of τ is not in general a cocycle over the action φ_* but $\alpha(\ell + \ell') = \alpha(\ell) + \varphi_*(\ell)\alpha(\ell') + k(\ell, \ell')$, $k(\ell, \ell') \in \mathbb{Z}^q$. The action φ can be lift to a $\mathbb{Z}^p$-action $\tilde{\varphi} = \varphi_* + \alpha$ if and only if the affine $\mathbb{Z}^p$-action φ is isotopic to its linear part φ_0.

Proposition 2.1. *If an affine $\mathbb{Z}^p$-action φ on T^q is minimal then $\sigma(\varphi_*) = \{1\}$.*

Proof. We first show that $1 \in \sigma(\varphi_*(\ell))$ for all $\ell \in \mathbb{Z}^p$. In fact, if $1 \notin \sigma(\varphi_*(\ell_0)$ for some $\ell_0 \in \mathbb{Z}^p$ then $\varphi(l_0)$ has a fixed point $x_0 = \pi(\tilde{x}_0)$ where $\tilde{x}_0 = (id - \varphi_*(\ell_0))^{-1}\alpha(\ell_0)$, $\alpha : \mathbb{Z}^p \longrightarrow \mathbb{R}^q$ being any lifting of the translation cocycle of φ and $\pi : \mathbb{R}^q \longrightarrow T^q$ be the covering map. So the closure of the φ-orbit of x_0 is left fixed by $\varphi(\ell_0)$ and as φ is minimal then $\varphi(\ell_0)$ is the identity map of T^q, giving a contradiction.

Assume now there exist $\ell_0 \in \mathbb{Z}^p$ such that $\sigma(\varphi_*(\ell_0)) \neq \{1\}$. Let $p(x) = (x-1)^m q(x)$ be the primary decomposition of the minimal polynomial of $\varphi_*(\ell_0)$ over $\mathbb{Q}[x]$.

Thus there is decomposition $\mathbb{R}^q = E_1 \oplus E_2$, where $E_1 = \ker(id - \varphi_*(\ell_0))^m$ and $E_2 = \ker q(\varphi_*(\ell_0))$ are invariant by the action φ_*, and E_1 has a basis in $\mathbb{Z}^p$. Decompose a lifting of the translation cocycle of φ as $\alpha(\ell) = \alpha_1(\ell) + \alpha_2(\ell),\ \ell \in \mathbb{Z}^p$. Now it is easy to verify that the translation $\tilde{T}_\zeta = id + \zeta, \zeta = (id - \varphi_*(\ell_0))^{-1}\alpha_1(\ell_0)$ conjugate φ with the action φ_1 given on the covering $\mathbb{R}^q$by $\tilde{\varphi}_1(\ell) = \varphi_*(\ell) + \alpha_1(\ell) + r(\ell),\ r(\ell) \in E_2 \cap \mathbb{Q}^q$, $r(\ell_0) = 0$, i.e. $\varphi_1(\ell) = T_\sigma^{-1} \circ \varphi(\ell) \circ T_\sigma,\ \ell \in \mathbb{Z}^p$ and φ_1 is not minimal since the orbit of $\pi(0)$ lies in finitely many tori of dimension less than q if degree of $q(x) > 1$. □

For $p = 1$ the above proposition is a result of F.J. Hahn [5].

There is an algebraic characterization of minimal actions.

Let $\Gamma \subset \mathbb{Z}^q$ be the set of fixed points of φ^*, i.e.,

$$\Gamma = \left\{k \in \mathbb{Z}^q \mid \varphi^*(\ell)k = k,\ \text{for all } \ell \in \mathbb{Z}^p\right\}.$$

We say that φ *satisfies the irrationality condition on* Γ if for each $k \in \Gamma - \{0\}$ there exist $\ell \in \mathbb{Z}^p$ such that

$$\langle k, \alpha(\ell)\rangle \notin \mathbb{Z}$$

where $\tilde{\varphi}(\ell) = \varphi_*(\ell) + \alpha(\ell)$ is any lift of $\varphi(\ell)$ to the covering $\mathbb{R}^q$.

We prove

Theorem 2. *Let φ be an affine $\mathbb{Z}^p$-action on T^q with $\sigma(\varphi_*) = \{1\}$. The following data are equivalent*

1. *φ satisfies the irrationality condition on Γ.*
2. *φ is ergodic with respect to the Haar measure on T^q.*
3. *φ is uniquely ergodic.*
4. *φ has a dense orbit.*
5. *φ is a minimal action.*

Proof. **1.** $\Rightarrow$ **2.** It is sufficient to show that any function $f \in L^2(T^q, m)$ such that

$$f \circ \varphi(\ell) = f \text{ a.e.}$$

for all $\ell \in \mathbb{Z}^p$ is constant a.e. In fact, the Fourier coefficients of f satisfy

$$\widehat{f \circ \varphi(\ell)}(k) = e^{2\pi i \langle k, (\varphi_*(\ell))^{-1}\alpha(\ell)\rangle} \widehat{f}_{(\varphi^*(\ell))^{-1}k} = \widehat{f(k)} \tag{1}$$

If $k \in \Gamma - \{0\}$, then from (1) we obtain that $\widehat{f_k} = e^{2\pi i \langle k, \alpha(\ell)\rangle} \widehat{f_k}$ and $e^{2\pi i \langle k, \alpha(\ell)\rangle} \neq 1$ since φ satisfies the irrationality condition on Γ. Thus $\widehat{f_k} = 0$, for all $k \in \Gamma - \{0\}$.

If $k \notin \Gamma - \{0\}$ there exist $\ell_0 \in \mathbb{Z}^p$ such that $\varphi^*(\ell_0)^{-1}k \neq k$. As $\sigma(\varphi^*(\ell_0)) = \{1\}$, the sequence $\{\varphi^*(n\ell_0)k\}$ is infinite. For each $n \in \mathbb{Z}$ we take $\ell = n\ell_0$ in (1), as $\lim_{n\to\pm\infty} |\varphi^*(n\ell_0)k| = \lim_{n\to\pm\infty} |\varphi^*(\ell_0)^n k| = \infty$. By The Riemann - Lebesgue Theorem gives $\widehat{f_k} = 0$.

2. $\Rightarrow$ **4.** It follows from the fact that φ is ergodic with respect to the Haar measure on T^q.

4. $\Rightarrow$ **5.** If $q = 1$, a dense orbit trivially implies minimality because φ acts by rotations of the circle. Let us assume that the result is true for $q = n - 1$. We show it is true for $q = n$. Since $\sigma(\varphi_*) = \{1\}$, we may assume that the matrices $\varphi_*(\ell)$ are lower triangular and thus have

$$\varphi(\ell)(z_1, z_2) = (\varphi_1(\ell)(z_1), \beta(\ell)z_2 B(\ell)(z_1)) \tag{2}$$

where $z_1 \in T^{n-1}$, $z_2 \in T^1$, $\beta(\ell) \in T^1$ for all $\ell \in \mathbb{Z}^p$, φ_1 is an action of $\mathbb{Z}^p$ on the affine group of T^{n-1}, and $B(\ell)$ is an homomorphism of T^{n-1} in T^1. Observe that if the φ-orbit of (z_1^0, z_2^0) by φ is dense in T^n, then the φ_1-orbit of z_1^0 is dense in T^{n-1}, so by the induction assumption φ_1 is minimal. Therefore for each $z = (z_1, z_2) \in T^{n-1} \times T^1$ we have

$$\Lambda(z) = \overline{\{\varphi(\ell)(z) \mid \ell \in \mathbb{Z}^p\}} \cap (\{z_1^0\} \times T^1) \neq \phi.$$

As the orbit of φ at every point of $\{z_1^0\} \times T^1$ is dense in T^n, then $\Lambda(z) = T^n$.

3. $\Rightarrow$ **5.** This follows from the fact that the only invariant measure is the Haar measure.

2. $\Rightarrow$ **3.** If $q = 1$, then φ acts as rotations of the circle, thenφ is uniquely ergodic. Let us assume that it is true for $q = n - 1$. We show that it is true for $q = n$.

From (2) we see that if φ is ergodic then φ_1 is ergodic and by the induction hypothesis φ_1 is uniquely ergodic,therefore φ is uniquely ergodic [21].

5. $\Rightarrow$ **1.** If φ does not satisfies the irrationality condition on Γ, then there exist k_0 so that $\langle k_0, \alpha(\ell)\rangle \in \mathbb{Z}$ for all $\ell \in \mathbb{Z}^p$ and the function $\cos(2\pi\langle k_0, x\rangle)$ is invariant by φ therefore φ is not minimal. $\square$

Problem 1. Let A a homomorphism of $\mathbb{Z}^p$ into $GL(q,\mathbb{Z})$ with spectrum 1. Give a necessary and sufficient condition for the existence of a minimal affine $\mathbb{Z}^p$ -action on T^q whose induced action on the first homology group of T^q is precisely A.

3 The first cohomology group of affine $\mathbb{Z}^p$-actions on tori

A function $f : \mathbb{Z}^p \longrightarrow C^\infty(T^q)$ is a 1-*cocycle* of an affine $\mathbb{Z}^p$- action φ on T^q if $f(\ell+\ell') = f(\ell)+f(\ell')\circ\varphi(\ell)$, for all ℓ and ℓ' in $\mathbb{Z}^p$. The set $Z^1(\mathbb{Z}^p, C^\infty_\varphi(T^q))$ of all 1-cocycles $f : \mathbb{Z}^p \longrightarrow C^\infty(T^q)$ of an affine $\mathbb{Z}^p$-action φ on $T^q$is an abelian group. A 1-cocycle f is *trivial* or a 1-*coboundary* if $f(\ell) = h - h \circ \varphi(\ell)$ for some $h \in C^\infty(T^q)$ and all $\ell \in \mathbb{Z}^p$. The set $B^1(\mathbb{Z}^p, C^\infty_\varphi(T^q))$ of all 1-coboundaries of φ is a subgroup of $Z^1(\mathbb{Z}^p, C^\infty_\varphi(T^q))$ and $H^1(\mathbb{Z}^p, C^\infty_\varphi(T^q)) = Z^1(\mathbb{Z}^p, C^\infty(T^q))/B^1(\mathbb{Z}^p, C^\infty(T^q))$ is the *first cohomology group of* φ.

Since the Haar measure m is invariant by φ then to each 1-cocycle f of φ there corresponds a homomorphism $m(f) : \mathbb{Z}^p \longrightarrow \mathbb{R}$, $m(f)(\ell) = \int_{T^q} f(\ell)dm$, $\ell \in \mathbb{Z}^p$. Thus we have the decomposition

$$Z^1(\mathbb{Z}^p, C^\infty_\varphi(T^q)) = Hom(\mathbb{Z}^p,\mathbb{R}) \oplus \ker m \tag{3}$$

where $\ker m = \{f \in Z^1(\mathbb{Z}^p, C^\infty_\varphi(T^q)) \mid m(f)(\ell) = 0 \text{ for all } \ell \in \mathbb{Z}^p\}$.

We now prove the Theorem 1.

Proof. The implication 2 ⇒ 3 is trivial. The main implication 1 ⇒ 2 is proved in Section 5; we now prove the implication 3 ⇒ 1.

Suppose that φ does not verify any Diophantine condition on the fixed points of φ^*. We show that the first cohomology group of φ is non-Hausdorff. In fact there exist j, $1 \le j \le p$ and a sequence $\{k_s\}_{s\ge1}$ in Γ such that $|k_s| \to \infty$ as $s \to \infty$

$$0 < \|\langle k_s, \alpha(e_j)\rangle\| = \max_{1\le i\le p} \|\langle k_s, \alpha(e_i)\rangle\| < |k_s|^{-s}. \tag{4}$$

We construct a non-trivial cocycle sequence $f : \mathbb{Z}^p \longrightarrow C^\infty(T^q)$ of φ where as before $\tilde{\varphi}(\ell) = \varphi_*(\ell) + \alpha(\ell)$ is a lifting of $\varphi(\ell)$ to the covering $\mathbb{R}^q$ which is the limit in the C^∞ topology of coboundaries. For each $\ell \in \mathbb{Z}^p - \{0\}$ we consider the function $f(\ell)$ given by the Fourier series

$$f(\ell) = \sum_{s=1}^{\infty} \left[\left(1 - e^{-2\pi i\langle k_s,\alpha(\ell)\rangle}\right)\phi_{-k_s} + \left(1 - e^{2\pi i <k_s,\alpha(\ell)\rangle}\right)\phi_{k_s} \right]$$

where as usual $\phi_k = e^{2\pi i\langle k,x\rangle}$.

From (4) we get

$$0 < ||\langle k_s, \alpha(\ell)\rangle|| \leq |\ell| \cdot ||\langle k_s, \alpha(e_j)\rangle|| \leq |\ell| \cdot |k_s|^{-s} \tag{5}$$

which shows that the above Fourier series converges in the C^∞ topology to a C^∞ function $f(\ell) : T^q \longrightarrow \mathbb{R}$.

From (5) we see that $f : \mathbb{Z}^p \longrightarrow C^\infty(T^q)$, $\ell \mapsto f(\ell)$ is a cocycle since the sequence $\{k_s\}$ consists of fixed points of the action φ^*. The partial sums $S_n(\ell)$ of the Fourier series of $f(\ell)$ also define cocycles $S_n(\ell) : \mathbb{Z}^p \longrightarrow C^\infty(T^q)$ which are clearly coboundaries. Now we show that f is not a coboundary. Suppose that $f(\ell) = h - h \circ \varphi(\ell)$ for some C^∞ function h and all $\ell \in \mathbb{Z}^p$.

Thus for $s \geq 1$ we have

$$\widehat{f(\ell)}(k_s) = (1 - e^{2\pi i \langle k_s, \alpha(\ell)\rangle})\widehat{h}(k_s)$$

and $\widehat{h}(k_s) = 1$, contradicting the Riemann-Lebesgue theorem. □

The distributions on T^q which are invariant by φ play an important role on the computation of the cohomology of φ, and are discussed in the next section.

4 Invariant distributions and the first cohomology group

Let φ be an affine $\mathbb{Z}^p$- action on T^q and I be an isotropy group of the induced action φ^*. Choose a subgroup H of $\mathbb{Z}^p$ so that $I \cap H = \{0\}$. Fix $k \in \mathbb{Z}^q$ so that $I(k) = I$. Since $\varphi^*(\ell)k \neq k$ for all $\ell \in H$ and the Fourier series of any $h \in C^\infty(T^q)$ is absolutely convergent, see [14], the series

$$\sum_{\ell \in H} (\widehat{h \circ \varphi(\ell)})(k)$$

is also absolutely convergent. Thus

$$\rho_k^H(h) = \sum_{\ell \in H} (\widehat{h \circ \varphi(\ell)})(k), \ h \in C^\infty(T^q) \tag{6}$$

defines a distribution ρ_k^H invariant by the restriction of φ to H i.e.

$$\rho_k^H(h \circ \varphi(\ell)) = \rho_k^H(h), \text{ for each } h \in C^\infty(T^q) \text{ and for all } \ell \in H.$$

Moreover a routine computation shows that

$$\rho_k^H(h \circ \varphi(\ell)) = e^{2\pi i \langle \varphi^*(-\ell)k, \alpha(\ell)\rangle} \rho_{\varphi^*(-\ell)k}^H(h), \quad \text{for all } \ell \in \mathbb{Z}^p. \tag{7}$$

Thus if $\ell \in I$, then $\rho_k^H(h - h \circ \varphi(\ell)) = (1 - e^{2\pi i \langle k, \alpha(\ell)\rangle})\rho_k^H(h)$.

We denote by ρ_k^ℓ the distribution corresponding to the cyclic group H generated by $\ell \in \mathbb{Z}^p - I$. This distribution is invariant by $\varphi(\ell)$.

Proposition 4.1. *Let φ be an affine $\mathbb{Z}^p$-action on T^q with $\sigma(\varphi_*) = \{1\}$. If the first cohomology group of φ is finite dimensional, then φ satisfies the irrationality condition.*

Proof. Suppose that φ does not satisfies the irrationality condition, then there exist $k \in \mathbb{Z}^q - \{0\}$ so that $dim I(k) \geq p - 1$ and

$$\langle k, \alpha(\ell) \rangle \in \mathbb{Z}, \text{ for all } \ell \in I(k). \tag{8}$$

We show that the first cohomology of φ is infinite dimensional. We first notice that $I(nk) = I(k)$ and $\langle nk, \alpha(\ell) \rangle \in \mathbb{Z}$ for all $n \in \mathbb{Z}$, $n \neq 0$.

We consider two cases.

Case 1. $I(k) = \mathbb{Z}^p$.

We construct an infinite sequence of cocycles f_n, $n \in \mathbb{Z} - \{0\}$ giving linearly independent cohomology classes in the first cohomology group of φ. Since $\langle nk, \varphi(\ell)x \rangle - \langle nk, x \rangle$ is integer for all $\ell \in \mathbb{Z}^p$ and each n in $\mathbb{Z}$ then $f_n(\ell) = \ell_1 \cos 2\pi \langle nk, \cdot \rangle$, $\ell = (\ell_1, \ldots, \ell_p) \in \mathbb{Z}^p$ defines a cocycle for each $n \in \mathbb{Z}$. Moreover for any smooth function $h : T^q \longrightarrow \mathbb{R}$ we have that

$$(\widehat{h \circ \varphi(\ell)})(nk) = \widehat{h}(nk), \text{ for all } \ell \in \mathbb{Z}^p \text{ and all } n \in \mathbb{Z}.$$

From this we see easily that the cocycles f_n give an infinite linearly independent sequence of cohomology classes.

Case 2. $\dim I(k) = p - 1$.

In this case there is, $1 \leq i \leq p$ so that $e_i \notin I(k)$. Thus $\mathbb{Z}^p = I(k) \oplus H$ where H is the subgroup generated by e_i. Let $\rho_{nk}^{e_i}$ as in (6) be the distributions invariant by $\varphi(e_i)$. Now we choose functions $h_n \in C^\infty(T^q)$, such that $\widehat{h_n}(k) = 0$ if k does not belong to the orbit $\varphi^*(je_i)nk$, $j \in \mathbb{Z}$ of nk for each $n \in \mathbb{Z} - \{0\}$. We now define an infinite sequence of cocycles f_n of φ as follows $f_n(je_i) = h_n + h_n \circ \varphi(e_i) + \cdots + h_n \circ \varphi((j-1)e_i)$, for $j \geq 0$ and $f_n(je_i) = -(h_n + h_n \circ \varphi(-e_i) + \cdots + h_n \circ \varphi(-(j-1)e_i))$, for $j < 0$ for all $\ell \in I(k)$. Thus $f_n(je_i) \circ \varphi(k) = f_n(je_i)$. We see from (8) that $\langle \varphi^*(\ell)nk, \alpha(\ell) \rangle = n\langle k, \alpha(\ell) \rangle$ is an integer. The mappings $f_n : \mathbb{Z}^p \longrightarrow C^\infty(T^q)$ given by $f_n(\ell) = f_n(P(\ell))$ where P is the projection onto H, $\mathbb{Z}^p = I(k) \oplus H$ are easily seen to be linearly independent cohomology classes in the first cohomology of φ. □

Let φ be an affine $\mathbb{Z}^p$-action on the torus T^q. For each isotropy group of φ^* we consider the submodule $M_I = \{f \in C^\infty(T^q) | \widehat{f}(k) = 0 \text{ for all } k,\ I(k) \neq I\}$ of $C^\infty_\varphi(T^q)$. For each $f \in C^\infty(T^q)$, f_I denotes the projection of f onto M_I.

Lemma 4.2. *Let φ be an affine $\mathbb{Z}^p$-action on the torus T^q and $I \neq \mathbb{Z}^p$ be an isotropy group of the action φ^*. Suppose that for some $\ell \notin I$ and $f \in C^\infty(T^q)$ we have the equations*

$$\rho_k^\ell(f) = 0 \textit{ for all } k \in \mathbb{Z}^q, \textit{ satisfying } I(k) = I. \tag{9}$$

Then there is a function $h_I \in M_I$ so that

$$h_I - h_I \circ \varphi(\ell) = f_I.$$

Moreover for each $r \geq 1$ the Fourier coefficients of h_I satisfy the inequality

$$|\widehat{h_I}(k)| \leq C(r)|k|^{-r} \textit{ for all } k, \textit{ such that } I(k) = I$$

where the constant $C(r)$ dependent only on r, f and ℓ.

Proof. Consider the number given by (9) we have

$$\widehat{h}(k) = \sum_{n\geq 0} (f \widehat{\circ \varphi(n\ell)})(k) = -\sum_{n\geq 1} (f \widehat{\circ \varphi(-n\ell)})(k) \tag{10}$$

for all $k \in \mathbb{Z}^q$, $I(k) = I$.

Now we show that the Fourier series $\sum_{k\in\mathbb{Z}^q, I(k)=I} \widehat{h}(k)\phi_k$ defines a function in M_I i.e., for each $m > 0$ there exist $C(m) > 0$ such that $|\widehat{h}(k)| \cdot |k|^m \leq C(m)$, for all $k \in \mathbb{Z}^q - \{0\}$, $I(k) = I$. For each $k \in \mathbb{Z}^q - \{0\}$, $I(k) = I$ there exist $k_0 \in \mathbb{Z}^q$ in the orbit of k by $\varphi^*(\ell)$ such that

$$|\varphi^*(n\ell)k_0| \geq |k_0|, \text{ for all } n \in \mathbb{Z}.$$

By Lemma A in the appendix there is a constant $E > 0$ so that $|\varphi^*(n\ell)k_0| \geq E|n|$ for all $n \in \mathbb{Z}$. Consider $n_0 \in \mathbb{Z}$ such that $k = \varphi^*(n_0\ell)k_0$. Since $f \in C^\infty(T^q)$ then for each $s \geq 0$ there exist $C'(s) > 0$ so that $|\widehat{f}(k)| \cdot |k|^s \leq C'(s)$. There two possibilities.

If $n_0 \geq 0$ we choose in (10) the equation

$$\widehat{h}(k) = -\sum_{n\geq 1} (f \widehat{\circ \varphi(-n\ell)})(k) = \sum_{n\geq 1} e^{2\pi i\langle k, \alpha(n\ell)\rangle} \widehat{f}(\varphi^*(n\ell)k).$$

Taking $s = m + r$, where r will be chosen later, we have that

$$|\widehat{h}(k)| \cdot |k|^m \leq C'(m+r) \sum_{n\geq 1} \frac{|k|^m}{|(\varphi^*(n\ell))k|^{m+r}} = C'(m+r) \sum_{n\geq 1} \frac{|\varphi^*(n_0\ell)k_0|^m}{|\varphi^*((n+n_0)\ell)k_0|^{m+r}}.$$

As $\sigma(\varphi^*(\ell)) = \{1\}$ then $N(\ell) = \varphi^*(\ell) - id$ is a nilpotent matrix, thus $N(\ell)^d = 0$, for some $1 \leq d \leq q$. Thus there is $C_0 > 0$ so that for every $n_0 \in \mathbb{Z}$

$$|\varphi^*(n_0\ell)| \leq C_0 \max(1, |n_0|^{d-1}).$$

Hence $|\varphi^*(n_0\ell)k_0| \leq C_0 \max(1, |n_0|^{d-1})|k_0|$ and

$$\begin{aligned} |\widehat{h}(k)| \cdot |k|^m &\leq C'(m+r)C_0^m \sum_{n\geq 1} \tfrac{\max(1,|n_0|^{d-1})^m |k_0|^m}{|\varphi^*((n+n_0)\ell)k_0|^{m+r}} \\ &\leq C'(m+r)C_0^m E^{-1} \sum_{n\geq 1} \tfrac{\max(1,|n_0|^{d-1})^m}{(n+n_0)^r}. \end{aligned}$$

Choosing $r = m(d-1) + 1 + \varepsilon$, for some $\varepsilon > 0$ we have that

$$\frac{n_0^{(d-1)m}}{(n+n_0)^r} \leq \frac{1}{(n+n_0)^{1+\varepsilon}}.$$

Thus $|\widehat{h}(k)| \cdot |k|^m \leq C(m)$, where

$$C(m) = C'(md + 1 + \varepsilon)C_0^m E^{-1} \sum_{n\geq 1} \frac{1}{n^{1+\varepsilon}}$$

depends on ℓ but not on I.

The case $n_0 < 0$ follows in a similar way considering that

$$\widehat{h}(k) = \sum_{n\geq 0} \widehat{f \circ \varphi(n\ell)}(k) = \sum_{n\geq 0} e^{2\pi i \langle k, \alpha(-n\ell)\rangle} \widehat{f}_{\varphi^*(-n\ell)k}.$$

From (10) we easily that $h_I - h_I \circ \varphi(\ell) = f_I$. □

5 Computing the first cohomology group

Lemma 5.1. *Let $\varphi : \mathbb{Z}^p \longrightarrow$ Affine(T^q) be a minimal action and I be an isotropy group of φ^*. Let $\ell \notin I$ and $h \in L^1(T^q, \mathbb{R})$. If $\partial^\ell h = h - h \circ \varphi(\ell) = 0$ then $\widehat{h}(k) = 0$ for all $k \in \mathbb{Z}^q$ such that $I(k) = I$.*

Proof. The equation $\partial^\ell h = 0$ implies that $h - h \circ \varphi(n\ell) = 0$ for all $n \in \mathbb{Z}$. Thus

$$\widehat{h}(k) = e^{2\pi i \langle \varphi^*(-n\ell)k, \alpha(n\ell)\rangle} \widehat{h}_{\varphi^*(-n\ell)k}.$$

then since $\ell \notin I$ $\lim_{n\to\infty} |\varphi^*(-n\ell)k| = \infty$ and since $h \in L^1(T^q, \mathbb{R})$ we conclude that $\widehat{h}(k) = 0$. □

Let $f \in Z^1(\mathbb{Z}^p, \ker m)$ be an 1-cocycle. We prove that for each isotropy group I of φ^* the cocycle $f_I : \mathbb{Z}^p \longrightarrow M_I$, $f_I(\ell) = f(\ell)_I$ as in Lemma 4.2 is trivial, i.e., there is $h_I \in M_I$ so that

$$f_I(\ell) = h_I - h_I \circ \varphi(\ell), \text{ for all } \ell \in \mathbb{Z}^p$$

where $f(\ell)_I$ denotes the projection of $f(\ell) \in \ker m$ onto M_I. Moreover we prove that for all $r \geq 1$ there is $C(r) > 0$ independent of I such that

$$|\widehat{(h_I)}_k| \leq C(r)|k|^{-r}.$$

So $h = \sum_{\mathcal{I}} h_I$ defines a function in M_0 such that for all $\ell \in \mathbb{Z}^p$

$$h - h \circ \varphi(\ell) = f(\ell).$$

This proves that $f \in Z^1(\mathbb{Z}^p, \ker m)$ is a coboundary and $H^1(\mathbb{Z}^p, \ker m) = 0$.

Proof of the Theorem 1. It remains to prove the implication $1 \Rightarrow 2$. Let I be an isotropy group and let $\{e_1, \ldots, e_p\}$ be the canonical base of $\mathbb{Z}^p$. Let f be a 1-cocycle.

Case 1. $I = \mathbb{Z}^p$. For $k \in \Gamma$ exist ℓ' such that $e^{2\pi i \langle k, \alpha(\ell')\rangle} \neq 1$. Define

$$\widehat{h}(k) = \frac{\widehat{f(\ell')}(k)}{1 - e^{2\pi i \langle k, \alpha(\ell')\rangle}}.$$

From the cocycle equation

$$f(\ell + \ell') = f(\ell) + f(\ell') \circ \varphi(\ell) = f(\ell') + f(\ell) \circ \varphi(\ell')$$

we get

$$f(\ell') - f(\ell') \circ \varphi(\ell) = f(\ell) - f(\ell) \circ \varphi(\ell') \text{ for all } \ell \in \mathbb{Z}^p$$

thus

$$\widehat{f(\ell')}(k)(1 - e^{2\pi i \langle k, \alpha(\ell)\rangle}) = \widehat{f(\ell)}(k)(1 - e^{2\pi i \langle k, \alpha(\ell')\rangle})$$

and we have $\widehat{h}(k)(1 - e^{2\pi i \langle k, \alpha(\ell)\rangle}) = \widehat{f(\ell)}(k)$.

By the Diophantine condition there exist $\beta > 0$ and $C > 0$ such that for each of $k \in \Gamma$, there is j, $1 \leq j \leq p$ so that

$$\|\langle k, \alpha(e_j)\rangle\| \geq C|k|^{-\beta}.$$

and for all $r \geq 1$ there exist $C(r) > 0$ such that $\|\langle k, \alpha(e_j)\rangle\| \geq \frac{C(r)}{|k|^r} > 0$.

Thus $\widehat{h}(k) = \dfrac{\widehat{f(e_j)}(k)}{1 - e^{2\pi i \langle k, \alpha(e_j)\rangle}}$ and therefore

$$|\widehat{h}(k)| = \frac{|\widehat{f(e_j)}(k)|}{|1 - e^{2\pi i \langle k, \alpha(e_j)\rangle}|} \leq \frac{C(r)C^{-1}}{|k|^{r-\beta}}.$$

Thus the function $h_I = \sum_\Gamma \widehat{h}(k)\phi(k) \in M_I$ satisfies $h_I - h_I \circ \varphi(\ell) = f(\ell)_I$, for all $\ell \in I = \mathbb{Z}^p$, where $f(\ell)_I \in M_I$ is the projection of $f(\ell) \in M_0$ onto M_I.

Case 2. $\dim I = p - 1$. Let $1 \leq i \leq p$ such that $e_i \notin I$. By the, irrationality condition, for each $k \in \mathbb{Z}^q - \{0\}$ such that $I(k) = I$ there is $\ell' \in I$ so that $e^{2\pi i \langle k, \alpha(\ell')\rangle} \neq 1$. From the cocycle equation

$$f(\ell') - f(\ell') \circ \varphi(e_i) = f(e_i) - f(e_i) \circ \varphi(\ell')$$

we get $\rho_k^{e_i}(\partial^{\ell'} f(e_i)) = \rho_k^{\ell'}(\partial^{e_i} f(\ell')) = 0$. From (7) and the irrationality condition we see that $\rho_k^{e_i}(\partial^{\ell'} f(e_i)) = 0$. By Lemma 4.2 there is $h_I \in M_I$ such that $f(e_i)_I = h_I - h_I \circ \varphi(e_i)$.

Projecting the cocycle equation onto M_I we get

$$\begin{aligned} f_I(\ell) - f_I(\ell) \circ \varphi(e_i) &= f(e_i)_I - f(e_i)_I \circ \varphi(\ell) = \\ &= h_I - h_I \circ \varphi(\ell) - (h_I - h_I \circ \varphi(\ell)) \circ \varphi(e_i) \end{aligned}$$

Thus by Lemma 5.1 we see that $f(\ell)_I = h_I - h_I \circ \varphi(\ell)$ for all $\ell \in \mathbb{Z}^p$.

Note that Lemma 4.2 implies that for every $r \geq 1$ there is $C(r) > 0$, only depending in $f(e_1), \ldots, f(e_p)$ and independent of I, such that

$$|\widehat{h_I}(k)| < C(r)|k|^{-r}.$$

Case 3. $\dim I < p - 1$. There are e_i and e_j so that $H \cap I = \{0\}$, where H is subgroup generated by e_i and e_j. Thus for each $k \in \mathbb{Z}^q$, $I(k) = I$ we have that

$$\varphi^*(\ell)k \neq k \text{ for all } \ell \in H. \tag{11}$$

From the cocycle equation

$$f(e_j) - f(e_j) \circ \varphi(e_i) = f(e_i) - f(e_i) \circ \varphi(e_j)$$

a routine computation shows that

$$\rho_k^{e_i}(f(e_i)) = \rho_k^{e_i}(f(e_i) \circ \varphi(ne_j)) \text{ for all } n \in \mathbb{Z}$$

from (7) we get

$$\rho_k^{e_i}(f(e_i)) = e^{2\pi i\langle\varphi^*(-ne_j)k,\alpha(ne_j)\rangle}\rho^{e_i}_{\varphi^*(-ne_j)k}(f(e_i)) \text{ for all } n\in\mathbb{Z}. \tag{12}$$

Since the series in (6) is absolutely convergent from (11) we see that

$$\lim_{n\to\infty}\rho^{e_i}_{\varphi^*(-ne_j)k}(f(e_i)) = 0.$$

Now from (12) we get $\rho_k^{e_i}(f(e_i)) = 0$ and we proceed as in the Case 2 above.

In both case 2 and 3 Lemma 4.2 implies that for every $r \geq 1$ there is $C(r) > 0$, only depending on $f(e_1), \dots, f(e_p)$ and independent of I, such that

$$|\widehat{h_I}(k)| < C(r)|k|^{-r}. \qquad \square$$

Appendix

Lemma A. *Let $A \in GL(q,\mathbb{Z})$ such that $\sigma(A) = \{1\}$. Then there is $C > 0$ such that for all $n \in \mathbb{Z}$ and all $k \in \mathbb{Z}^q$ satisfying $Ak \neq k$, either $|A^nk| \geq C|n|$ or $|k| \geq C|n|$.*

Proof. By Jordan canonical form theorem there is a $P \in M(q,\mathbb{Z})$ such that PAP^{-1} is in upper triangular Jordan form, which means there is a decomposition of PAP^{-1} into a direct sums of Jordan blocks. It is enough to prove the lemma for each Jordan block.

If J is a $m\times m$ Jordan block then there is $\epsilon(m)$ so that

$$|\frac{1}{j!}(1-\frac{1}{n})\cdots(1-\frac{j-1}{n})| \geq \epsilon(m) \text{ for all } n\in\mathbb{Z}$$

and j, $1 \leq j \leq m-1$ such that $n > j$ or equivalently

$$|\binom{n}{j}| \geq \epsilon(m)|n|^j \text{ for all } n\in\mathbb{Z} \text{ and } 1\leq j\leq m-1 \text{ such that } n > j. \tag{13}$$

Take $C = \frac{\epsilon}{m+1}$, suppose that $|k| \leq C|n|$ for some $k \in \mathbb{Z}^q - \{0\}$ such that $Jk \neq k$ and $n \in \mathbb{Z} - \{0\}$, we show that $|J^nk| \geq C|n|$. In fact, the first coordinate of J^nk is equal to

$$k_1 + nk_2 + \cdots + \binom{n}{j}k_j + \cdots + \binom{n}{m-1}k_m \text{ where } k = (k_1,\dots,k_m).$$

Let j, $0 \leq j \leq m-1$ be the largest integer so that $\binom{n}{j}k_j \neq 0$. Since $Jk \neq k$ then $j > 0$.

Note that,

$$|k_1 + \cdots + \binom{n}{j-1} k_{j-1}| < m|n|^{j-1} \frac{\varepsilon}{m+1}|n| = \frac{m}{m+1}\varepsilon|n|^j.$$

Now from the inequality (13) we get

$$|k_1 + k_2 + \cdots + \binom{n}{j} k_j| \geq |\binom{n}{j} k_j| - \frac{m}{m+1}\varepsilon|n|^j \geq \frac{\varepsilon}{m+1}|n|.$$

Thus $|J^n k| \geq \frac{\varepsilon}{m+1}|n|$ and the lemma follows for J with $C = \frac{\varepsilon(m)}{m+1}$. □

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References

[1] Arraut, J.L. and dos Santos, N.M. Linear foliations of T^n, Bol. Soc. Brasil Mat. **21** (1991), 189–204.

[2] Atiyah, M. F. and Wall, C.T.C. Cohomology of groups, in Algebraic number theory, ed. J.W.S. Cassels and A. Fröhlich. Acad. Press (1967), 95–115.

[3] Furstenberg, H. Strict ergodic and transformations of torus. American J. of Math., **83** (1961), 573–601.

[4] Haefliger, A. Some remarks on foliations with minimal leaves. J. Differ. Geom. **15** (1980), 269–284.

[5] Hahn, F.J. On affine transformations of compact abelian groups. American J. of Math., **85** (1963), 428–446.

[6] Herman, M. R. Sur la conjugaison différentiable des différentiable de cercle á des rotations. Publ. IHES **49** (1979), 5–233.

[7] Hurder, S. Problems in Rigidity of Groups Actions and Cocycles, Ergod. Th. and Dyn. Sys. **5** (1985), 473–484.

[8] Katok, A. and Nitica, V. and A. Torok, A. Nonabelian cohomology of abelian Anosov actions, Ergod. Th. and Dyn. Sys. **20** (2000), 259–288.

[9] Katok, A. and Schmidt, K. The cohomology of expansive $\mathbb{Z}^d$–actions by automorphisms of abelian groups, Pacif J. Math. **170** (1995), 105–142.

[10] Katok, A. and Spazier, R.J. First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity. Pub. IHES **79** (1994), 131–156.

[11] Katok, A. and Spazier, R.J. Differential rigidity of Anosov actions of higher rank abelian groups and algebraic lattice actions, Proc of Steklov Inst. Math. **216** (1997), 292–319.

[12] Katok, A. and Spazier, R.J. Subelliptic estimates of polinomial differential operators and applications to rigidity, Math. Res. Letters **1** (1994), 193–202.

[13] Kolmogorov, A.N. On dynamical systems with an integral invariant on the torus. Dokl. Acad. Sci., SSSR **93** (1953), 763–766.

[14] Kolmogorov, A.N. and Fomin, S. V. Introductory real analysis. Prentice-Hall, Inc., Englewood Cliffs, N.Y. 1970.

[15] Luz, R.U. Actions of $\mathbb{Z}^p$ on the Affine Group of T^q, Tese de Doutorado, PUC-Rio (1993).

[16] Luz, R.U. and dos Santos, N.M. Cohomology-free diffeomorphisms of low-dimension tori. Ergod. Th. and Dyn. Sys. **18** (1998), 985–1006.

[17] MacLane, S. Homology, Springer-Verlag, 1963.

[18] Mañe, R. Ergodic theory dynamics and differentiable dynamics, Springer-Verlag.

[19] Matsumoto, S. and Mitsumatsu, Y. Leafwise cohomology and rigidity of certain Lie group actions. To appear in Ergod. Th. and Dyn. Sys.

[20] Moser, J. On commuting circle mappings and simultaneous Diophantine approximations. Math. Z. **205** (1990), 105–121.

[21] Parry, W. Topics in ergodic theory. Cambridge University Press, 1981.

[22] Pereira, M.S. and dos Santos, N.M. On the cohomology of foliated bundles. Proyecciones, Vol. 21 N°2, pp. 173–195 August 2002, Universidad Católica del Norte, Antofagasta-Chile.

[23] Schmidt, K. Cohomological rigidity of algebraic Z^d–actions. Ergod. Th. and Dyn. Sys. **15** (1995), 759–805.

[24] Schmidt, K. On the cohomology of algebraic Z^d–actions with values in compact Lie groups. in: *Lie Groups and Ergodic Theory* (Proceed. of Internat. Colloq. at the Tata Institute, Mumbai 1996), ed. S.G. Dani, Narosa Publishing House, New Delhi, (1998), 203–228.

[25] dos Santos, N.M. Differentiable conjugation and isotopies of actions of $\mathbb{R}^p$. Geometric Study of Foliations ed. by T. Mizutani. World Scientific, Singapore, (1994), 181–191.

[26] dos Santos, N.M. Cohomologically rigid Z^p–action on low-dimension tori. Preprint 2001.

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