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Harmonic mean curvature lines on surfaces immersed in $\mathbb{R}^3$

Ronaldo Garcia and Jorge Sotomayor

Abstract. Consider oriented surfaces immersed in $\mathbb{R}^3$. Associated to them, here are studied pairs of transversal foliations with singularities, defined on the *Elliptic* region, where the Gaussian curvature $\mathcal{K}$, given by the product of the principal curvatures k_1, k_2 is positive. The leaves of the foliations are the *lines of harmonic mean curvature,* also called *characteristic* or *diagonal lines*, along which the normal curvature of the immersion is given by $\mathcal{K}/\mathcal{H}$, where $\mathcal{H} = (k_1 + k_2)/2$ is the arithmetic mean curvature. That is, $\mathcal{K}/\mathcal{H} = ((1/k_1 + 1/k_2)/2)^{-1}$ is the *harmonic mean* of the principal curvatures k_1, k_2 of the immersion. The singularities of the foliations are the *umbilic points* and *parabolic curves*, where $k_1 = k_2$ and $\mathcal{K} = 0$, respectively.
Here are determined the structurally stable patterns of *harmonic mean curvature lines* near the *umbilic points*, *parabolic curves* and *harmonic mean curvature cycles*, the periodic leaves of the foliations. The genericity of these patterns is established.
This provides the three essential local ingredients to establish sufficient conditions, likely to be also necessary, for *Harmonic Mean Curvature Structural Stability* of immersed surfaces. This study, outlined towards the end of the paper, is a natural analog and complement for that carried out previously by the authors for the *Arithmetic Mean Curvature* and the *Asymptotic Structural Stability* of immersed surfaces, [13, 14, 17], and also extended recently to the case of the *Geometric Mean Curvature Configuration* [15].

Keywords: umbilic point, parabolic point, harmonic mean curvature cycle, harmonic mean curvature lines.

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1 Introduction

In this paper are studied the *harmonic mean curvature configurations* associated to immersions of oriented surfaces into $\mathbb{R}^3$. They consist on the *umbilic points* and *parabolic curves*, as singularities, and of the *lines of harmonic mean curvature* of the immersions, as the leaves of the two transversal foliations in the configurations. The normal curvature of the immersion along these lines is given by the *harmonic mean* of the principal curvatures, defined by $\mathcal{K}/\mathcal{H} = ((1/k_1 + 1/k_2)/2)^{-1}$, in terms of the standard curvature functions: *principal curvatures* k_1, k_2, *arithmetic mean curvature* $\mathcal{H} = (k_1 + k_2)/2$ and *Gaussian curvature* $\mathcal{K} = k_1 k_2$.

The two transversal foliations, called here *harmonic mean curvature foliations*, are well defined and regular only on the non-umbilic part of the elliptic region of the immersion, where the Gaussian Curvature is positive. In fact, there they are the integral curves of smooth quadratic differential equations. The set where the Gaussian Curvature vanishes, the parabolic set, is generically a regular curve which is the border of the elliptic region; see [3]. The umbilic points are those at which the principal curvatures coincide, generically are isolated and disjoint from the parabolic curve. See section 2 for precise definitions.

This study is a natural development and extension of previous results about the Arithmetic Mean Curvature and Asymptotic Configurations, dealing with the qualitative properties of the lines along which the normal curvature is the arithmetic mean of the principal curvatures (i.e. is the standard Mean Curvature) or is null. This has been considered previously by the authors; see [13, 17] and [14], and has also been extended recently to the case of the *Geometric Mean Curvature* [15].

The point of departure of this line of research, however, can be found in the classical works of Euler, Monge, Dupin and Darboux, concerned with the lines of principal curvature and umbilic points of immersions. See [9, 31, 32] for an initiation on the basic facts on this subject; see [19, 21] for a discussion of the classical contributions and for their analysis from the point of view of structural stability of differential equations. A modern general presentation of structural stability of dynamical systems can be found in [25].

This paper establishes sufficient conditions, likely to be also necessary, for the structural stability of *harmonic mean curvature configurations* under small perturbations of the immersion. See section 7 for precise statements.

This extends to the harmonic mean curvature setting the main theorems on structural stability for the arithmetic and geometric mean curvature configurations and for the asymptotic configurations, proved in [13, 14, 15, 17].

Three local ingredients are essential for this extension: the umbilic points, endowed with their harmonic mean curvature separatrix structure, the harmonic mean curvature cycles, with the calculation of the derivative of the Poincaré return map, through which is expressed the hyperbolicity condition and the parabolic curve, together with the parabolic tangential singularities and associated separatrix structure.

The conclusions of this paper, on the elliptic region, are complementary to results valid independently on the hyperbolic region (on which the Gaussian curvature is negative), where the separatrix structure near the parabolic curve and the asymptotic structural stability has been studied in [13, 17].

The parallel with the conditions for principal, arithmetic mean curvature and asymptotic structural stability is remarkable. This can be attributed to the unifying role played by the notion of Structural Stability of Differential Equations and Dynamical Systems, coming to Geometry through the seminal work of Andronov and Pontrjagin [1] and Peixoto [28].

The interest on lines of harmonic mean curvature appears in the paper of Raffy [29]; see also Eisenhart [12], section 55. The work of Ogura [27] regards these lines in terms of his unifying notion *T-Systems* and makes a local analysis of the expressions of the fundamental quadratic forms in a chart whose coordinate curves are lines of harmonic mean curvature. A comparative study of these expressions with those corresponding to other lines of geometric interest, such as the *principal, asymptotic, arithmetic* and *geometric mean curvature lines* is carried out by Ogura in the context of *T-Systems*, away from singularities. In the paper of Occhipinti [26] is established the following interesting projective relationship: *a line of harmonic mean curvature divides harmonically those of geometric mean curvature (both) and that (one) of arithmetic mean curvature* . See [4], chapter 6.

For being more descriptive and coherent with that of previous recent papers already cited, we adopt in this work the denomination of *harmonic mean curvature lines* instead of *characteristic or diagonal lines*, also found in the literature.

No global examples, or even local ones around singularities, of harmonic mean curvature configurations seem to have been considered in the literature on differential equations of classic differential geometry, in contrast with the situations for the principal and asymptotic cases mentioned above. See also the work of Anosov, for the global structure of the geodesic flow [2], and that of Banchoff, Gaffney and McCrory [3] for the parabolic and asymptotic lines.

This paper is organized as follows:

Section 2 is devoted to the general study of the differential equations and general properties of Harmonic Mean Curvature Lines. Here are given the precise definitions of the Harmonic Mean Curvature Configuration and of the two transversal Harmonic Mean Curvature Foliations with singularities into which it splits. The definition of Harmonic Mean Curvature Structural Stability focusing on the preservation of the qualitative properties of the foliations and the configuration under small perturbations of the immersion, will be given at the end of this section.

In Section 3 the equation of lines of harmonic mean curvature is written in a Monge chart. The condition for umbilic harmonic mean curvature stability is explicitly stated in terms of the coefficients of the third order jet of the function which represents the immersion in a Monge chart. The local harmonic mean curvature separatrix configurations at stable umbilics is established for C^4 immersions and resemble the three Darbouxian patterns of principal and arithmetic mean curvature configurations [10, 19]. These patterns have been also recently established for the case of geometric mean curvature configurations [15].

In Section 4 the derivative of first return Poincaré map along a harmonic mean curvature cycle is established. It consists of an integral expression involving the curvature functions along the cycle.

In Section 5 are studied the foliations by lines of harmonic mean curvature near the parabolic set of an immersion, which typically is a regular curve. Three singular tangential patterns exist generically in this case: the *folded node* the *folded saddle* and the *folded focus*. However, these types alternate with the patterns established for the asymptotic lines on the hyperbolic region. The following is established and made precise here: an elliptic harmonic (resp. asymptotic) saddle goes adjacent with a hyperbolic (resp. harmonic) asymptotic node or focus. See subsection 5.1 and the pertinent bifurcation diagram. Notice also that it has been proved that in the geometric mean curvature case the *folded focus* is absent generically [15].

Section 6 presents new examples of Harmonic Mean Curvature Configurations on the Torus of revolution and the quadratic Ellipsoid, presenting non-trivial recurrences. This situation, impossible for lines of principal curvature, has been established, with different technical details, for arithmetic and geometric mean curvature configurations in [14, 15].

In Section 7 the results presented in Sections 3, 4 and 5 are put together to provide sufficient conditions for Harmonic Mean Curvature Structural Stability. The density of these conditions is formulated and discussed at the end of this section, however its rather technical proof will be postponed to another paper.

Section 8 contains an initial discussion motivated by this and previous related papers. We inquire about the possibility and interest of developing a unifying general Theory for Mean Curvature Configurations, valid for those already studied and also for possible "new" mean curvature functions.

2 Differential equations of harmonic mean curvature lines

Let $\alpha : \mathbb{M}^2 \to \mathbb{R}^3$ be a C^r, $r \geq 4$, immersion of an oriented smooth surface $\mathbb{M}^2$ into $\mathbb{R}^3$. This means that $D\alpha$ is injective at every point in $\mathbb{M}^2$.

The space $\mathbb{R}^3$ is oriented by a once for all fixed orientation and endowed with the Euclidean inner product $<, >$.

Let N be a vector field orthonormal to α. Assume that (u, v) is a positive chart of $\mathbb{M}^2$ and that $\{\alpha_u, \alpha_v, N\}$ is a positive frame in $\mathbb{R}^3$.

In the chart (u, v), the *first fundamental form* of an immersion α is given by:

$$I_\alpha =< D\alpha, D\alpha >= Edu^2 + 2Fdudv + Gdv^2,$$

with

$$E =< \alpha_u, \alpha_u >, \quad F =< \alpha_u, \alpha_v >, \quad G =< \alpha_v, \alpha_v >$$

The *second fundamental form* is given by:

$$II_\alpha =< N, D^2\alpha >= edu^2 + 2fdudv + gdv^2.$$

The normal curvature at a point p in a tangent direction $t = [du : dv]$ is given by:

$$k_n = k_n(p) = \frac{II_\alpha(t, t)}{I_\alpha(t, t)}.$$

The lines of harmonic mean curvature of α are regular curves γ on $\mathbb{M}^2$ having normal curvature equal to the harmonic mean curvature of the immersion, i.e., $k_n = \frac{\mathcal{K}}{\mathcal{H}}$, where $\mathcal{K} = \mathcal{K}_\alpha$ and $\mathcal{H} = \mathcal{H}_\alpha$ are the Gaussian and Arithmetic Mean curvatures of α.

Therefore the pertinent differential equation for these lines is given by:

$$\frac{edu^2 + 2fdudv + gdv^2}{Edu^2 + 2Fdudv + Gdv^2} = \frac{\mathcal{K}}{\mathcal{H}}$$

Or equivalently by

$$\left[g - \frac{\mathcal{K}}{\mathcal{H}}G\right]dv^2 + 2\left[f - \frac{\mathcal{K}}{\mathcal{H}}F\right]dudv + \left[e - \frac{\mathcal{K}}{\mathcal{H}}E\right]du^2 = 0. \qquad (1)$$

Also, as remarked by Occhipinti in [26], the equation of harmonic curvature lines can be written as

$$Jac(Jac(II, I), II) = 0,$$

which leads to:

$$\begin{aligned} &Ldv^2 + Mdudv + Ndu^2 = 0, \\ &L = g(gE - eG) + 2f(gF - fG) \\ &M = 2g(fE - eF) + 2e(fG - gF) \\ &N = e(eG - gE) + 2f(fE - eF) \end{aligned} \tag{2}$$

This equation is defined only on the closure of the *Elliptic region*, $\mathbb{EM}^2{}_\alpha$, of α, where $\mathcal{K} > 0$. It is bivalued and C^{r-2}, $r \geq 4$, smooth on the complement of the umbilic, $\mathcal{U}_\alpha$, and parabolic, $\mathcal{P}_\alpha$, sets of the immersion α. In fact, on $\mathcal{U}_\alpha$, where the principal curvatures coincide, i.e where $\mathcal{H}^2 - \mathcal{K} = 0$, the equation vanishes identically; on $\mathcal{P}_\alpha$, it is univalued.

The developments above allow us to organize the lines of harmonic mean curvature of immersions into the *harmonic mean curvature configuration,* as follows:

Through every point $p \in \mathbb{EM}^2{}_\alpha \setminus (\mathcal{U}_\alpha \cup \mathcal{P}_\alpha)$, pass two harmonic mean curvature lines of α. Under the orientability hypothesis imposed on $\mathbb{M}$, the harmonic mean curvature lines define two foliations: $\mathbb{H}_{\alpha,1}$, called the *minimal harmonic mean curvature foliation*, along which the geodesic torsion is negative (i.e $\tau_g = -\sqrt{\mathcal{K}}\sqrt{\mathcal{H}^2 - \mathcal{K}}/|\mathcal{H}|$), and $\mathbb{H}_{\alpha,2}$, called the *maximal harmonic mean curvature foliations*, along which the geodesic torsion is positive (i.e $\tau_g = \sqrt{\mathcal{K}}\sqrt{\mathcal{H}^2 - \mathcal{K}}/|\mathcal{H}|$).

By comparison with the arithmetic mean curvature directions, making angle $\pi/4$ with the minimal principal directions, the harmonic ones are located between them and the principal ones, making an angle θ_h such that $tan\theta_h = \pm\sqrt{\frac{k_1}{k_2}}$, as follows from Euler's Formula. The particular expression for the geodesic torsion given above results from the formula $\tau_g = (k_2 - k_1) sin\theta cos\theta$ [32], is found in the work of Occhipinti [26]. See also Lemma 1 in Section 4 below. In [26, 15] is also proved that geometric mean curvature lines are between the harmonic and arithmetic mean curvature ones, making an angle θ_g such that $tan\theta_g = \pm\sqrt{\frac{k_1}{k_2}}$.

With this data, Occhipinti [26], has proved that the two lines of mean geometric curvature, that of mean harmonic and geometric curvature form a harmonic quadruple of lines.

The quadruple $\mathbb{H}_\alpha = \{\mathcal{P}_\alpha, \mathcal{U}_\alpha, \mathbb{H}_{\alpha,1}, \mathbb{H}_{\alpha,2}\}$ is called the *harmonic mean curvature configuration* of α.

It splits into two foliations with singularities:

$$\mathbb{G}^i_\alpha = \{\mathcal{P}_\alpha, \mathcal{U}_\alpha, \mathbb{H}_{\alpha,i}\}, i = 1, 2.$$

Let $\mathbb{M}^2$ be also compact. Denote by $\mathcal{M}^{r,s}(\mathbb{M}^2)$ be the space of C^r immersions of $\mathbb{M}^2$ into the Euclidean space $\mathbb{R}^3$, endowed with the C^s topology.

An immersion α is said *C^s-local harmonic mean curvature structurally stable at a compact set $C \subset \mathbb{M}^2$* if for any sequence of immersions α_n converging to α in $\mathcal{M}^{r,s}(\mathbb{M}^2)$ there is a neighborhood V_C of C, sequence of compact subsets C_n and a sequence of homeomorphisms mapping C to C_n converging to the identity of $\mathbb{M}^2$ such that on V_C it maps umbilic and parabolic points and arcs of the harmonic mean curvature foliations $\mathbb{H}_{\alpha,i}$ to those of $\mathbb{H}_{\alpha_n,i}$ for $i = 1, 2$.

An immersion α is said to be *C^s-harmonic mean curvature structurally stable* if the compact C above is the closure of $\mathbb{EM}^2{}_\alpha$.

Analogously, α is said to be *i- C^s-harmonic mean curvature structurally stable* if only the preservation of elements of *i-th, i=1,2* foliation with singularities is required.

A general study of the structural stability of quadratic differential equations (not necessarily derived from normal curvature properties) has been carried out by Guíñez [18]. See also the work of Bruce and Fidal [6] Bruce and Tari [7], [8] and Davydov [11] for the analysis of umbilic points for general quadratic and also implicit differential equations.

For a study of the topology of foliations with non-orientable singularities on two dimensional manifolds, see the works of Rosenberg and Levitt [30, 24]. In these works the leaves are not defined by normal curvature properties.

3 Harmonic mean curvature lines near umbilic points

Let 0 be an umbilic point of a C^r, $r \geq 4$, immersion α parametrized in a Monge chart (x, y) by $\alpha(x, y) = (x, y, z(x, y))$, where

$$h(x, y) = \frac{k}{2}(x^2 + y^2) + \frac{a}{6}x^3 + \frac{b}{2}xy^2 + \frac{c}{6}y^3 + O(4). \qquad (3)$$

This reduced form is obtained by means of a rotation of the x, y-axes. See [19, 21].

According to Darboux [10, 19], the differential equation of principal curvature lines is given by:

$$-[by + P_1]dy^2 + [(b - a)x + cy + P_2]dxdy + [by + P_3]dx^2 = 0. \qquad (4)$$

As an starting point, recall the behavior of principal lines near Darbouxian umbilics in the following proposition.

Proposition 1. [19, 21] *Assume the notation established in 3. Suppose that the transversality condition $T : b(b - a) \neq 0$ holds and consider the following situations:*

D_1) $\Delta_P > 0$

D_2) $\Delta_P < 0$ *and* $\dfrac{a}{b} > 1$

D_3) $\dfrac{a}{b} < 1$

Here $\Delta_P = 4b(a-2b)^3 - c^2(a-2b)^2$

Then each principal foliation has in a neighborhood of 0*, one hyperbolic sector in the D_1 case, one parabolic and one hyperbolic sector in D_2 case and three hyperbolic sectors in the case D_3. These points are called principal curvature Darbouxian umbilics.*

Proposition 2. *Assume the notation established in 3. Suppose that the transversality condition $T_h : kb(b-a) \neq 0$ holds and consider the following situations:*

H_1) $\Delta_h > 0$

H_2) $\Delta_h < 0$ *and* $\dfrac{a}{b} > 1$

H_3) $\dfrac{a}{b} < 1.$

Here $\Delta_h = 4c^2(2a-b)^2 - [3c^2 + (a-5b)^2][3(a-5b)(a-b)+c^2].$

Then each harmonic mean curvature foliation has in a neighborhood of 0*, one hyperbolic sector in the H_1 case, one parabolic and one hyperbolic sector in H_2 case and three hyperbolic sectors in the case H_3. These umbilic points are called harmonic mean curvature Darbouxian umbilics.*

The harmonic mean curvature foliations $\mathbb{H}_{\alpha,i}$ near an umbilic point of type H_k has a local behavior as shown in Figure 1. The separatrices of these singularities are called umbilic separatrices.

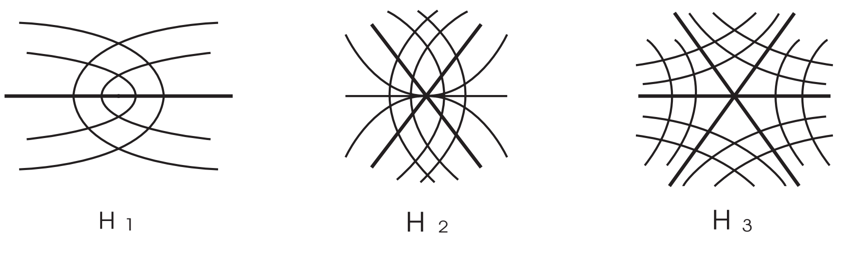

Figure 1: Harmonic mean curvature lines near the umbilic points H_i and their separatrices.

Proof. Near 0, the functions $\mathcal{K}$ and $\mathcal{H}$ have the following Taylor expansions.

$$\mathcal{K} = k^2 + (a+b)kx + cky + O_1(2), \quad \mathcal{H} = k + \frac{1}{2}(a+b)x + \frac{1}{2}cy + O_2(2).$$

The differential equation of the harmonic mean curvature lines

$$\left[g - \frac{\mathcal{K}}{\mathcal{H}}G\right]dv^2 + 2\left[f - \frac{\mathcal{K}}{\mathcal{H}}F\right]dudv + \left[e - \frac{\mathcal{K}}{\mathcal{H}}E\right]du^2 = 0 \tag{5}$$

is given by:

$$\begin{aligned} &[(b-a)x + cy + M_1(x,y)]dy^2 + [4by + M_2(x,y)]dxdy \\ &-[(b-a)x + cy + M_3(x,y)]dx^2 = 0 \end{aligned} \tag{6}$$

where M_i, $i = 1, 2, 3$, represent functions of order $O((x^2+y^2))$.

Thus, at the level of first jet, the differential equation 6 is the same as that of the arithmetic mean curvature lines given by

$$[g - \mathcal{H}G]dv^2 + 2[f - \mathcal{H}F]dudv + [e - \mathcal{H}E]du^2 = 0,$$

as follows from the obvious fact that $\mathcal{H}$ and $\frac{\mathcal{K}}{\mathcal{H}}$ have the same $1-$jet at 0.

The conditions on Δ_h coincide with those on Δ_H, established to characterize the arithmetic mean curvature Darbouxian umbilics studied in detail in [14]. Thus reducing the analysis of the umibilic points to that of the hyperbolicity saddles and nodes whose phase portrait is determined only by the first jet of the equation. □

Theorem 1. *An immersion $\alpha \in \mathcal{M}^{r,s}(\mathbb{M}^2)$, $r \geq 4$, is C^3-local harmonic mean curvature structurally stable at $\mathcal{U}_\alpha$ if and only if every $p \in \mathcal{U}_\alpha$ is one of the types H_i, $i = 1, 2, 3$ of proposition 2.*

Proof. Clearly proposition 2 shows that the condition H_i, $i = 1, 2, 3$ together with $T_h : kb(b-a) \neq 0$ imply the C^3-local harmonic mean curvature structural stability. This involves the construction of the homeomorphism (by means of canonical regions), mapping simultaneously minimal and maximal harmonic mean curvature lines around the umbilic points of α onto those of a C^4 slightly perturbed immersion.

We will discuss the necessity of the condition $T_h : k(b-a)b \neq 0$ and of the conditions H_i, $i = 1, 2, 3$. The first one follows from its identification with a transversality condition that guarantees the persistent isolatedness of the umbilic points of α and its separation from the parabolic set, as well as the persistent regularity of the Lie-Cartan surface $\mathcal{G}$, obtained from the projectivization of the equation 5. Failure of T_h condition has the following implications:

a) $b(b-a)=0$; in this case the elimination or splitting of the umbilic point can be achieved by small perturbations.

b) $k=0$ and $b(b-a)\neq 0$; in this case a small perturbation separates the umbilic point from the parabolic set.

The necessity of condition H_i follows from its dynamic identification with the hyperbolicity of the equilibria along the projective line of the vector field obtained lifting equation (5) to the surface $\mathcal{G}$. Failure of this condition would make possible to change the number of harmonic mean curvature umbilic separatrices at the umbilic point by means a small perturbation of the immersion. □

4 Periodic harmonic mean curvature lines

Let $\alpha : \mathbb{M}^2 \to \mathbb{R}^3$ be an immersion of a compact and oriented surface and consider the foliations $\mathbb{H}_{\alpha,i}$, $i=1,\ 2$, given by the *harmonic mean curvature lines*.

In terms of geometric invariants, here is established an integral expression for the first derivative of the return map of a periodic harmonic mean curvature line, called *harmonic mean curvature cycle*. Recall that the return map associated to a cycle is a local diffeomorphism with a fixed point, defined on a cross section normal to the cycle by following the integral curves through this section until they meet again the section. This map is called holonomy in Foliation Theory and Poincaré Map in Dynamical Systems, [25].

A harmonic mean curvature cycle is called *hyperbolic* if the first derivative of the return map at the fixed point is different from one.

The harmonic mean curvature foliations $\mathbb{H}_{\alpha,i}$ has no harmonic mean curvature cycles such that the return map reverses the orientation. Initially, the integral expression for the derivative of the return map is obtained in class C^6; see Lemma 2 and Proposition 3. Later on, in Remark 4 it is shown how to extend it to class C^3.

The characterization of hyperbolicity of harmonic mean curvature cycles in terms of local structural stability is given in Theorem 2 of this section.

Lemma 1. *Let $c : I \to \mathbb{M}^2$ be a harmonic mean curvature line parametrized by arc length. Then the Darboux frame is given by:*

$$\begin{aligned} T' &= k_g N\wedge T + \frac{\mathcal{K}}{\mathcal{H}} N \\ (N\wedge T)' &= -k_g T + \tau_g N \\ N' &= -\frac{\mathcal{K}}{\mathcal{H}} T - \tau_g N\wedge T \end{aligned}$$

where $\tau_g = \pm\sqrt{\mathcal{K}}\frac{\sqrt{\mathcal{H}^2-\mathcal{K}}}{|\mathcal{H}|}$. *The sign of* τ_g *is positive (resp. negative) if* c *is maximal (resp. minimal) harmonic mean curvature line.*

Proof. The normal curvature k_n of the curve c is by the definition the harmonic mean curvature $\frac{\mathcal{K}}{\mathcal{H}}$. From the Euler equation $k_n = k_1\cos^2\theta + k_2\sin^2\theta = \frac{\mathcal{K}}{\mathcal{H}}$, get $\tan\theta = \pm\sqrt{\frac{k_1}{k_2}}$. Therefore, by direct calculation, the geodesic torsion is given by $\tau_g = (k_2 - k_1)\sin\theta\cos\theta = \pm\sqrt{\mathcal{K}}\frac{\sqrt{\mathcal{H}^2-\mathcal{K}}}{\mathcal{H}}$. □

Remark 1. The expression for the geodesic curvature k_g will not be needed explicitly in this work. However, it can be given in terms of the principal curvatures and their derivatives using a formula due to Liouville [32].

Lemma 2. *Let* $\alpha : \mathbb{M} \to \mathbb{R}^3$ *be an immersion of class* C^r, $r \geq 6$, *and* c *be a mean curvature cycle of* α, *parametrized by arc length and of length* L. *Then the expression,*

$$\begin{aligned}\alpha(s, v) &= c(s) + v(N \wedge T)(s) + \\ &+ \left[(2\mathcal{H}(s) - \frac{\mathcal{K}}{\mathcal{H}}(s))\frac{v^2}{2} + \frac{A(s)}{6}v^3 + v^3 B(s, v)\right]N(s)\end{aligned}$$

where $B(s, 0) = 0$, *defines a local chart* (s, v) *of class* C^{r-5} *in a neighborhood of* c.

Proof. The curve c is of class C^{r-1} and the map $\alpha(s, v, w) = c(s) + v(N \wedge T)(s) + wN(s)$ is of class C^{r-2} and is a local diffeomorphism in a neighborhood of the axis s. In fact $[\alpha_s, \alpha_v, \alpha_w](s, 0, 0) = 1$. Therefore there is a function $W(s, v)$ of class C^{r-2} such that $\alpha(s, v, W(s, v))$ is a parametrization of a tubular neighborhood of $\alpha \circ c$. Now for each s, $W(s, v)$ is just a parametrization of the curve of intersection between $\alpha(\mathbb{M})$ and the normal plane generated by $\{(N \wedge T)(s), N(s)\}$. This curve of intersection is tangent to $(N \wedge T)(s)$ at $v = 0$ and notice that $k_n(N \wedge T)(s) = 2\mathcal{H}(s) - \frac{\mathcal{K}}{\mathcal{H}}(s)$. Therefore,

$$\begin{aligned}\alpha(s, v, W(s, v)) &= c(s) + v(N \wedge T)(s) \\ &+ \left[(2\mathcal{H}(s) - \frac{\mathcal{K}}{\mathcal{H}}(s))\frac{v^2}{2} + \frac{A(s)}{6}v^3 + v^3 B(s, v)\right]N(s),\end{aligned} \tag{7}$$

where A is of class C^{r-5} and $B(s, 0) = 0$. □

We now compute the coefficients of the first and second fundamental forms in the chart (s, v) constructed above, to be used in proposition 3.

$$N(s,v) = \frac{\alpha_s \wedge \alpha_v}{\mid \alpha_s \wedge \alpha_v \mid} = [-\tau_g(s)v + O(2)]T(s)$$
$$-[(2\mathcal{H}(s) - \frac{\mathcal{K}}{\mathcal{H}}(s))v + O(2)](N \wedge T)(s) + [1 + O(2)]N(s).$$

Therefore it follows that $E = <\alpha_s, \alpha_s>$, $F = <\alpha_s, \alpha_v>$, $G = <\alpha_v, \alpha_v>$, $e = <N, \alpha_{ss}>$, $f = <N, \alpha_{sv}>$ and $g = <N, \alpha_{vv}>$ are given by

$$\begin{aligned}
E(s,v) &= 1 - 2k_g(s)v + h.o.t \\
F(s,v) &= 0 + 0.v + h.o.t \\
G(s,v) &= 1 + 0.v + h.o.t \\
e(s,v) &= \frac{\mathcal{K}}{\mathcal{H}}(s) + v[\tau_g'(s) - 2k_g(s)\mathcal{H}(s)] + h.o.t \\
f(s,v) &= \tau_g(s) + \{[2\mathcal{H}(s) - \frac{\mathcal{K}}{\mathcal{H}}(s)]' + k_g(s)\tau_g(s)\}v + h.o.t \\
g(s,v) &= 2\mathcal{H}(s) - \frac{\mathcal{K}}{\mathcal{H}}(s) + A(s)v + h.o.t
\end{aligned} \tag{8}$$

Proposition 3. *Let $\alpha : \mathbb{M} \to \mathbb{R}^3$ be an immersion of class C^r, $r \geq 6$ and c be closed harmonic line c of α, parametrized by arc length s and of total length L. Then the derivative of the Poincaré map π_α associated to c is given by:*

$$ln\pi_\alpha'(0) = \int_0^L \left[\frac{[\frac{\mathcal{K}}{\mathcal{H}}]_v}{2\tau_g} + \frac{k_g}{\tau_g}(\mathcal{H} - \frac{\mathcal{K}}{\mathcal{H}}) \right] ds.$$

Here $\tau_g = \pm\frac{\sqrt{\mathcal{K}}}{\mathcal{H}}\sqrt{\mathcal{H}^2 - \mathcal{K}}$.

Proof. The Poincaré map associated to c is the map $\pi_\alpha : \Sigma \to \Sigma$ defined in a transversal section to c such that $\pi_\alpha(p) = p$ for $p \in c \cap \Sigma$ and $\pi_\alpha(q)$ is the first return of the harmonic mean curvature line through q to the section Σ, choosing a positive orientation for c. It is a local diffeomorphism and is defined, in the local chart (s, v) introduced in Lemma 2, by $\pi_\alpha : \{s = 0\} \to \{s = L\}$, $\pi_\alpha(v_0) = v(L, v_0)$, where $v(s, v_0)$ is the solution of the Cauchy problem

$$\left(g - \frac{\mathcal{K}}{\mathcal{H}}\right)dv^2 + 2\left(f - \frac{\mathcal{K}}{\mathcal{H}}F\right)dsdv + \left(e - \frac{\mathcal{K}}{\mathcal{H}}E\right)ds^2 = 0, \quad v(0, v_0) = v_0.$$

Direct calculation gives that the derivative of the Poincaré map satisfies the following linear differential equation:

$$\frac{d}{ds}\left(\frac{dv}{dv_0}\right) = -\frac{N_v}{M}\left(\frac{dv}{dv_0}\right) = -\frac{[e - \frac{\mathcal{K}}{\mathcal{H}}(s)E]_v}{2[f - \frac{\mathcal{K}}{\mathcal{H}}(s)F]}\left(\frac{dv}{dv_0}\right)$$

Therefore, using equation 8 it results that

$$\frac{[e - \frac{\mathcal{K}}{\mathcal{H}}(s)E]_v}{2[f - \frac{\mathcal{K}}{\mathcal{H}}(s)F]} = -\frac{\tau_g'}{2\tau_g} - \frac{[\frac{\mathcal{K}}{\mathcal{H}}(s)]_v}{2\tau_g} - \frac{k_g}{\tau_g}\left(\mathcal{H} - \frac{\mathcal{K}}{\mathcal{H}}\right).$$

Integrating the equation above along an arc $[s_0, s_1]$ of harmonic mean curvature line, it follows that:

$$\frac{dv}{dv_0}|_{v_0=0} = \frac{(\tau_g(s_1))^{\frac{-1}{2}}}{(\tau_g(s_0))^{\frac{-1}{2}}} \exp[\int_{s_0}^{s_1} \left[\frac{[\frac{\mathcal{K}}{\mathcal{H}}]_v}{2\tau_g} + \frac{k_g}{\tau_g}(\mathcal{H} - \frac{\mathcal{K}}{\mathcal{H}})\right] ds. \qquad (9)$$

Applying 9 along the harmonic mean curvature cycle of length L, obtain

$$\frac{dv}{dv_0}|_{v_0=0} = \exp[\int_0^L \left[\frac{[\frac{\mathcal{K}}{\mathcal{H}}]_v}{2\tau_g} + \frac{k_g}{\tau_g}(\mathcal{H} - \frac{\mathcal{K}}{\mathcal{H}})\right] ds.$$

From the equation $\mathcal{K} = (eg - f^2)/(EG - F^2)$ evaluated at $v = 0$ it follows that $\mathcal{K} = \frac{\mathcal{K}}{\mathcal{H}}[2\mathcal{H} - \frac{\mathcal{K}}{\mathcal{H}}] - \tau_g^2$. Solving this equation it follows that $\tau_g = \pm\frac{\sqrt{\mathcal{K}}}{\mathcal{H}}\sqrt{\mathcal{H}^2 - \mathcal{K}}$. This ends the proof. □

Remark 2. At this point we show how to extend the expression for the derivative of the hyperbolicity of harmonic mean curvature cycles established for class C^6 to class C^3 (in fact we need only class C^4).

The expression 9 is the derivative of the transition map for a harmonic mean curvature foliation (which at this point is only of class C^1), along an arc of harmonic mean curvature line. In fact, this follows by approximating the C^3 immersion by one of class C^6. The corresponding transition map (now of class C^4) whose derivative is given by expression 9 converges to the original one (in class C^1) whose expression must given by the same integral, since the functions involved there are the uniform limits of the corresponding ones for the approximating immersion.

Proposition 4. *Let $\alpha : \mathbb{M} \to \mathbb{R}^3$ be an immersion of class C^r, $r \geq 6$, and c be a maximal harmonic mean curvature cycle of α, parametrized by arc length and* *of length L. Consider a chart (s, v) as in lemma 2 and consider the deformation*

$$\beta_\epsilon(s,v) = \beta(\epsilon, s, v) = \alpha(s,v) + \epsilon\left[\frac{A_1(s)}{6}v^3\right]\delta(v)N(s)$$

where $\delta = 1$ in neighborhood of $v = 0$, with small support and $A_1(s) = \tau_g(s) > 0$.

Then c is a harmonic mean curvature cycle of β_ϵ for all ϵ small and c is a hyperbolic harmonic mean curvature cycle for β_ϵ, $\epsilon \neq 0$.

Proof. In the chart (s, v), for the immersion β_ϵ, it is obtained that:

$$\begin{aligned}
E_\epsilon(s,v) &= 1 - 2k_g(s)v + h.o.t\\
F_\epsilon(s,v) &= 0 + 0.v + h.o.t\\
G_\epsilon(s,v) &= 1 + 0.v + h.o.t\\
e_\epsilon(s,v) &= \frac{\mathcal{K}}{\mathcal{H}}(s) + v[\tau_g'(s) - 2k_g(s)\mathcal{H}(s)\ )] + h.o.t\\
f_\epsilon(s,v) &= \tau_g(s) + [(2\mathcal{H}(s) - \frac{\mathcal{K}}{\mathcal{H}}(s))' + k_g\tau_g]v + h.o.t\\
g_\epsilon(s,v) &= 2\mathcal{H}(s) - \frac{\mathcal{K}}{\mathcal{H}}(s) + v[A(s) + \epsilon A_1(s)] + h.o.t
\end{aligned}$$

In the expressions above $E_\epsilon =< \beta_s, \beta_s >$, $F_\epsilon =< \beta_s, \beta_v >$, $G_\epsilon =< \beta_v, \beta_v >$, $e_\epsilon =< \beta_{ss}, N >$, $f_\epsilon =< N, \beta_{sv} >$, $g_\epsilon =< N, \beta_{vv} >$, where $N = N_\epsilon = \beta_s \wedge \beta_v / \mid \beta_s \wedge \beta_v \mid$.

For all ϵ small it follows that:

$$\begin{aligned}
(e_\epsilon - \frac{\mathcal{K}_\epsilon}{\mathcal{H}_\epsilon}E_\epsilon)(s,0,\epsilon) &= 0\\
\mathcal{K}_{\epsilon v}(s,0,\epsilon) &= \epsilon\frac{\mathcal{K}_\epsilon}{\mathcal{H}_\epsilon}A_1(s) + f_1(k_g,\tau_g,\mathcal{K},\mathcal{H})(s)\\
\mathcal{H}_{\epsilon v}(s,0,\epsilon) &= \frac{1}{2}\epsilon A_1(s) + f_2(k_g,\tau_g,\mathcal{K},\mathcal{H})(s)\\
\frac{d}{d\epsilon}\Big[\frac{\mathcal{K}_\epsilon}{\mathcal{H}_\epsilon}\Big]_v\Big|_{\epsilon=0} &= \frac{1}{2}\frac{\mathcal{K}}{\mathcal{H}^2}A_1(s).
\end{aligned}$$

Therefore c is a maximal harmonic mean curvature cycle for all β_ϵ.

Assuming that $A_1(s) = 4\tau_g(s) > 0$, it results that

$$\frac{d}{d\epsilon}(ln\pi'(0))|_{\epsilon=0} = \int_0^L \frac{d}{d\epsilon}\left(\frac{(\frac{\mathcal{K}_\epsilon}{\mathcal{H}_\epsilon})_v}{2\tau_g} + \frac{k_g}{\tau_g}(\mathcal{H}_\epsilon - \frac{\mathcal{K}_\epsilon}{\mathcal{H}_\epsilon})\right)ds = \int_0^L \frac{\mathcal{K}}{\mathcal{H}^2}ds > 0.$$

□

As a synthesis of propositions 3 and 4, the following theorem is obtained.

Theorem 2. *An immersion $\alpha \in \mathcal{M}^{r,s}(\mathbb{M}^2)$, $r \geq 6$, is C^6-local harmonic mean curvature structurally stable at a harmonic mean curvature cycle c if only if,*

$$\int_0^L \left[\frac{[\frac{\mathcal{K}}{\mathcal{H}}]_v}{2\tau_g} + \frac{k_g}{\tau_g}(\mathcal{H} - \frac{\mathcal{K}}{\mathcal{H}}) \right] ds \neq 0.$$

Proof. Using propositions 3 and 4, the local topological character of the foliation can be changed by small perturbation of the immersion, when the cycle is not hyperbolic. □

5 Harmonic mean curvature lines near the parabolic curve

Let 0 be a parabolic point of a C^r, $r \geq 6$, immersion α parametrized in a Monge chart (x, y) by $\alpha(x, y) = (x, y, z(x, y))$, where

$$\begin{aligned} z(x, y) =& \frac{k}{2}y^2 + \frac{a}{6}x^3 + \frac{b}{2}xy^2 + \frac{d}{2}x^2y + \frac{c}{6}y^3 \\ &+ \frac{A}{24}x^4 + \frac{B}{6}x^3y + \frac{C}{4}x^2y^2 + \frac{D}{6}xy^3 + \frac{E}{24}y^4 + O(5) \end{aligned} \tag{10}$$

The coefficients of the first and second fundamental forms are given by:

$$\begin{aligned} E(x, y) &= 1 + O(4) \\ F(x, y) &= +O(3) \\ G(x, y) &= 1 + k^2y^2 + O(3) \\ e(x, y) &= ax + dy + \frac{A}{2}x^2 + Bxy + \frac{C}{2}y^2 + O(3) \\ f(x, y) &= dx + by + \frac{B}{2}x^2 + Cxy + \frac{D}{2}y^2 + O(3) \\ g(x, y) &= k + bx + cy + \frac{C}{2}x^2 + Dxy + \frac{1}{2}(E - k^3)y^2 + O(3) \end{aligned} \tag{11}$$

The Gaussian and the Arithmetic Mean curvatures are given by

$$\begin{aligned}\mathcal{K}(x, y) &= k(ax + dy) + \frac{1}{2}(Ak + 2ab - 2d^2)x^2 + (Bk + ac - bd)xy \\ &\quad + \frac{1}{2}(Ck + 2cd - 2b^2)y^2 + O(3), \\ \mathcal{H}(x, y) &= \frac{1}{2}k + \frac{1}{2}(a + b)x + \frac{1}{2}(c + d)y + (A + C)\frac{x^2}{4} \\ &\quad + (B + D)xy + (E - 3k^3 + C)\frac{y^2}{4} + O(3)\end{aligned} \tag{12}$$

The coefficients of the quadratic differential equation 2 are given by

$$\begin{aligned}L &= k^2 + k(2b - a)x + k(2c - d)y \\ &\quad + (2kC - Ak + 2b^2 + 4d^2 - 2ab)\frac{x^2}{2} \\ &\quad + (3db - ac + 2kD - kB + 2cb)xy \\ &\quad + (2c^2 + 4b^2 + 2kE - 2cd - kC - 2k^4)\frac{y^2}{2} + O(3) \\ M &= 2k(d.x + b.y) + (4ad + 2kB + 4bd)\frac{x^2}{2} \\ &\quad + 2(b^2 + d^2 + ab + kC + cd)xy \\ &\quad + (4bd + 2kD + 4cb)\frac{y^2}{2} + O(3) \\ N &= -k(ax + dy) + (2a^2 + 4d^2 - 2ab - Ak)\frac{x^2}{2} \\ &\quad + (2ad - kB + 3bd - ac)xy \\ &\quad + (2d^2 + 4b^2 - kC - 2cd)\frac{y^2}{2} + O(3)\end{aligned} \tag{13}$$

Lemma 3. *Let* 0 *be a parabolic point and consider the parametrization* $(x, y, h(x, y))$ *as above. If* $k > 0$ *and* $a^2 + d^2 \neq 0$ *then the set of parabolic points is locally a regular curve normal to the vector* (a, d) *at* 0.

If $a \neq 0$ *the parabolic curve is transversal to the minimal principal direction* $(1, 0)$.

If $a = 0$ *then the parabolic curve is tangent to the principal direction given by* $(1, 0)$ *and has quadratic contact with the corresponding minimal principal curvature line if* $dk(Ak - 3d^2) \neq 0$.

Proof. If $a \neq 0$, from the expression of $\mathcal{K}$ given by equation 12 it follows that the parabolic line is given by $x = -\frac{d}{a}y + O_1(2)$ and so is transversal to the principal direction $(1, 0)$ at $(0, 0)$.

If $a = 0$, from the expression of $\mathcal{K}$ given by equation 12 it follows that the parabolic line is given by

$$y = \frac{2d^2 - Ak}{2dk}x^2 + O_2(3) \quad \text{and that} \quad y = -\frac{d}{2k}x^2 + O_3(3)$$

is the principal line tangent to the principal direction $(1, 0)$. Now the condition of quadratic contact $\frac{2d^2 - Ak}{2dk} \neq -\frac{d}{2k}$ is equivalent to $dk(Ak - 3d^2) \neq 0$. □

Proposition 5. *Let* 0 *be a parabolic point and the Monge chart* (x, y) *as above.*

If $a \neq 0$ *then the mean harmonic curvature lines are transversal to the parabolic curve and the mean curvatures lines are shown in the picture below, the cuspidal case.*

If $a = 0$ *and* $\sigma = k^2(Ak - 3d^2) \neq 0$ *then the mean harmonic curvature lines are shown in the picture below. In fact, if* $\sigma > 0$ *then the mean harmonic curvature lines are folded saddles. Otherwise, if* $\sigma < 0$ *then the mean harmonic curvature lines are folded nodes or folded focus according to* $\delta = -23d^2 + 8Ak$ *be positive or negative. The two separatrices of these tangential singularities, folded saddle and folded node, as illustrated in the Figure 2, are called parabolic separatrices.*

Proof. Consider the quadratic differential equation

$$H(x, y, [dx : dy]) = Ldy^2 + Mdxdy + Ndx^2 = 0$$

and the Lie-Cartan line field X of class C^{r-3} defined by

$$\begin{aligned} x' &= H_p \\ y' &= pH_p \\ p' &= -(H_x + pH_y), \quad p = \frac{dy}{dx} \end{aligned}$$

where L, M and N are given by equation 13.

If $a \neq 0$ the vector Y is regular and therefore the mean harmonic curvature lines are transversal to the parabolic line and at parabolic points these lines are tangent to the principal direction $(1, 0)$.

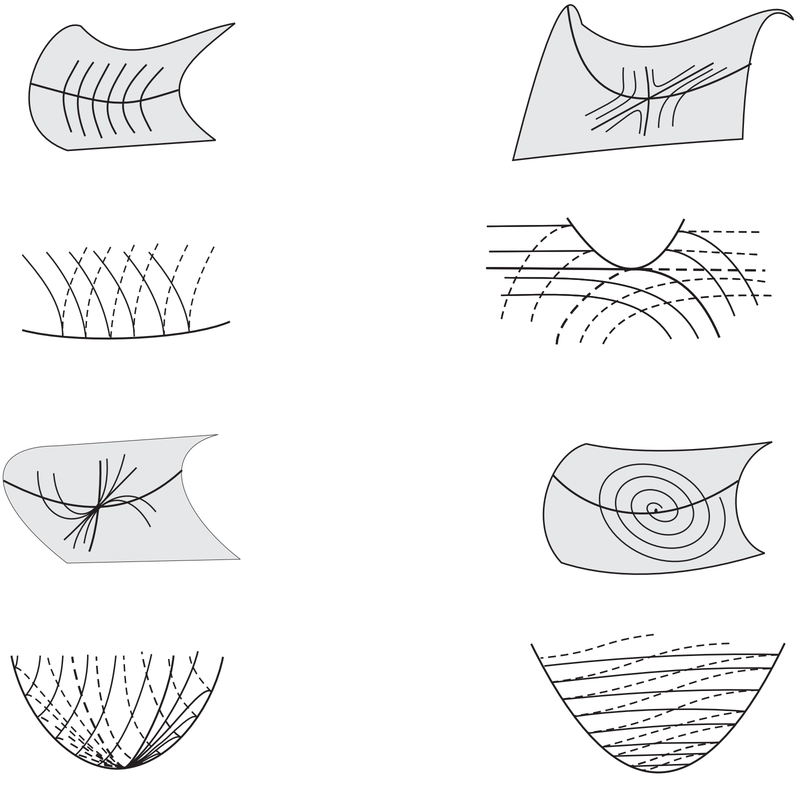

Figure 2: Harmonic mean curvature lines near a parabolic point (cuspidal, folded saddle, folded node and folded focus) and their separatrices.

If $a = 0$, direct calculation gives $H(0) = 0$, $H_x(0) = 0$, $H_y(0) = -kd$, $H_p(0) = 0$.

$$DX(0) = \begin{pmatrix} 2kd & 2kb & 2k^2 \\ 0 & 0 & 0 \\ Ak - 4d^2 & kB - 3bd & -kd \end{pmatrix} \tag{14}$$

The non vanishing eigenvalues of $DX(0)$ are

$$\lambda_1 = (\frac{1}{2}d + \frac{1}{2}\sqrt{-23d^2 + 8Ak})k, \quad \lambda_2 = (\frac{1}{2}d - \frac{1}{2}\sqrt{-23d^2 + 8Ak})k$$

Therefore, $\lambda_1\lambda_2 = -2k^2(Ak - 3d^2)$.

It follows that 0 is a hyperbolic singularity provided $\sigma(Ak - 3d^2)kd \neq 0$. If $\sigma > 0$ then the mean harmonic curvature lines are folded saddles and if $\sigma < 0$

then the mean harmonic curvature lines are folded nodes ($8Ak - 23d^2 > 0$) or folded focus ($8Ak - 23d^2 < 0$). See Figure 2. □

Theorem 3. *An immersion $\alpha \in \mathcal{M}^{r,s}(\mathbb{M}^2)$, $r \geq 6$, is C^6-local harmonic mean curvature structurally stable at a tangential parabolic point p if only if, the condition $\sigma\delta \neq 0$ in proposition 5 holds.*

Proof. Direct from Lemma 3 and proposition 5, the local topological character of the foliation can be changed by small perturbation of the immersion when $\delta\sigma = 0$. □

5.1 Asymptotic lines near a parabolic curve

Proposition 6. *Let 0 be a parabolic point and the Monge chart (x, y) as above.*

If $a \neq 0$ then the mean asymptotic lines are transversal to the parabolic curve and are shown in the picture below, the cuspidal case.

If $a = 0$ and $\sigma = k^2(Ak - 3d^2) \neq 0$ then the asymptotic are shown in the picture below. In fact, if $\sigma < 0$ then the asymptotic lines are folded saddles. Otherwise, if $\sigma > 0$ then the asymptotic lines are folded nodes or folded focus according to $\delta_a = 25d^2 - 8Ak$ be positive or negative. The two separatrices of these tangential singularities, folded saddle and folded node, as illustrated in Figure 3, are called parabolic separatrices.

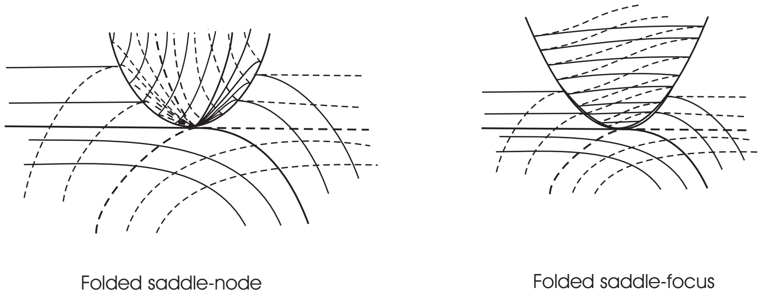

Figure 3: Harmonic Node and Focus adjacent to Asymptotic Saddle.

Proof. The proof follows from direct calculations similar to those performed in proposition 5. In fact, considering the implicit differential equation

$$\mathcal{A}(x, y, p) = gp^2 + 2fp + e = 0, \quad p = \frac{dy}{dx}$$

where e, f and g are given by equation 11 and the Lie-Cartan line field

$$Y = \mathcal{A}_p \frac{\partial}{\partial x} + p\mathcal{A}_p \frac{\partial}{\partial y} - (\mathcal{A}_x + p\mathcal{A}_y)\frac{\partial}{\partial x},$$

it follows that

$$DY(0) = \begin{pmatrix} 2d & 2b & 2k \\ 0 & 0 & 0 \\ -A & -B & -3d \end{pmatrix} \tag{15}$$

The non vanishing eigenvalues of $DY(0)$ are

$$r_1 = \frac{1}{2}d + \frac{1}{2}\sqrt{25d^2 - 8Ak}, \quad r_2 = \frac{1}{2}d - \frac{1}{2}\sqrt{25d^2 - 8Ak}$$

Therefore, $r_1 r_2 = 2(Ak - 3d^2)$.

It follows that 0 is a hyperbolic singularity provided $Ak - 3d^2 \neq 0$. If $Ak - 3d^2 < 0$ then the mean harmonic curvature lines are folded saddles; if $Ak - 3d^2 > 0$ then the mean harmonic curvature lines are folded nodes $(25d^2 - 8Ak > 0)$ or folded focus $(25d^2 - 8Ak < 0)$. See Figure 3. □

Remark 3. The geometric conditions of asymptotic folded saddles, nodes and focus near a parabolic line was obtained in [13].

Remark 4. In the plane $k = 1$ the diagram of folded saddles, folded nodes and folded focus for harmonic mean curvature lines and asymptotic lines is as shown in Figure 4.

6 Examples of harmonic mean curvature configurations

As mentioned in the Introduction, no examples of harmonic mean curvature foliations are given in the literature, in contrast with the principal and asymptotic foliation. In this section are studied the harmonic mean curvature configurations in two classical surfaces: The Torus and the Ellipsoid. In contrast with the principal case [31, 32] (but in concordance with the arithmetic mean curvature one [14]) non-trivial recurrence can occur here.

Proposition 7. *Consider a torus of revolution $T(r, R)$ obtained by rotating a circle of radius r around a line in the same plane and at a distance R, $R > r$, from its center. Define the function ρ of $a = \frac{r}{R}$, as follows:*

$$\rho = \rho(a) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{a}{\cos s(1 + a\cos s)}}\, ds.$$

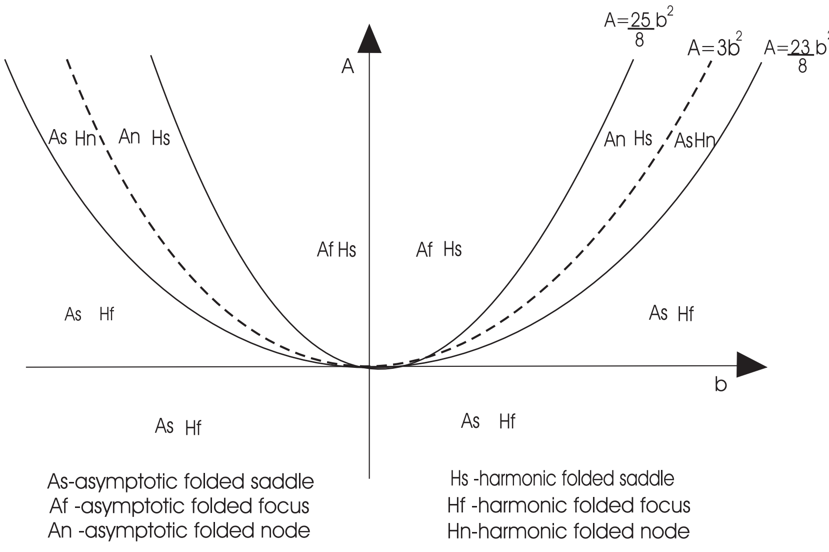

Figure 4: Bifurcation diagram of asymptotic and harmonic mean curvature lines in the plane b×A.

Consider the regular curves (folded extended harmonic lines) defined as the union of harmonic lines and parabolic points (a harmonic line of one foliation that arrive at the parabolic set at a given point is continued through the line of the other foliation leaving the parabolic set at this point and so on). Then the folded extended harmonic mean curvature lines on $T(r, R)$*, defined in the elliptic region are all closed or all recurrent according to* $\rho \in \mathbb{Q}$ *or* $\rho \in \mathbb{R} \setminus \mathbb{Q}$*. Furthermore, both cases occur for appropriate* (r, R)*.*

Proof. The torus of revolution $T(r, R)$ is parametrized by

$$\alpha(s, \theta) = ((R + r\cos s)\cos\theta, (R + r\cos s)\sin\theta,\ r\sin s).$$

Direct calculation shows that $E = r^2$, $F = 0$, $G = [R + r\cos s]^2$, $e = -r$, $f = 0$ and $g = -\cos s(R + r\cos s)$. Clearly (s, θ) is a principal chart.

The differential equation of the harmonic mean curvature lines, in the principal chart (s, θ), is given by $eds^2 - gd\theta^2 = 0$. This is equivalent to

$$-\cos s(1 + a\cos s)d\theta^2 + ads^2 = 0, \quad a = \frac{r}{R}$$

Solving the equation above it follows that,

$$\int_{\theta_0}^{\theta_1} d\theta = \pm \int_{s_0}^{s_1} \sqrt{\frac{a}{\cos s(1 + a\cos s)}} ds.$$

So the two Poincaré maps, $\pi_\pm : \{s = -\frac{\pi}{2}\} \to \{s = \frac{\pi}{2}\}$, defined by $\pi_\pm(\theta_0) = \theta_0 \pm 2\pi\rho(\frac{r}{R})$ have rotation number equal to $\pm\rho(\frac{r}{R})$. Direct calculations gives that $\rho(a) \neq 0$ and $\rho'(a) \neq 0$ for $a > 0$. Therefore, both the rational and irrational cases occur. This ends the proof. □

Proposition 8. *Consider the ellipsoid $\mathbb{E}_{a,b,c}$ with three axes $a > b > c > 0$. Then $\mathbb{E}_{a,b,c}$ have four umbilic points located in the plane of symmetry orthogonal to middle axis; they are of the type H_1 for harmonic mean curvature lines and of type D_1 for the principal curvature lines.*

Proof. This follows from proposition 2 and the fact that the arithmetic mean curvature lines have this configuration, as established in [14]. □

Proposition 9. *Consider the ellipsoid $\mathbb{E}_{a,b,c}$ with three axes $a > b > c > 0$. On the ellipse $\Sigma \subset \mathbb{E}_{a,b,c}$, containing the four umbilic points, p_i, $i = 1, \cdots, 4$, oriented counterclockwise, denote by $S_1 = \int_{-b^2}^{-c^2} \frac{1}{\sqrt{h(v)}} dv$ the distance between the adjacent umbilic points p_1 and p_2 and by $S_2 = \int_{-a^2}^{-b^2} \frac{1}{\sqrt{h(u)}} du$ the distance between the adjacent umbilic points p_1 and p_4, where $h(x) = (x + a^2)(x + b^2)(x + c^2)$. Define $\rho = \frac{S_2}{S_1}$.*

Then if $\rho \in \mathbb{R} \setminus \mathbb{Q}$ (resp. $\rho \in \mathbb{Q}$) all the harmonic mean curvature lines are recurrent (resp. all, with the exception of the harmonic mean curvature umbilic separatrices, are closed).

Proof. The ellipsoid $\mathbb{E}_{a,b,c}$ belongs to the triple orthogonal system of surfaces defined by the one parameter family of quadrics, $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1$ with $a > b > c > 0$, see also [31] and [32].

The following parametrization of $\mathbb{E}_{a,b,c}$.

$$\alpha(u, v) = \Big(\pm\sqrt{\frac{M(u, v, a)}{W(a, b, c)}}, \pm\sqrt{\frac{M(u, v, b)}{W(b, a, c)}}, \pm\sqrt{\frac{M(u, v, c)}{W(c, a, b)}}\Big)$$

where,

$$M(u, v, w) = w^2(u + w^2)(v + w^2) \quad \text{and} \quad W(a, b, c) = (a^2 - b^2)(a^2 - c^2),$$

define the ellipsoidal coordinates (u, v) on $\mathbb{E}_{a,b,c}$, where $u \in (-b^2, -c^2)$ and $v \in (-a^2, -b^2)$.

The first fundamental form of $\mathbb{E}_{a,b,c}$ is given by:

$$I = ds^2 = Edu^2 + Gdv^2 = \frac{1}{4}\frac{(u-v)u}{h(u)}du^2 + \frac{1}{4}\frac{(v-u)v}{h(v)}dv^2$$

The second fundamental form is given by

$$II = edu^2 + gdv^2 = \frac{abc(u-v)}{4\sqrt{uv}h(u)}du^2 + \frac{abc(v-u)}{4\sqrt{uv}h(v)}dv^2,$$

where $h(x) = (x+a^2)(x+b^2)(x+c^2)$. The four umbilic points are

$$(\pm x_0, 0, \pm z_0) = (\pm a\sqrt{\frac{a^2-b^2}{a^2-c^2}}, 0, \pm c\sqrt{\frac{c^2-b^2}{c^2-a^2}}).$$

The differential equation of the harmonic mean curvature lines is given by:

$$\frac{(du)^2}{h(u)} - \frac{(dv)^2}{h(v)} = 0$$

Define $d\sigma_1 = \frac{1}{\sqrt{h(u)}}du$ and $d\sigma_2 = \frac{1}{\sqrt{h(v)}}dv$. By integration, this leads to the chart (σ_1, σ_2), in which the differential equation of the harmonic mean curvature lines is given by

$$d\sigma_1^2 - d\sigma_2^2 = 0.$$

On the ellipse $\Sigma = \{(x, 0, z) | \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1\}$ the distance between the umbilic points $p_1 = (x_0, 0, z_0)$ and $p_4 = (x_0, 0, -z_0)$ is given by $S_1 = \int_{-b^2}^{-c^2} \frac{1}{\sqrt{h(v)}}dv$ and that between the umbilic points $p_1 = (x_0, 0, z_0)$ and $p_2 = (-x_0, 0, z_0)$ is given by

$$S_2 = \int_{-a^2}^{-b^2} \frac{1}{\sqrt{h(u)}}du.$$

It is clear that the ellipse Σ is the union of four umbilic points and the four principal umbilical separatrices for the principal foliations. So $\Sigma \backslash \{p_1, p_2, p_3, p_4\}$ is a transversal section of both harmonic mean curvature foliations. The differential equation of the harmonic mean curvature lines in the principal chart (u, v) is given by $edu^2 - gdv^2 = 0$, which amounts to $d\sigma_1 = \pm d\sigma_2$. Therefore near the umbilic point p_1 the harmonic mean curvature lines with a harmonic mean curvature umbilic separatrix contained in the region $\{y > 0\}$ define a the transition map $\sigma_+ : \Sigma \to \Sigma$ which is an isometry, reversing the orientation, with $\sigma_+(p_1) = p_1$. This follows because in the principal chart (u, v) this map is defined by $\sigma_+ : \{u = -b^2\} \to \{v = -b^2\}$ which satisfies the differential equation $\frac{d\sigma_2}{d\sigma_1} = -1$. By analytic continuation it results that σ_+ is an orientation reversing isometry, with two fixed points $\{p_1, \ p_3\}$. The harmonic reflection σ_-, defined in the region $y < 0$ have the two umbilics $\{p_2, \ p_4\}$ as fixed points.

So on the ellipse parametrized by arclength defined by σ_i, the Poincaré return map $\pi_1 : \Sigma \to \Sigma$ (composition of two isometries σ_+ and σ_-) is a rotation with rotation number given by $\frac{S_2}{S_1}$.

Analogously for the other harmonic mean curvature foliation, with the Poincaré return map given by $\pi_2 = \tau_+ \circ \tau_-$, where τ_+ and τ_- are two isometries having respectively $\{p_2, p_4\}$ and $\{p_1, p_3\}$ as fixed points. □

7 On Harmonic mean curvature structural stability

In this section the results of sections 3, 4 and 5 are put together to provide sufficient conditions for harmonic mean curvature stability, outlined below.

Theorem 4. *The set of immersions $\mathcal{A}_i(\mathbb{M}^2)$, $i = 1, \ 2$ which satisfy conditions i), ... , v) below are i-C^s-mean curvature structurally stable and $\mathcal{A}_i$, $i = 1, \ 2$ is open in $\mathcal{M}^{r,s}(\mathbb{M}^2)$, $r \geq s \geq 6$.*

i) *The parabolic curve is regular: $\mathcal{K} = 0$ implies $d\mathcal{K} \neq 0$ and the tangential singularities are saddles and nodes.*

ii) *The umbilic points are of type H_i, $i = 1, \ 2, \ 3$.*

iii) *The harmonic mean curvature cycles of $\mathbb{H}_{\alpha,i}$ are hyperbolic.*

iv) *The harmonic mean curvature foliations $\mathbb{H}_{\alpha,i}$ has no separatrix connections. This means that there is no harmonic mean curvature line joing two umbilic or tangential parabolic singularities and being separatrices at both ends. See propositions 2 and 5*

v) *The limit set of every leaf of $\mathbb{H}_{\alpha,i}$ is a parabolic point, umbilic point or a harmonic mean curvature cycle.*

Proof. The openness of $\mathcal{A}_i(\mathbb{M}^2)$ it follows from the local structure of the harmonic mean curvature lines near the umbilic points of types H_i, $i = 1, 2, 3$, near the harmonic mean curvature cycles and by the absence of umbilic harmonic mean curvature separatrix connections and the absence of recurrences. The equivalence can be performed by the method of canonical regions and their continuation as was done in [19, 21] for principal lines, and in [17], for asymptotic lines. □

Notice that Theorem 4 can be reformulated so as to give the mean harmonic stability of the configuration rather than that of the separate foliations. To this end it is necessary to consider the folded extended lines, that is to consider the line of one foliation that arrive at the parabolic set at a given transversal point as continuing through the line of the other foliation leaving the parabolic set at this point, in a sort of "billiard". This gives raise to the extended folded cycles and separatrices that must be preserved by the homeomorphism mapping simultaneously the two foliations.

Therefore the third, fourth and fifth hypotheses above should be modified as follows:

iii') the extended folded periodic cycles should be hyperbolic,

iv') the extended folded separatrices should be disjoint,

v') the limit set of extended lines should be umbilic points, parabolic singularities and extended folded cycles.

The class of immersions which verify the extended five conditions i), ii), iii'), iv'), v') of a compact and oriented manifold $\mathbb{M}^2$ will be denoted by $\mathcal{A}(\mathbb{M}^2)$.

This procedure has been adopted by the authors in the case of asymptotic lines by the suspension operation in order to pass from the foliations to the configuration and properly formulate the stability results. See [17].

Remark 5. In the space of convex immersions $\mathcal{M}_c^{r,s}(\mathbb{S}^2)$ ($\mathcal{K}_\alpha > 0$), the sets $\mathcal{A}(\mathbb{S}^2)$ and $\mathcal{A}_1(\mathbb{S}^2) \cap \mathcal{A}_2(\mathbb{S}^2)$ coincide.

The genericity result involving the five conditions above is formulated now.

Theorem 5. *The sets* $\mathcal{A}_i$, $i = 1, 2$ *are dense in* $\mathcal{M}^{r,2}(\mathbb{M}^2)$, $r \geq 6$.
In the space $\mathcal{M}_c^{r,2}(\mathbb{S}^2)$ *the set* $\mathcal{A}(\mathbb{S}^2)$ *is dense.*

The main ingredients for the proof of this theorem are the Lifting and Stabilization Lemmas, essential for the achievement of condition five. The conceptual background for this approach goes back to the works of Peixoto and Pugh.

The elimination of non-trivial recurrences – the so called "Closing Lemma Problem"– as a step to achieve condition v)is by far the most difficult of these details. See the book of Palis and Melo, [25], for a presentation of these ideas in the case of vector fields on surfaces.

The proof of theorem 5 will be postponed to a forthcoming paper [16]. It involves technical details that are closer to those of the proofs of genericity theorems given by Gutierrez and Sotomayor, [20, 21], for principal curvature lines and by Garcia and Sotomayor, [14], for arithmetic curvature lines.

8 Additional comments and a related problem

The study of families of curves on surfaces defined by normal curvature properties and their singularities has attracted the interest of generations of mathematicians, among whom can be mentioned Euler, Monge, Dupin, Gauss, Cayley, Darboux, Gullstrand, Caratheodory, Hamburger. See [22, 32] for references.

On the other hand, the ideas on the *"Qualitative Theory of Differential Equations"* initiated by Poincaré and culminating with the study of the Structural Stability and Genericity of differential equations on surfaces, made systematic from 1937 to 1962 due to the seminal work of Andronov Pontrjagin and Peixoto, were assimilated by Gutierrez, Garcia and Sotomayor and, reformulated, were applied to principal curvature lines [19] as well as to other differential equations of classical geometry: asymptotic lines [13, 17], arithmetic and geometric mean curvature lines [14, 15], and harmonic mean curvature lines studied here.

Thus, progress in Differential Equations and Geometry led to delineate a fruitful field of interaction of Geometry and Analysis.

The work of Monge, on the principal configuration of the Ellipsoid; that of Dupin, on Triply Orthogonal families of surfaces and the study of Darboux, on umbilic points on a surface, are the classical geometric paradigms of this field of interaction.

An overview of the ensemble of recent contributions of the authors and others, cited here, reveals that there is a common ground. In fact, they share an analogy in purpose, problems and methods of analysis. It seems, therefore, appropriate to inquire here for the common mathematical features they enjoy and for the discrepancies they present.

In principle any expression such as $\mu = \mu(k_1, k_2) \in [k_1, k_2]$, involving the principal curvatures, could be rightly called a "mean curvature".

The situations that appear in the works quoted above correspond to the Principal Curvatures: $\mu = k_1$ or $\mu = k_2$, Arithmetic, Geometric and Harmonic Mean Curvatures: $\mu = \mathcal{H}$, $\mu = \mathcal{K}^{1/2}$ and $\mu = \frac{\mathcal{K}}{\mathcal{H}}$. The asymptotic lines correspond to

$\mu = 0$. To these five functions we will refer to as the *"classical" mean curvature functions.*

At this point, a pertinent problem is proposed to provoke the discussion.

Problem 1. Formulate and prove a general theorem from which the results obtained before for the "classical" mean curvature functions would follow and that also would include an interesting class of "new" curvature functions $\mu = \mu(k_1, k_2)$ and associated differential equations.

There are a great number of *means* that are of interest in Analysis. For instance the *Holder Means*

$$H_r(k_1, k_2) = [(k_1^r + k_2^r)/2]^{1/r},$$

which contains, in the form of a one parameter family, the classical means. In fact, Arithmetic, corresponds to $r = 1$; Geometric, corresponds to $r = 0$, understood as a limit as $r \to 0$; Harmonic, corresponds to $r = -1$. See Hardy et al. [23].

There are also more subtle, non-algebraic means, such as the AGM-mean, obtained from the limit of the Arithmetic and Geometric. This limit was studied classically by Gauss and Legendre. See the book of Borwein and Borwein [5] for the connections of these means with differential equations and the number π. The interest of this mean for Geometry seems to have been overlooked so far.

A satisfactory answer to Problem 1 involves an analysis of the limits of the methods introduced in the recent papers and their adaptability to deal with configurations associated to "new" mean curvature functions. Bearing in mind the " closing lemma" difficulties mentioned at the end of the previous section, this analysis will be postponed to another work, [16], which contains a proposal for a partial solution.

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Jorge Sotomayor
Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão 1010, Cidade Universitária,
CEP 05508-090, São Paulo, SP.
BRAZIL
E-mail: sotp@ime.usp.br

Ronaldo Garcia
Instituto de Matemática e Estatística
Universidade Federal de Goiás
CEP 74001-970, Caixa Postal 131, Goiânia, GO
BRAZIL
E-mail: ragarcia@mat.ufg.br