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The homotopical reduction of a nearest neighbor random walk*

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Abstract. Consider a nearest neighbor random walk on a graph G and discard all the segments of its trajectory that are homotopically equivalent to a single point. We prove that if the lift of the random walk to the covering tree of G is transient, then the resulting "reduced" trajectories induce a Markov chain on the set of oriented edges of G. We study this chain in relation with the original random walk. As an intermediate result, we give a simple proof of the Markovian structure of the harmonic measure on trees.

Keywords: Graphs, nearest neighbor random walk, harmonic measure on trees.

Mathematical subject classification: 60J45, 60G50.

1 Introduction

Let $G = (V, E)$ be a countable non-oriented graph, where V is the set of vertices and E the set of non-oriented edges. We write $x \sim y$ if $x, y \in V$ are neighbors, and we assume that the degree of every $x \in V$, $deg(x) = |\{y \in V : x \sim y\}|$ satisfies $2 \leq deg(x) < \infty$.

A *path* in G is a finite sequence $(y_0, \dots, y_n)$ of vertices such that $y_i \sim y_{i+1}$ for all $i = 1, \dots n-1$, and we say that it connects y_0 with y_n. We will assume that G is connected (i.e. any pair of vertices is connected by a path) and further, that G has neither loops nor repeated edges. A path $(y_0, ..., y_n)$ is *reduced* if $y_i \neq y_{i+2}$ for all $i = 0, ..., n-2$, and it is *closed* if $y_0 = y_n$. Every path contains a unique reduced path. Two paths with same starting and end points y_0 and y_n are said to be *homotopically equivalent* if they contain the same reduced path.

Let (Y_n) be a nearest neighbor random walk on G. Denote by $\tau_{Y_0}^{hom}$ the stopping time corresponding to the first moment at which (Y_n) comes back to the start-

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ing point, in such way that the closed path $(Y_0, Y_1, ..., Y_{\tau_{Y_0}^{hom}})$ is homotopically equivalent to the zero-length path (Y_0). The random walk is recurrent if $\tau_{y_0}^{hom}$ is finite $\mathbb{P}_{y_0}$-a.s., but nothing can be said in general if $\mathbb{P}_{y_0}\{\tau_{y_0}^{hom} < \infty\} < 1$. We will interpret this probability as the return probability of the "lifted random walk of (Y_n)", which is a random walk (X_n) on the covering tree of G that "projects" onto (Y_n).

We will show that if $\mathbb{P}\{\tau_{Y_0}^{hom} < \infty\} < 1$ holds, then the trajectories of (Y_n) can be almost surely simplified or "reduced", by discarding the segments of the infinite path $(Y_0, Y_1, ...Y_n, ...)$ which are homotopically equivalent to a single point. Hence, the resulting trajectories do never backtrack, and we will prove that they define a Markov chain $(\widetilde{Y}_m)$ on the set of oriented edges $\overrightarrow{E} = \{(x, y) \in V^2 : x \sim y\}$ of G. This chain will be called the "homotopical reduction of (Y_n)".

To compute the transition probabilities of $(\widetilde{Y}_m)$, we will use the results of Cartier [1] on transient nearest neighbor random walks (X_n) on infinite trees. We will give an elementary proof that the associated harmonic measure is Markovian, and compute its transition probabilities in terms of the hitting probabilities of (X_n). By using some elements of covering spaces theory in the graph setting, we will deduce the transition probabilities of the chain $(\widetilde{Y}_m)$. We shall also prove a simple characterization of irreducibility for $(\widetilde{Y}_m)$ (in terms of the limit set of the action of the fundamental group of G on its covering space), and in the irreducible case, we will prove that the type (recurrent or transient) of (Y_n) is preserved by $(\widetilde{Y}_m)$.

We notice that the trajectories of $(\widetilde{Y}_m)$ live in the space of "geodesic rays" of the graph G. In the case of a simple random walk (Y_n) on an homogeneous graph G, Coornaert and Papadopoulos proved in [2] that the harmonic measure of (X_n) corresponds to the Patterson-Sullivan measure of the geodesic flow on G, and that recurrence of (Y_n) is equivalent to the ergodicity of the geodesic flow. In that case, the chain $(\widetilde{Y}_m)$ corresponds to the one-sided shift associated to the flow. We think that $(\widetilde{Y}_m)$ is a natural object to take into account, to study the relation between geodesic flow dynamics and random walks on G in a more general setting than the one considered in [2].

Let us give some notation. Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space, S be a countable state set and $(X_n : \Omega \to S, n \in \mathbb{N})$ be an homogeneous Markov chain. By $\mathbb{P}_x$ we mean the law of (X_n) when issued from x and $\mathbb{E}_x$ denotes the associated expectation. Put $N_x = |\{n \in \mathbb{N} : X_n = x\}|$ and $\tau_x = \inf\{n > 0 \colon X_n = x\}$. We denote by $\mathcal{F}(x, y) := \mathbb{P}_x\{\tau_y < \infty\}$ the probability of hitting y, and by $\mathcal{G}(x, y) =: \mathbb{E}_x(N_y)$ the associated Green kernel. A state x is called *recurrent* if $\mathcal{G}(x, x) = \infty$ and *transient* if $\mathcal{G}(x, x) < \infty$. We will write $x \to y$ if $\mathcal{F}(x, y) > 0$ and $C(x) := \{x\} \cup \{y \in S : y \leftrightarrow x\}$.

2 Preliminaries

Let us consider a nearest neighbor random walk (Y_n) on $G = (V_G, E_G)$, starting from $y_0 \in V_G$. Our first aim is to study the stopping time $\tau_{y_0}^{hom}$ and the condition $\mathbb{P}_{y_0}\{\tau_{y_0}^{hom} < \infty\} < 1$. In this purpose we recall some topological facts about graphs. A graph $T = (V_T, E_T)$ satisfying the conditions of Section **1**, is a *tree* if, further, it does not contain closed paths of positive length. By $[x, y]$ we denote the unique reduced path connecting x and y in T, also called *geodesical segment* between x and y. Its length $|x - y|$ defines a distance on V_T. Similarly, a *geodesical ray* in T is a sequence of vertices $(x_0, x_1, ...)$ such that $x_i \sim x_{i+1}$ and $x_i \neq x_{i+2}$ for all $i \in \mathbb{N}$. A *geodesic* is a bi-infinite sequence $(...x_{-1}, x_0, x_1, ...)$ satisfying the same constraints.

Every connected graph $G = (V_G, E_G)$ has a *universal covering*, that is, a graph homomorphism $\nu : T \to G$, with $T = (V_T, E_T)$ a tree, ν surjective and such that for every $x \in V_T$ the restriction of ν to $\{x\} \cup \{y \in V_T : y \sim x\}$ is a bijection. We refer the reader to Massey [6], Ch. 5 and 6 for the following facts. The universal covering is unique up to graph isomorphism, and a realization of it is the following one. Choose and fix $y_0 \in V_G$. The set of vertices V_T of T is the set of reduced paths $(y_0, y_1, .., y_n)$ in G starting at y_0, and two vertices $x, y \in V_T$ are adjacent if and only if $x = (y_0, y_1, ..., y_n)$ and $y = (y_0, y_1, ...y_n, y_{n+1})$ for some $y_0, ..., y_{n+1} \in V_G$ or conversely. The projection ν is given here by $\nu(x) = x_n$. If $(y_0, y_1, ...y_m)$ is a path in G, for each $x_0 \in \nu^{-1}(y_0)$ there is a unique "lift" of it to a path $(x_0, x_1, ..., x_m)$ in T, such that $\nu(x_i) = y_i$. Two paths in G are homotopically equivalent if and only if their lifts to T (starting at the same given point) are homotopically equivalent (see [6] Ch. 5, Sect. 5).

Now, denote by Γ the group of isomorphisms of the covering $\nu : T \to G$ (that is, the group of isometries $\gamma : T \to T$ such that $\nu \circ \gamma = \nu$), and by $Orb(x) = \{\gamma x : \gamma \in \Gamma\}$ the orbit of $x \in V_T$. Every stabilizer $Est(x) = \{\gamma \in \Gamma : \gamma x = x\}$ is trivial. The quotient graph $\Gamma\backslash T$ is identified with G by mean of $y \in G \mapsto Orb(x) \in \Gamma\backslash T$, where $x \in \nu^{-1}(y)$ (see [6], Ch.5, Sect. 8, and Coornaert and Papadopoulos [2]). Since G has neither loops nor repeated edges, it follows that $|x - \gamma x| \geq 3$ for all $x \in V_T$ and every non trivial $\gamma \in \Gamma$. Let us recall that Γ is isomorphic to the fundamental group of G, $\Pi_1(G)$. Given a vertex $y_0 \in V_G$, $\Pi_1(G)$ is the quotient of the set of closed paths in G having extremes y_0, under the relation of homotopical equivalence. The product is induced by the concatenation of paths and the unity element is the class of the zero-length path (y_0). Up to isomorphism, $\Pi_1(G)$ is independent of the base point y_0, and it is a free group (see [6], Ch. 6, Sect. 5).

We introduce now the lift of the random walk (Y_n) to the universal covering T of G. Fix an arbitrary $x_0 \in \nu^{-1}(y_0)$. Define a mapping on $n-$length paths

$(y_0, y_1, ..., y_n)$ in G, by

$$\nu_{x_0}^{-1}(y_0, y_1, ..., y_n) = (x_0, x_1, ..., x_n),$$

where $(x_0, x_1, ..., x_n)$ is the unique lift of $(y_0, y_1, ..., y_n)$ to T starting at x_0. We also denote by $\nu_{x_0}^{-1}$ its natural extension to the set of infinite paths $(y_0, y_1, ...)$.

Under $\mathbb{P}_{y_0}$, the mapping

$$\nu_{x_0}^{-1} : (V_G)^{\mathbb{N}} \to (V_T)^{\mathbb{N}}$$

is well defined outside a null measure set, and it is measurable as it can be seen by considering cylinder sets. It is easy to check that

$$(X_n) := (\nu_{x_0}^{-1} \circ Y_n), \quad n \in \mathbb{N}, \tag{1}$$

is a Markov chain under $\mathbb{P}_{y_0}$, with transition probabilities given by

$$p(x, y) = \mathbb{P}_{\nu(x)}\{Y_1 = \nu(y)\} \text{ if } x \sim y,$$

and $p(x, y) = 0$ otherwise. (X_n) is hence a nearest neighbor random walk, that we call the "lift of (Y_n) to T". By definition of $\tau_{y_0}^{hom}$, on the event $\{Y_0 = y_0, \tau_{y_0}^{hom} = n\}$, a path $(Y_0, Y_1, ..., Y_k)$ with $k \leq n$ is homotopically equivalent to the zero-length path (y_0) if and only if $k = n$ (even though one can have $Y_k = y_0$ for some $0 < k < n$). We deduce the following result.

Lemma 2.1. *Writing* $\widehat{\mathbb{P}}_{x_0} := \nu_{x_0}^{-1}(\mathbb{P}_{y_0})$*, we have*

$$\mathbb{P}_{y_0}\{\tau_{y_0}^{hom} < \infty\} = \widehat{\mathbb{P}}_{x_0}\{\tau_{x_0} < \infty\}. \tag{2}$$

Proof. In the canonical space $\Omega = (V_G)^{\mathbb{N}}$, the event $B_n = \{Y_0 = y_0, \tau_{y_0}^{hom} = n\}$ is a disjoint union of cylinder sets, and it is the same for its image through $\nu_{x_0}^{-1}$. On the other hand, the path (y_0) lifts to (x_0). Then, on B_n, the path $\nu_{x_0}^{-1}(Y_0, Y_1, ..., Y_n)$ is homotopically equivalent to (x_0), so $X_n = x_0$. Also notice that $X_k \neq x_0$ if $1 \leq k < n$, because otherwise $(X_0, ..., X_k)$ would be homotopically equivalent to (x_0), and then $(Y_0, Y_1, ..., Y_k)$ would be homotopically equivalent to (y_0), contradicting the definition of $\tau_{y_0}^{hom}$. The statement follows directly from these considerations. □

Remark 2.1. In the case of an homogeneous graph, Coornaert and Papadopoulos in [2] have considered the lift of a random walk in order to establish alternative formulations of the ergodicity of the geodesic flow on the graph.

In the sequel we will assume that (X_n) is transient, that is, that the probability in (2) is strictly less that 1. This condition will allow us to define the "homotopical reduction" of (Y_n) in Section **4**. Before we do it, we will prove some elementary properties of the harmonic measure on trees.

3 Transient random walks on trees

In this section, $T = (V, E)$ is a given tree and (X_n) is some nearest neighbor random walk on it, and we assume that it is transient: $\mathcal{F}(x, x) < 1$ for some (or, equivalently all) $x \in V$. A classic result due to Cartier (see [1]) establishes in that case that, for all $x_0 \in V$, the random walk (X_n) converges $\mathbb{P}_{x_0}$-a.s. to the "boundary at infinity" of T.

The boundary at infinity of T or *hyperbolic boundary*, denoted ∂T, is a compact metric space consisting of all the "ends" of geodesical rays in T. For details on the construction of ∂T, see [1], or Coornaert and Papadopoulos [3], Ch. 1 (also for general facts on hyperbolic spaces). The endpoint ξ of a ray $r = (r_0, r_1, ...)$ is denoted by r_∞, and we shall usually write $r = [r_0, r_\infty)$.

We will keep in mind the following construction of ∂T. Fix a base point $x_0 \in V$. Then

$$\partial T = \{(y_n y_{n+1})_{n\in\mathbb{N}} \in (\overrightarrow{E})^{\mathbb{N}} \colon\, y_0 = x_0,\, y_n \sim y_{n+1},\, y_n \neq y_{n+2} \text{ for all } n \in \mathbb{N}\},$$

endowed with the product topology. Here, $(y_n y_{n+1})_{n\in\mathbb{N}}$ is the end point of the ray $r = (x_0, y_1, y_2, ...)$.

To define a topology on the set $V \cup \partial T$, we consider the Gromov product $(x.y)_{x_0} = \frac{1}{2}(|x - x_0| + |y - x_0| - |x - y|)$ defined on V^2, which in this case is equal the length of the common segment between $[x_0, x]$ and $[x_0, y]$. It extends naturally to $V \cup \partial T$. We define for $y \in V$ the sets $U_{x_0}(y) = \{z \in V \cup \partial T \colon (z.y)_{x_0} = |x_0 - y|\}$, and $O_{x_0}(y) = U_{x_0}(y) \cap \partial T$, which is a cylinder set of ∂T (the topology of ∂T is also induced by the distance $(\xi, \eta) \mapsto e^{-(\xi,\eta)_{x_0}}$). Hence, a neighborhood basis of $\xi = r_\infty \in V \cup \partial T$ is given by the family $U_{x_0}(r_n)$, $n \in \mathbb{N}$. The topology of V is the discrete one and it is an open dense subset in $V \cup \partial T$.

We will denote by $T \cup \partial T$ the set $V \cup \partial T$ endowed with this topology, called the *hyperbolic compactification* of T. Up to homeomorphism, the boundary and the compactification of T are independent of the base point x_0.

Following Cartier [1] the set

$$\Omega' = \{\omega \in \Omega :\ \text{there exists } \ \xi \in \partial T \text{ such that } X_n \to \xi\}$$

is of full measure, that is $\mathbb{P}_{x_0}(\Omega') = 1$, and the random variable $X_\infty = \lim_{n\to\infty} X_n$ is defined $\mathbb{P}_{x_0}$ - a.s. Furthermore, the family of measures $\mathbb{P}_{x_0}\{X_\infty \in \cdot\}$, $x_0 \in V$, defined on ∂T, is *harmonic*:

$$\mathbb{P}_{x_0}\{X_\infty \in \cdot\} = \sum_{x\sim x_0} p(x_0, x)\mathbb{P}_x\{X_\infty \in \cdot\}, \qquad \text{for all } x_0 \in V,$$

and one can identify ∂T with the Martin boundary of the transient chain X_n (see also [7], Ch. 4, Sect. 26).

Lemma 3.1. *Let $x_0, y \in V$ be different and $z \in V$ be the unique vertex such that $z \in [x_0, y]$, $z \sim y$. Then*

$$\mathbb{P}_{x_0}\{X_\infty \in O_{x_0}(y)\} = \mathcal{F}(x_0, y)\frac{1 - \mathcal{F}(y, z)}{1 - \mathcal{F}(z, y)\mathcal{F}(y, z)} .$$

Proof. First, we show that

$$\mathbb{P}_{x_0}\{X_\infty \in O_{x_0}(y)\} = \mathcal{G}(x_0, y)\mathbb{P}_y\{\tau_y = \infty, \tau_z = \infty\}. \tag{3}$$

Consider $F = \{X_\infty \in O_{x_0}(y), X_0 = x_0\}$ and $\omega \in F$. Denote by $N(\omega)$ the smallest $n(\omega)$ such that $X_k(\omega) \notin [x_0, y] \quad$ for all $k \geq n(\omega)$. Then, $X_{N(\omega)-1}(\omega) = y$ $a.s$. The sets $F_n = \{\omega \in F : N(\omega) = n\}$, with $n \geq 2$ define a partition of F. Writing $s = (s_1, \dots s_k) \in V^k$ and

$$W^k = \{s \in V^k : x_0 \sim s_1, s_k \sim y, s_i \sim s_{i+1} \text{ for all } i = 1, \dots k-1\},$$

we have $F_n = \bigcup_{s \in W^{n-2}} \{X_0 = x_0, X_1 = s_1, \dots X_{n-2} = s_{n-2}, X_{n-1} = y, X_k \notin [x_0, y] \text{ for all } k \geq n\}$. From the Markov property we get,

$$\begin{aligned} \mathbb{P}_{x_0}(F_n) = \sum_{s \in W^{n-2}} & \mathbb{P}\{X_k \notin [x_0, y] \quad \text{for all } k \geq n | X_{n-1} = y\} \\ & \times \mathbb{P}_{x_0}\{X_1 = s_1, \dots, X_{n-1} = y\} \\ = & \ \mathbb{P}_y\{ X_k \notin [x_0, y] \text{ for all } k \geq 1\}\mathbb{P}_{x_0}\{X_{n-1} = y\}. \end{aligned}$$

Since $x_0 \neq y$, we deduce that $\mathbb{P}_{x_0}(F) = \mathbb{P}_y\{ X_k \notin [x_0, y] \text{ for all } k \geq 1\}\mathcal{G}(x_0, y)$. On the other hand, as (X_n) is of nearest neighbor type and T is a tree, we get the almost sure equality

$$\{X_0 = y, X_k \notin [x_0, y] \quad \text{for all } k \geq 1\} = \{X_0 = y, \tau_y = \infty, \tau_z = \infty\},$$

and we conclude (3).

Now, we have

$$\mathbb{P}_y\{\tau_y = \infty, \tau_z = \infty\} = 1 - \mathcal{F}(y, y) - \mathcal{F}(y, z) + \mathbb{P}_y\{\tau_y < \infty, \tau_z < \infty\}. \tag{4}$$

On another side,

$$\begin{aligned} \mathbb{P}_y\{\tau_y < \infty, \tau_z < \infty\} &= \mathbb{P}_y\{\tau_y < \tau_z < \infty\} + \mathbb{P}_y\{\tau_z < \tau_y < \infty\} \\ &= \mathbb{E}_y(1_{\{\tau_y < \tau_z\}} 1_{\{\tau_y < \infty\}} \mathbb{E}(1_{\{\tau_y < \tau_z < \infty\}} | \mathcal{F}^{\tau_y})) \\ &\quad + \mathbb{E}_y(1_{\{\tau_z < \tau_y\}} 1_{\{\tau_z < \infty\}} \mathbb{E}(1_{\{\tau_z < \tau_y < \infty\}} | \mathcal{F}^{\tau_z})), \end{aligned}$$

and then, by the strong Markov property we find

$$\mathbb{P}_y\{\tau_y < \infty, \tau_z < \infty\} = \mathbb{P}_y\{\tau_y < \tau_z\}\mathcal{F}(y,z) + \mathbb{P}_y\{\tau_z < \tau_y\}\mathcal{F}(z,y) .$$

Since $\{X_0 = y, \tau_z < \tau_y\} = \{X_0 = y, X_1 = z\}$ a.s., we have $\mathbb{P}_y\{\tau_z < \tau_y\} = p(y,z)$ and then

$$\begin{aligned}\mathbb{P}_y\{\tau_y < \infty, \tau_z < \infty\} &= \mathbb{P}_y\{\tau_y < \tau_z\}\mathcal{F}(y,z) + p(y,z)\mathcal{F}(z,y)\\ &= (1 - \mathbb{P}_y\{\tau_y = \infty, \tau_z = \infty\}\\ &\qquad - p(y,z))\mathcal{F}(y,z) + p(y,z)\mathcal{F}(z,y).\end{aligned}$$

By replacing this expression in relation (4), we obtain

$$\mathbb{P}_y\{\tau_y = \infty, \tau_z = \infty\} = \frac{1 - \mathcal{F}(y,y) + p(y,z)(\mathcal{F}(z,y) - \mathcal{F}(y,z))}{1 + \mathcal{F}(y,z)} . \tag{5}$$

Now, it is proven in [1] that for nearest neighbor random walks on trees the following relation holds

$$p(y,z)\mathcal{G}(y,y) = \frac{1}{\mathcal{F}(y,z)^{-1} - \mathcal{F}(z,y)} . \tag{6}$$

By using (3), (5), (6) and the identities $1 - \mathcal{F}(y,y) = (\mathcal{G}(y,y))^{-1}$ and $\mathcal{F}(x_0,y)\,\mathcal{G}(y,y) = \mathcal{G}(x_0,y)$, we conclude that

$$\begin{aligned}\mathbb{P}_{x_0}\{X_\infty \in O_{x_0}(y)\} =&\mathcal{G}(x_0,y)\left[\frac{1 - \mathcal{F}(y,y) + p(y,z)(\mathcal{F}(z,y) - \mathcal{F}(y,z))}{1 + \mathcal{F}(y,z)}\right]\\ =&\frac{\mathcal{F}(x_0,y)}{1 + \mathcal{F}(y,z)}\left(1 + \frac{\mathcal{F}(y,z)(\mathcal{F}(z,y) - \mathcal{F}(y,z))}{1 - \mathcal{F}(z,y)\mathcal{F}(y,z)}\right)\\ =&\mathcal{F}(x_0,y)\left(\frac{1 - \mathcal{F}(y,z)}{1 - \mathcal{F}(z,y)\mathcal{F}(y,z)}\right) .\end{aligned}$$ □

Now, let us describe the way (X_n) determines X_∞, as n tends to ∞. Define inductively a sequence of random times $k_m \in \mathbb{N}$ and random variables $\widehat{X}_m \in V$ by

- $k_0 = \sup\{k \in \mathbb{N} \colon X_k = X_0\} + 1, \quad \widehat{X}_0 = X_{k_0},$
- $k_{m+1} = \sup\{k \in \mathbb{N} \colon X_k = \widehat{X}_m\} + 1, \quad \widehat{X}_{m+1} = X_{k_{m+1}}, \quad m \geq 1.$

Since (X_n) is transient, the variable k_m is finite, and by an induction argument the variables k_m and $\widehat{X}_m$ are measurable, for every $m \in \mathbb{N}$. By construction, the sequence $(\widehat{X}_0, \widehat{X}_1, ..., \widehat{X}_m, ...)$ is a geodesic ray issued from X_0 with end point $\xi = (\widehat{X}_m \widehat{X}_{m+1})_{m\in\mathbb{N}}$, and

$$X_n \in U_{x_0}(\widehat{X}_m) \quad \text{for all } n \geq k_m.$$

Thus, X_n has a limit point $X_\infty \in \partial T$ equal to ξ.

Now, we set $\widetilde{X}_0 = (X_0 \widehat{X}_0)$ and $\widetilde{X}_m = (\widehat{X}_{m-1} \widehat{X}_m)$ for all $m \geq 1$, and by $\widetilde{\mathbb{P}}^{x_0}$ we mean the probability measure induced on $(\overrightarrow{E})^{\mathbb{N}}$ by $(\widetilde{X_m})$ when $X_0 = x_0$, so $\widetilde{\mathbb{P}}^{x_0} = \mathbb{P}_{x_0}\{X_\infty \in \cdot\}$.

Proposition 3.1. $((\widetilde{X}_m), \widetilde{\mathbb{P}}^{x_0})$ *is a Markov chain on E with initial distribution $\widetilde{p} = (\widetilde{p}_{(xy)})$ and transition matrix $\widetilde{P} = (\widetilde{p}((xy), (zw)))$ given respectively by*

$$\widetilde{p}_{(xy)} = \begin{cases} \mu(x_0 y) & \textit{if } x = x_0 \\ 0 & \textit{otherwise} \end{cases}, \quad \widetilde{p}((xy), (wz)) = \begin{cases} \frac{\mu(yz)}{1-\mu(yx)} & \textit{if } w = y, x \neq z \\ 0 & \textit{otherwise} \end{cases},$$

where for each $(xy) \in \overrightarrow{E}$, $\mu(xy)$ is defined by

$$\mu(xy) = \frac{\mathcal{F}(x, y)(1 - \mathcal{F}(y, x))}{1 - \mathcal{F}(x, y)\mathcal{F}(y, x)}. \tag{7}$$

Proof. For $x \sim y$ it holds $\mathbb{P}_{x_0}$–a.s. that

$$\begin{aligned}\{X_0 = x, \widehat{X}_0 = y\} &= \{\text{ There exists } N \in \mathbb{N}\colon X_{N-1} = x, X_N = y, \\ &\qquad \text{for all } n \geq N \ X_n \neq x\} \\ &= \{X_\infty \in O_x(y)\}.\end{aligned}$$

Thus, by Lemma 3.1 we get $\widetilde{\mathbb{P}}^{x_0}\{\widetilde{X}_0 = (x_0 y)\} = \mathbb{P}_{x_0}\{X^\infty \in O_{x_0}(y)\} = \mu(x_0 y)$, so $\widetilde{P}$ is a stochastic matrix and $\widetilde{p}$ a probability vector. It is clear that $\widetilde{\mathbb{P}}^{x_0}\{\widetilde{X}_{m+1} = (xy) | \widetilde{X}_m = (uv)\} = 0$ except if $y = u$ and $x \neq v$. Denote by $(x_0, y_0, ... y_{m-2}, x, y)$ the reduced path connecting x_0 and y. Then,

$$\{X_0 = x_0, \widetilde{X}_m = (xy)\} = \{\widetilde{X}_0 = (x_0 y_0), \widetilde{X}_1 = (y_0 y_1), ..., \widetilde{X}_m = (xy)\}.$$

We deduce that if $\widetilde{\mathbb{P}}^{x_0}\{\widetilde{X}_m = (xy)\} > 0$ then

$$\begin{aligned}&\widetilde{\mathbb{P}}^{x_0}\{\widetilde{X}_{m+1} = (yz) | \widetilde{X}_m = (xy)\} \\ &\quad = \frac{\widetilde{\mathbb{P}}^{x_0}\{\widetilde{X}_{m+1} = (yz), \widetilde{X}_m = (xy), \widetilde{X}_{m-1} = (y_{m-1}x), ..., \widetilde{X}_0 = (x_0 y_0)\}}{\widetilde{\mathbb{P}}^{x_0}\{\widetilde{X}_m = (xy), \widetilde{X}_{m-1} = (y_{m-1}x), ..., \widetilde{X}_0 = (x_0 y_0)\}}.\end{aligned} \tag{8}$$

Hence

$$\widetilde{\mathbb{P}}^{x_0}\{\widetilde{X}_{m+1} = (yz)|\widetilde{X}_m = (xy)\} =$$
$$\widetilde{\mathbb{P}}^{x_0}\{\widetilde{X}_{m+1}=(yz)|\widetilde{X}_m=(xy),\, \widetilde{X}_{m-1}=(y_{m-1}x),\, ...,\, \widetilde{X}_0 = (x_0y_0)\},$$

proving that $\widetilde{\mathbb{P}}^{x_0}$ is Markovian. Now,

$$\{\widetilde{X}_0 = (x_0y_0), \widetilde{X}_1 = (y_0y_1), \ldots, \widetilde{X}_m = (xy)\} = \{X_0 = x_0\} \cap \{X_\infty \in O_{x_0}(y)\}$$

$\mathbb{P}_{x_0}$-a.s., which, together with (8), yields

$$\widetilde{\mathbb{P}}^{x_0}\{\widetilde{X}_{m+1} = (yz)|\widetilde{X}_m = (xy)\} = \frac{\mathbb{P}_{x_0}\{X_\infty \in O_{x_o}(z)\}}{\mathbb{P}_{x_0}\{X_\infty \in O_{x_o}(y)\}}. \tag{9}$$

Since T is a tree, one has $\mathcal{F}(x_0, z) = \mathcal{F}(x_0, y)\mathcal{F}(y, z)$, and from (9) and Lemma 3.1 we conclude that

$$\widetilde{p}((xy), (yz)) = \frac{\mathcal{F}(y, z)\frac{1-\mathcal{F}(z,y)}{1-\mathcal{F}(y,z)\mathcal{F}(z,y)}}{\frac{1-\mathcal{F}(y,x)}{1-\mathcal{F}(x,y)\mathcal{F}(y,x)}} = \frac{\mu(yz)}{1-\mu(yx)}. \qquad \square$$

Remark 3.1. The previous statement extends to arbitrary trees the result of Dynkin and Malyutov [4] on the harmonic measure on free groups of finite rank. See also Ledrappier [5].

At this point, we can define the homotopical reduction of the nearest neighbor random walk (X_n) on T as the $\overrightarrow{E}$ valued Markov chain $(\widetilde{X}_m)$. Our aim is to extend this definition to general graphs G.

4 The homotopical reduction of (Y_n)

In this paragraph and in the next lemma, Γ is a group acting by the left on a given set S. A matrix A indexed by S is said to be Γ-invariant if $A(x, y) = A(\gamma x, \gamma y)$ for all $x, y \in S$, for all $\gamma \in \Gamma$. If $P = (p(x, y) : x, y \in S)$ is a Γ-invariant stochastic matrix, it is the same for P^n, $\mathcal{G}$, $\mathcal{F}$; the associated Markov chain (Z_n) is said to be Γ-*invariant*.

Let $\overline{S} = \Gamma\backslash S$ be the quotient space and denote by $\nu\colon\ S \to \overline{S}$ the canonical projection. For $x \in S$ we denote by $\overline{x} \in \overline{S}$ its orbit or equivalence class.

Lemma 4.1. *Let (Z_n) be a Γ-invariant Markov chain on S.*

(i) $\overline{Z}_n = \nu \circ X_n$ *defines a Markov chain on* $\overline{S}$ *with transition probabilities given by* $\overline{p}(\overline{x}, \overline{y}) = \sum_{y' \in \overline{y}} p(x, y')$ *for all* $\overline{x}, \overline{y} \in \overline{S}$ *and initial distribution* $\overline{p}_{\overline{x}_0} = \sum_{y \in \overline{x}_0} p_y$ *(these quantities are independent of the choice of* $x \in \overline{x}$*).*

(ii) *Let* $\mathcal{G}$ *and* $\mathcal{F}$ *denote respectively denote the Green kernel and the hitting probabilities of* (Z_n)*, and* $\overline{\mathcal{G}}$ *and* $\overline{\mathcal{F}}$ *the corresponding functions for* $(\overline{Z}_n)$*.*

(a) *If* x *is a recurrent state for* (Z_n)*, then* $\overline{x}$ *is a recurrent state for* $(\overline{Z}_n)$*.*

(b) *If* x *is transient for* (Z_n)*, then* $\overline{x}$ *is recurrent for* $(\overline{Z}_n)$ *if and only if* $\sum_{y' \in \overline{x}} \mathcal{F}(x, y') = \infty$ *.*

(c) *Let* $\overline{y}$ *be a transient state and* $\overline{x} \to \overline{y}$*. Then*

$$\overline{\mathcal{F}}(\overline{x}, \overline{y}) = \frac{\sum_{z \in \overline{x}} \mathcal{F}(x, z)}{1 + \sum_{z' \in \overline{y} \setminus \{y\}} \mathcal{F}(y, z')},$$

$$\overline{\mathcal{F}}(\overline{y}, \overline{y}) = \frac{\sum_{z \in \overline{y}} \mathcal{F}(y, z)}{1 + \sum_{z' \in \overline{y} \setminus \{y\}} \mathcal{F}(y, z')}.$$

(d) *If* y *is transient and* $\overline{x} \to \overline{y}$*, then* $\overline{y}$ *is recurrent if and only if* $\sum_{z \in \overline{y}} \mathcal{F}(x, z) = \infty$*.*

Proof. Part (i) is standard. Let us check (ii).

(a): It is obvious from the relation $\overline{\mathcal{G}}(\overline{x}, \overline{y}) = \sum_{y' \in \overline{y}} \mathcal{G}(x, y')$ for all $\overline{x}, \overline{y} \in \overline{S}$.

(b): We use $\overline{\mathcal{G}}(\overline{x}, \overline{x}) = \sum_{y' \in \overline{x}} \mathcal{G}(x, y') = \mathcal{G}(x, x) + \sum_{y' \in \overline{x} \setminus \{x\}} \mathcal{F}(x, y')\mathcal{G}(y', y')$. Since $\mathcal{G}$ is Γ- invariant, $\overline{\mathcal{G}}(\overline{x}, \overline{x}) = (\mathcal{G}(x, x))(1 + \sum_{y' \in \overline{x} \setminus \{x\}} \mathcal{F}(x, y'))$, and the equivalence follows from $0 < \mathcal{G}(x, x) < \infty$.

(c): Take $\overline{x} \neq \overline{y}$. We have $\overline{\mathcal{G}}(\overline{x}, \overline{y}) = \sum_{z \in \overline{y}} \mathcal{G}(x, z)$, and the second identity in the proof of (b) yields

$$\overline{\mathcal{G}}(\overline{x}, \overline{y}) = \overline{\mathcal{F}}(\overline{x}, \overline{y})(\mathcal{G}(y, y))(1 + \sum_{z' \in \overline{y} \setminus \{y\}} \mathcal{F}(y, z')).$$

Since $0 < \mathcal{G}(y, y) < \infty$, we obtain

$$\overline{\mathcal{F}}(\overline{x}, \overline{y})(1 + \sum_{z' \in \overline{y} \setminus \{y\}} \mathcal{F}(y, z')) = \sum_{z \in \overline{y}} \frac{\mathcal{G}(x, z)}{\mathcal{G}(y, y)} = \sum_{z \in \overline{y}} \frac{\mathcal{G}(x, z)}{\mathcal{G}(z, z)} = \sum_{z \in \overline{y}} \mathcal{F}(x, z) , \quad (10)$$

the latter holding because $x \neq z$, for every $z \in \overline{y}$. The first relation in (c) follows. For the second one, notice that y is transient because $\overline{y}$ is, so $\mathcal{G}(y, y) = \frac{1}{1 - \mathcal{F}(y, y)}$

(a similar relation holds for $\overline{\mathcal{G}}(\overline{y}, \overline{y})$). Therefore, the second identity in the proof of (b) yields

$$\frac{1}{1 - \overline{\mathcal{F}}(\overline{y}, \overline{y})} = \frac{1 + \sum_{z' \in \overline{y} \setminus \{y\}} \mathcal{F}(y, z')}{1 - \mathcal{F}(y, y)},$$

and the asserted relation for $\overline{\mathcal{F}}(\overline{y}, \overline{y})$ is obtained.

(d): It follows from (10). □

Remark 4.1. By induction $\overline{p}^{(n)}(\overline{x}, \overline{y}) = \sum_{y' \in \overline{y}} p^{(n)}(x, y')$. It follows that $C(\overline{x}) = \{\overline{y} \in \overline{S} :$ there exist $y', y'' \in \overline{y}$ with $x \to y'$ and $y'' \to x\}$ and $C(\overline{x}) \supseteq \nu(C(x))$. In particular, (Z_n) irreducible implies $(\overline{Z}_n)$ irreducible.

Let us consider again the random walk (Y_n) on the graph $G = (V_G, E_G)$ as in Section **2**. The lifted random walk (X_n) defined in (1) is easily seen to be invariant for the group Γ of isomorphisms of the covering $\nu : T \to G$. Further, with the notation of Lemma 4.1 one has $\overline{X}_n = Y_n$. However, we will apply Lemma 4.1 in a different way. Indeed, the group Γ also acts on the left on $\overrightarrow{E}_T$ by $\gamma(xy) = (\gamma x\ \gamma y)$ and the quotient space $\Gamma \backslash \overrightarrow{E}_T$ is identified with the set $\overrightarrow{E}_G$ of oriented edges of G by $Orb((xy)) \mapsto \nu((xy)) := (\nu(x)\nu(y))$. We can now state our main result.

Theorem 4.1. *Let (Y_n) be a nearest neighbor random walk on the graph $G = (V_G, E_G)$ and assume that*

$$\mathbb{P}_{y_0}\{\tau_{y_0}^{hom} < \infty\} < 1$$

for some (or all) $y_0 \in V_G$. With each sample path $(Y_0, Y_1, ..., Y_n ...)$ we associate a sequence

$$(Y_0, \widehat{Y}_0, \widehat{Y}_1, ..., \widehat{Y}_m ...)$$

of vertices of V_G by erasing the segments of the original path which are homotopically equivalent to a zero-length path. The mapping $(Y_n)_{n \in \mathbb{N}} \mapsto (\widehat{Y}_m)_{m \in \mathbb{N}}$ is measurable, and if we set

$$\widetilde{Y}_0 := (Y_0 \widehat{Y}_0), \quad \widetilde{Y}_m := (\widehat{Y}_{m-1} \widehat{Y}_m), \quad m \geq 1,$$

then $(\widetilde{Y}_m)$ is a Markov chain with values in $\overrightarrow{E}_G$. Let $\mu(uv)$ be defined as in (7) in terms of the hitting probabilities $\mathcal{F}(u, v)$ of the lifted random (X_n) associated with (Y_n). Then, conditioned to $Y_0 = y_0$, the initial distribution and transition probabilities of $(\widetilde{Y}_m)$ are given respectively by:

$$\widetilde{p}_{(xy)} = \begin{cases} \mu(x_0\, y') & \textit{if } x = x_0 \\ 0 & \textit{otherwise} \end{cases},$$

where $x_0 \in \nu^{-1}(y_0)$ is arbitrary and $y' \in V_T$ satisfies $y' \sim x_0$ and $\nu(y') = y$; and

$$\widetilde{p}((xy),(wz)) = \begin{cases} \frac{\mu(y'z')}{1-\mu(y'x')} & \text{if } w = y, \quad x \neq z \\ 0 & \text{otherwise} \end{cases},$$

where $y' \in \nu^{-1}(y)$ is arbitrary, and $x', z' \in V_T$ satisfy $x', z' \sim y'$ and $\nu(x') = x$, $\nu(z') = z$. This chain $(\widetilde{Y}_m)$ will be called the **homotopical reduction of** *(Y_n).*

Proof. Consider the homotopical reduction $(\widetilde{X}_m)$ of the lift (X_n) of (Y_n), as defined in the previous section. Notice (with the notation therein) that the paths $(X_0, ..., X_{k_1-1})$ and $(\widehat{X}_m, X_{k_{m+1}-1})$, $m \geq 0$, are homotopically equivalent to the zero-length paths (X_0) and $(\widehat{X}_m)$. Define $\widehat{Y}_m := \nu(\widehat{X}_m)$ for all $m \in \mathbb{N}$. Then, $(Y_0, ..., Y_{k_1-1})$ and $(\widehat{Y}_m, Y_{k_{m+1}-1})$, $m \geq 0$, are paths in G homotopically equivalent to the zero-length paths (Y_0) and $(\widehat{Y}_m)$ respectively. On the other hand, as $|X_0 - \widetilde{X}_2| = |\widetilde{X}_m - \widetilde{X}_{m+2}| = 2$, we have $Y_0 \neq \widetilde{Y}_2$, $\widetilde{Y}_m \neq \widetilde{Y}_{m+2}$, and for all m the path $(Y_0, \widehat{Y}_0, \widehat{Y}_1, ..., \widehat{Y}_m)$ is reduced. By construction, $(\widetilde{Y}_m)$ is a measurable transformation of the trajectories of (Y_n). Now, from the properties fulfilled by ν, we get that for each pair $(xy), (wz) \in \overrightarrow{E}_G$ and any $(x'y') \in \overrightarrow{E}_T$ with $(\nu(x')\nu(y')) = (xy)$, there exists a unique $(w'z') \in \nu^{-1}((wz))$ such that $z' \sim y'$. The result follows from this observation, Lemma 4.1 applied to $Z = \widetilde{X}$, and Proposition 3.1. $\square$

5 Some examples

In this section we supply an example, concerning a question put by the referee. We notice that computing explicitly the transition probabilities of $(\widetilde{Y}_m)$ (or equivalently of $(\widetilde{X}_m)$) might not be possible in general. Clearly this should be easier in presence of symmetry. For instance, let X_n be is a simple random walk on a regular tree T^k, (with $deg(x) = k$ for all $x \in V_T$), or on a bi-regular tree $T^{k,l}$ (that is, $deg(x) = k$ or l for all $x \in V_T$, and $x \sim y$ implies $deg(x) \neq deg(y)$). We readily see that in these cases $\widetilde{X}_m$ has associated probabilities $\widetilde{p}_{(xy)} = \frac{1}{deg(x)}$ and $\widetilde{p}((xy)(yz)) = \frac{1}{deg(y)-1}$. The same is valid for the simple random walk on $\mathbb{Z}^d$ (as follows from the case of the regular tree T^{2d}).

In these examples however, symmetry has simplified things too much. Indeed, here we could have obtained the same random walks with reduced trajectories in a more "naive" way: at each step, simply choose with equal probability one neighbor among those being different from the vertex visited at the previous step. (More generally, this could be seen as choosing a neighbor conditioned to not backtracking.)

In general, even in presence of symmetry, the homotopical reduction we have introduced may not coincide with the previous construction. We will now give a simple example of this on a tree.

Let T be a tree with V_T partitioned in two subsets, say $V_T = V_1 \cup V_2$. Assume that each vertex $x \in V_1$ (respectively V_2) has $deg(x) = k_1$ (respectively $deg(x) = k_2$), with $k_i \geq 3$ and $k_1 \neq k_2$. Further, every vertex in V_1 is connected to $k_1 - 1$ vertices in V_1 and to one vertex in V_2. On the other side, every vertex of V_2 is connected to k_2 vertices of V_1.

We consider a simple random walk X_n on T. For the sake of concreteness we shall assume $k_1 = 3$, $k_2 = 4$. Let u, u' be in V_1 and v be in V_2 and such that u' and v are neighbors of u. Let us write $a := \mathcal{F}(u, u')$. By symmetry we have $\mathcal{F}(u, u') = \mathcal{F}(u', u)$ and then from (7) we obtain

$$\mu(uu') = \frac{a - a^2}{1 - a^2}. \tag{11}$$

On the other hand, also by symmetry one has $\mathbb{P}_u\{X_\infty \in \partial T\} = 1 = 2\mu(uu') + \mu(uv)$, and we deduce that

$$\mu(uv) = \frac{1 - a}{1 + a}\ . \tag{12}$$

By similar reasons, it is obtained $\mu(vu) = \frac{1}{4}$.

Now, by the harmonic property of $\mathbb{P}_u\{X_\infty \in \cdot\}$, it holds that

$$\begin{aligned}\mu(uv) = \mathbb{P}_u\{X_\infty \in O_u(v)\} &= \frac{1}{3}\mathbb{P}_v\{X_\infty \in O_v^c(u)\} + 2 \cdot \frac{1}{3}\mathbb{P}_{u'}\{X_\infty \in O_{u'}(v)\} \\ &= \frac{1}{4} + \frac{2}{3}\mu(u'u)\frac{\mu(uv)}{1 - \mu(uu')} = \frac{1}{4} + \frac{2}{3}a\mu(uv). \end{aligned} \tag{13}$$

We have used here the facts that $\mathbb{P}_v\{X_\infty \in O_v^c(u)\} = 3\mathbb{P}_v\{X_\infty \in O_v(u)\}$ and $\frac{\mu(u'u)}{1-\mu(uu')} = \mathcal{F}(u, u') = a$. From (12) and (13) we conclude that a is the unique solution in $]0, 1[$ of $8x^2 - 23x + 9 = 0$ (in particular $a \neq \frac{1}{2}$).

Now we can easily check that the transition probabilities of the homotopical reduction $\widetilde{X}_m$ are different from those of the "naive" reduction. In fact, if they would coincide, we should have

$$\frac{\mu(uv)}{1 - \mu(uu')} = \frac{\mu(u\bar{u})}{1 - \mu(uu')},$$

where $\bar{u} \in V_1$ is the neighbor of u which is different from u' and v, and we deduce that $\mu(uu') = \mu(uv)$. This together with (11) and (12) imply that $a = \frac{1}{2}$, a contradiction.

6 Irreducibility and recurrence

For $u, x, y \in V_T$ let us denote $u <_x y$ if $u \in [x, y]$ and $u \neq y$. The structure of the Markov chain $(\widetilde{X}_m)$ is very simple: for two different edges $(xy), (wz) \in \overrightarrow{E}_T$ one has $(xy) \to (wz)$ if and only if $y <_x w <_x z$, which is also equivalent to $\widetilde{\mathbb{P}}^x_{(xy)}\{\widetilde{X}_m = (wz)\} > 0$, where $m = |x - w| = |y - z|$. Of course, the additional complexity of $(\widetilde{Y}_m)$ comes from the "folding " of some geodesic segments of T into closed reduced paths in G, and it is entirely determined by the action of the group Γ on T. Let Λ denote the limit set of Γ,

$$\Lambda := Adh\{Orb(x)\} \cap \partial T$$

(which is independent of $x \in V_T$). We introduce the notation $rk(\Gamma)$ for the rank of Γ,

$$\overline{x} := \nu(x) \text{ and } (\overline{xy}) := (\nu(x)\nu(y)) \text{ for every } x, y \in V_T.$$

We will show the following result.

Proposition 6.1. *Assume $rk(\Gamma) \geq 2$. The following properties are equivalent*
(a) $(\widetilde{Y}_m)$ is irreducible.
(b) for all $(\overline{xy}) \in \overrightarrow{E}_G$, $(\overline{yx}) \in C(\overline{xy})$.
(c) $\Lambda = \partial T$.

For its proof we will first state some elementary facts. In this purpose we introduce some new notation. We call e the unity element of Γ. For each $x \in V_T$ and $\gamma \in \Gamma\backslash\{e\}$, let $x_\gamma \in V_T$ be the neighbor of x such that $x_\gamma \in [x, \gamma x]$. The vertices γx_γ and γx are adjacent, and one can either have

$$\mathbf{(1)} : \gamma x <_x \gamma x_\gamma, \quad \text{or} \quad \mathbf{(2)} : \gamma x_\gamma <_x \gamma x.$$

We will write for each $x \in V_T$ and for $i = 1, 2$,

$$\Gamma_i^x := \{\gamma \in \Gamma : x_\gamma \text{ satifies the condition } \mathbf{(i)}\}.$$

We remind that the group of graph isomorphisms Γ acts without fixed points, and further, $|x - \gamma x| \geq 3$ for all $x \in V_T$ and $\gamma \in \Gamma\backslash\{e\}$. Also notice that $r <_s u$ and $u <_r w$ imply $r, u <_s w$.

Lemma 6.1.

(a) Let $\gamma \in \Gamma\backslash\{e\}$. Then $\gamma \in \Gamma_2^x$ iff $x_\gamma = x_{\gamma^{-1}}$, and $\Gamma_i^x = (\Gamma_i^x)^{-1}$ for $i = 1, 2$.

(b) For $x \in V_T$, $\gamma \in \Gamma_1^x$, it is verified $\gamma \in \Gamma_1^z$ for all $z \in [x, \gamma x]$.

(c) For $i = 1, 2$, $\gamma \in \Gamma_i^x \implies \gamma^n \in \Gamma_i^x$ for all $n \in \mathbb{Z}\backslash\{0\}$.

Proof.

(a): Since $\gamma \in \Gamma_2^x$ is equivalent to $x_\gamma \in [\gamma^{-1}x, x]$, the statement follows easily.

(b): Consider $z = x_\gamma$. One has $\gamma x \in [x, \gamma z]$. If we had $\gamma \in \Gamma_2^z$, then $\gamma z_\gamma = \gamma x$ and consequently, $z_\gamma = x$ and $x \in [z, \gamma z]$. But also $z \in [x, \gamma z]$, so we would obtain $x = z$, a contradiction. One can repeat this argument with $z' = z_\gamma$, and along the whole segment $[x, \gamma x]$.

(c): From (a) we only need to prove it for $n \in \mathbb{N}$. First consider $\gamma \in \Gamma_1^x$. Notice that

$$(\gamma^n x)_\gamma = \gamma^n x_\gamma \text{ for all } n \in \mathbb{N}\backslash\{0\}. \tag{14}$$

By definition, for $n = 1$ we have that $x_{\gamma^n} = x_\gamma$ and $\gamma^n x_\gamma \notin [x, \gamma^n x]$. If this property is true for some $n \geq 1$, from (14) we get $(\gamma^n x)_\gamma \in [\gamma^n x, \gamma^{n+1}x]$, and then $\gamma^n x \in [x, \gamma^{n+1}x]$. Thus, $x_{\gamma^{n+1}} = x_\gamma$. Since $\gamma x_\gamma \notin [x, \gamma x]$, we have $\gamma^{n+1}x_\gamma \in [\gamma^n x, \gamma^{n+1}x]$, and then $\gamma^{n+1}x_\gamma \notin [x, \gamma^{n+1}x]$, which proves the property for $n+1$. Therefore, $\gamma^{n+1} \in \Gamma_1^x$.

Now, let us consider $\gamma \in \Gamma_2^x$. The equality (14) also holds in this case. Take $m = |x - \gamma x|$, which satisfies $m \geq 5$. Assume for a while that there exists $z \in [x, \gamma x]$ such that $|z - x| \leq k := \lfloor (m-3)/2 \rfloor$ and $\gamma \in \Gamma_1^z$, and take a vertex z with such properties minimizing the distance to x . Let $y \in [x, z]$ be such that $y \sim z$. Since $\gamma^n z_\gamma = (\gamma^n z)_\gamma$, one has $\gamma \in \Gamma_1^{\gamma^n z}$ for every $n \in \mathbb{Z}$, so from (b) and (a) we deduce that $[\gamma^n x, \gamma^n y] \cap [\gamma^l z, \gamma^m z] = \emptyset$ for all $m, n, l \in \mathbb{Z}$.

Now, let $n \in \mathbb{Z}\backslash\{0\}$ and write $\alpha = \gamma^n$. The previous set of equalities imply that $(x.\alpha z)_z = 0$, and $(x.\alpha^{-1}z)_z = 0$. It follows that $(\alpha x.z)_{\alpha z} = 0$. We deduce that $[z, \alpha z] \subseteq [x, \alpha x]$, and since $x_\gamma \in [x, z]$ and $\alpha x_\gamma \in [\alpha x, \alpha z]$, we find that $[x_\gamma, \alpha x_\gamma] \subseteq [x, \alpha x]$, and then $\alpha \in \Gamma_2^x$.

It only remains us to prove that the required z exists. Suppose that this is not true. Let $w \in [x, \gamma x]$ be such that $|x - w| = k$ (k defined as above), and $x_0 = x, x_1, x_2, ..., x_m = \gamma x$ be the reduced path connecting x with γx. Then, $x_1 = x_\gamma$ and $x_{m-1} = \gamma x_1$. Since $\gamma \in \Gamma_2^{x_1}$, we also have $x_{m-2} = \gamma x_2$, and the same reasoning up to $x_k = w$ gives $\gamma x_i = x_{m-i}$, for $i = 0, ..., k+1$. On the other hand, for every $z \in [x, w]$ one has $|z - \gamma z| = |z_\gamma - \gamma z_\gamma| + 2$, which implies that $m = |x - \gamma x| = 2(k+1) + |x_{k+1} - \gamma x_{m-k-1}|$. We deduce that $|x_{k+1} - \gamma x_{m-k-1}| < 3$, which is a contradiction. □

Lemma 6.2. *Assume that $\Lambda = \partial T$ and $rk(\Gamma) \geq 2$. Then, for all $x, y \in V_T$, $x \sim y$, there exists $z \in U_x(y)$ such that $deg(z) \geq 3$.*

Proof. Consider $x \sim y$ and the nonempty set $\triangle_x := \{\gamma \in \Gamma\colon\ y = x_\gamma\}$. If there exists $\gamma \in \Gamma_2^x \cap \triangle_x$, from the proof of Lemma 6.1 (c) there exists $z \in [x, \gamma x]$ such that $\gamma \in \Gamma_z^1$. From Lemma 6.1 (a) we have $z_\gamma \neq z_{\gamma^{-1}}$, and since $\gamma^{-1} \in \Gamma_1^z$, Lemma 6.1 (b) implies that $u = z_{\gamma^{-1}}$ satisfies $\gamma \in \Gamma_1^u$. Thus, the neighbor $w \sim z$ in $[x, z]$ is different from z_γ and $z_{\gamma^{-1}}$.

Now, suppose that $x \sim y$ do not satisfy the assertion and $\triangle_x \subseteq \Gamma_1^x$. Then $U_x(y)$ is a geodesic ray. Assume $\alpha \in \triangle_x$ minimizes $\{|x - \gamma x|\colon \gamma \in \triangle_x\}$. Since $\alpha \in \Gamma_1^x$ and for all $n \in \mathbb{Z}$ the set $[\alpha^n x, \alpha^{n+1}]$ is isomorphic to $[x, \alpha x]$, T is equal to a geodesic and $|x - \alpha^n x| = |n||x - \alpha x|$. Furthermore, it is not hard to deduce that Γ is spanned by α, contradicting $rk(\Gamma) \geq 2$. $\square$

From Remark 4.1, $(\overline{wz}) \in C(\overline{xy})$ if and only if there are $\gamma, \gamma' \in \Gamma$ such that $(xy) \to (\gamma w\ \gamma z)$ and $(wz) \to (\gamma' x\ \gamma' y)$. Since $(\widetilde{X}_m)$ is $\Gamma-$invariant, this yields to $(\gamma w\ \gamma z) \to (\gamma\gamma' x\ \gamma\gamma' y)$, and then $(xy) \to (\hat{\gamma} x\ \hat{\gamma} y)$, with $\hat{\gamma} = \gamma\gamma'$. Therefore, we have $y <_x \gamma w <_x \gamma z <_x \hat{\gamma} x <_x \hat{\gamma} y$ and $\hat{\gamma} \in \Gamma_1^x$, and we can write

$$C(\overline{xy}) = \{(\overline{xy})\} \cup \Big\{ (\overline{wz}) \in \overrightarrow{E}_G\colon \text{ there exist } (w'z') \in (\overline{wz}) \text{ and } \gamma \in \Gamma_1^x \text{ such that } x_\gamma = y \text{ and } w' <_x z' <_x \gamma x \Big\}. \tag{15}$$

Proof of Proposition 6.1.

(b) $\Rightarrow$ (a): Since $y <_x w <_x z$ or $y <_x z <_x w$, the statement follows easily from the description of $C(\overline{xy})$ done in (15).

(a) $\Rightarrow$ (c): For $x, z \in V_T$, let $y \sim x, w \sim z$ be such that $y \in [x, z]$ and $w \notin [x, z]$. Since $(\overline{zw}) \in C(\overline{xy})$, there exist $\alpha, \beta \in \Gamma$ such that $x_\alpha = y$ and $\beta z <_x \beta w <_x \alpha x$ (notice that $\alpha \neq \beta$). Then, $\beta^{-1}\alpha x \in U_x(z)$, and (c) follows by taking for each $\xi \in \partial T$ a sequence $z_n \to \xi$.

(c) $\Rightarrow$ (b): It suffices to show that for each $(\overline{xy}) \in \overrightarrow{E}$ it holds $(\overline{xy}) \to (\overline{yx})$. Take $(xy) \in (\overline{xy})$ and $z \in U_y(x)$ satisfying $deg(z) \geq 3$ and minimizing the distance to x (z exists by Lemma 6.2). Let $z_1, z_2 \notin [x, z]$ be different neighbors of z. By hypothesis we can find $\alpha, \beta \in \Gamma$ verifying $\alpha x \in U_x(z_1)$ and $\beta x \in U_x(z_2)$, and we choose them so that $|u - z| \geq |\alpha x - z|$ for all $u \in U_x(z_1) \cap Orb(x)$ and $|u - z| \geq |\beta x - z|$ for all $u \in U_x(z_2) \cap Orb(x)$. Observe that $y = x_\alpha = x_\beta$.

If α or $\beta \in \Gamma_2^x$, the conclusion is easily obtained. Suppose now that α and β are both in Γ_1^x. From Lemma 6.1(a) and (b), one has $x = y_{\alpha^{-1}} = y_{\beta^{-1}}$ so

$l := (\alpha^{-1}y.\beta^{-1}y)_y \geq 1$. If $l < \min\{|\alpha^{-1}y - y|, |\beta^{-1}y - y|\}$, it follows that $\gamma^{-1}x \notin [y, \gamma^{-1}y]$ for $\gamma = \alpha, \beta$. This implies that $\alpha^{-1}x, \beta^{-1}x \notin [\alpha^{-1}y, \beta^{-1}y]$ and then $x, \alpha\beta^{-1}x \notin [y, \alpha\beta^{-1}y]$. Thus, $(xy) \to (\alpha\beta^{-1}y\ \alpha\beta^{-1}x)$.

If $l = \min\{|\alpha^{-1}y - y|, |\beta^{-1}y - y|\}$ we assume without loss of generality that $l = |\alpha y - y|$. Then, one has $[y, \alpha^{-1}y] \subseteq [y, \beta^{-1}y]$ and $[\alpha^{-1}x, x] \subseteq [\beta^{-1}x, x]$. As $|\alpha^m x - x| = |m||\alpha x - x|$ for every $m \in \mathbb{Z}$, there exists $n \in \mathbb{N}$ such that $\alpha^{-n}x <_x \beta^{-1}x <_x \alpha^{-n-1}x$ (we have $\alpha^{-n}x \neq \beta^{-1}x$ because otherwise $\alpha^n = \beta$ and $[x, \alpha x] \subseteq [x, \beta x]$, which contradicts the choice of α and β). Clearly the following relation holds

$$(\beta^{-1}x.\alpha^{-n-1}x)_x \leq |\beta^{-1}x - x| \ . \tag{16}$$

If the equality holds in (16), we deduce that $(\beta^{-1}y.\alpha^{-n-1}y)_y < |\beta^{-1}y - y|$, which together with $\alpha^{n+1}, \beta \in \Gamma_1^x$, yield to $(\alpha^{-n-1}x\ \alpha^{-n-1}y) \to (\beta^{-1}y\ \beta^{-1}x)$, and we conclude the result. If "$<$" holds in (16), then $\beta^{-1}x \in [\alpha^{-n-1}x, x]$, and from the choice of n we get $\beta^{-1}x \in [\alpha^{-n-1}x, \alpha^{-n}x]$. We also obtain $\beta^{-2}x \notin [\alpha^{-n-1}x, x]$. Since $\beta^{-1}x \in [\beta^{-2}x, x]$, we deduce that

$$|\beta^{-1}x - x| \leq (\alpha^{-n-1}x.\beta^{-2}x)_x \ . \tag{17}$$

Then, we must consider two subcases. In the subcase "$<$" of (17), the vertex $w \in [\alpha^{-n-1}x, x]$ such that $|w - x| = (\alpha^{-n-1}x.\beta^{-2}x)_x$, verifies $deg(w) \geq 3$ and $w \in [\alpha^{-n-1}x, \beta^{-1}x] \subseteq [\alpha^{-n-1}x, \alpha^{-n}x]$. Thus, $\alpha^{n+1}w \in [x, \alpha^{n+1}\beta^{-1}x] \subseteq [x, \alpha x]$. From the choice of α we must have $[x, \alpha^{n+1}\beta^{-1}x] \subseteq [x, z]$, and since $deg(\alpha^{n+1}w) \geq 3$ we get by definition of z that $\alpha^{n+1}w = x$ or $\alpha^{n+1}w = z$. The first relation is not feasible since $|w - x| < |\alpha^{-n-1}x - x|$. The second leads (with the definition of α) to $\alpha^{n+1}w = \alpha^{n+1}\beta^{-1}x$, and then $w = \beta^{-1}x$. This gives $(\beta^{-2}x\ \beta^{-2}y) \to (\alpha^{-n-1}y\ \alpha^{-n-1}x)$, and the result follows.

Finally, if in (17) the equality holds, one has $\alpha^{-n} <_x \beta^{-1} <_x \alpha^{-n-1} <_x \beta^{-2}$ and then $\beta^2\alpha^{-n-1}x \in [x, \beta x]$. From the choice of β we have $[x, \beta^2\alpha^{-n-1}x] \subseteq [x, z]$, so $z \in [\beta^2\alpha^{-n-1}x, \beta x]$. This implies that $\alpha^{n+1}\beta^{-2}z \in [x, \alpha^{n+1}\beta^{-1}x] \subseteq [x, \alpha x]$. From the choice of α, we necessarily have $[x, \alpha^{n+1}\beta^{-1}x] \subseteq [x, z]$, and since $deg(\alpha^{n+1}\beta^{-2}z) = deg(z) \geq 3$ we get $\alpha^{n+1}\beta^{-2}z = x$ or $\alpha^{n+1}\beta^{-2}z = z$. The latter contradicts the fact that $Est(z)$ is trivial (Γ is free). If the first relation holds, we deduce from $\beta^{-1}x \in [\alpha^{-n-1}x, \alpha^{-n}x]$ that $\alpha^{n+1}\beta^{-1}x = z$, and then $\alpha^{n+1}\beta^{-1}\alpha^{n+1}\beta^{-2}z = z$ and the same contradiction arises. This finishes the proof. □

Remark 6.1. *If there exists* $(\overline{xy})$ *such that* $(\overline{yx}) \in C(\overline{xy})$, *then it follows from Lemma 6.1 (c) that* $rk(\Gamma) \geq 2$.

Finally, we have the following result.

Proposition 6.2. *Assume that $(\widetilde{Y}_m)$ is irreducible. Then, it is recurrent if and only if (Y_n) is recurrent.*

Proof. Let $x \in V_T$ be fixed and denote by $\overline{\mathcal{G}}$ the Green function of $(\widetilde{Y}_m)$ and by $\mathcal{F}$ the hitting probabilities of (X_n). By Lemma 4.1, (Y_n) is recurrent if and only if $\sum_{\gamma\in\Gamma} \mathcal{F}(x, \gamma x) = \infty$. We will show that

$$\begin{aligned}
&\overline{\mathcal{G}}((\overline{xy}), (\overline{xy})) = \infty \text{ for every } y \sim x \text{ if and only if } \sum_{\gamma\in\Gamma_1} \mathcal{F}(x, \gamma x) = \infty;\\
&\text{and} \\
&\overline{\mathcal{G}}((\overline{xy}), (\overline{yx})) = \infty \text{ for every } y \sim x \text{ if and only if } \sum_{\gamma\in\Gamma_2} \mathcal{F}(x, \gamma x) = \infty.
\end{aligned} \tag{18}$$

For $y \sim x$, one has $\overline{\mathcal{G}}((\overline{xy}), (\overline{xy})) = \sum_{\gamma\in\Gamma} \mathcal{G}((xy), (\gamma x \gamma y))$. Then, if $n_\gamma = |x - \gamma x|$, we have

$$\overline{\mathcal{G}}((\overline{xy}), (\overline{xy})) = \mathcal{G}((xy)(xy)) + \sum_{\substack{\gamma\in\Gamma_1 \\ x\gamma = y}} \widetilde{\mathbb{P}}^x_{(xy)}\{\widetilde{X}_{n_\gamma} = (\gamma x\ \gamma y)\}.$$

Writing $n = n_\gamma$ and $(y, y_2, ..., y_{n-1}, \gamma x)$ for the reduced path connecting y and γx, we have

$$\widetilde{p}^{(n)}((xy), (\gamma x\ \gamma y)) = \frac{\mu(y_1 y_2)}{1 - \mu(yx)} \cdots \frac{\mu(y_{n-1}\gamma x)}{1 - \mu(y_{n-1} y_{n-2})} \frac{\mu(\gamma x\ \gamma y)}{1 - \mu(\gamma x\ y_{n-1})}$$

$$= \frac{\mu(\gamma x\ \gamma y)}{1 - \mu(yx)} \frac{\mu(y_1 y_2)}{1 - \mu(y_1 x)} \cdots \frac{\mu(y_{n-1}\gamma x)}{1 - \mu(y_{n-1} y_{n-2})} = \frac{\mu(xy)}{1 - \mu(yx)} \cdots \frac{\mu(y_{n-1}\gamma x)}{1 - \mu(y_{n-1} y_{n-2})}\ .$$

Now, one has $\frac{\mu(uv)}{1-\mu(vu)} = \mathcal{F}(u, v)$ for every $u \sim v$, so the previous expression is equal to

$$\mathcal{F}(x, y)\mathcal{F}(y_1, y_2) \cdots \mathcal{F}(y_{n-1}, \gamma x) = \mathcal{F}(x, \gamma x).$$

We deduce that $\sum_{y\sim x} \overline{\mathcal{G}}((\overline{xy}), (\overline{xy})) = deg(x) + \sum_{\gamma\in\Gamma_1} \mathcal{F}(x, \gamma x)$, and we conclude the first equivalence in (18).

Concerning the second equivalence, by using the notation $n_\gamma := |x - \gamma y| = |y - \gamma x|$, we have

$$\overline{\mathcal{G}}((\overline{xy}), (\overline{yx})) = \sum_{\substack{\gamma\in\Gamma_2 \\ y\gamma = y}} \widetilde{\mathbb{P}}^x_{(xy)}\{\widetilde{X}_{n_\gamma} = (\gamma y\ \gamma x)\}.$$

Now, if $(y, z_2, ..., z_{n-1}, \gamma y)$ is the reduced path connecting x and γy, for $n = n^\gamma$ we have

$$\widetilde{p}^{(n)}((xy), (\gamma y \gamma x)) = \frac{\mu(z_1 z_2)}{1-\mu(yx)} \cdots \frac{\mu(z_{n-1}\gamma y)}{1-\mu(z_{n-1}z_{n-2})} \quad \frac{\mu(\gamma y\, \gamma x)}{1-\mu(\gamma y\, z_{n-1})}$$

$$= \frac{\mu(\gamma y\, \gamma x)}{1-\mu(yx)} \frac{\mu(z_1 z_2)}{1-\mu(z_1 x)} \cdots \frac{\mu(z_{n-1}\gamma y)}{1-\mu(z_{n-1}z_{n-2})} = \frac{\mu(yx)}{1-\mu(yx)} \mathcal{F}(y, \gamma y)\ .$$

This expression is equal to

$$\frac{\mu(yx)}{\mu(xy)} \mathcal{F}(xy) \mathcal{F}(y, \gamma y) = \frac{1-\mu(xy)}{\mu(xy)} \mathcal{F}(y, x) \mathcal{F}(x, \gamma y) = \frac{1-\mu(xy)}{\mu(xy)} \mathcal{F}(x, \gamma x).$$

Therefore,

$$\overline{\mathcal{G}}((\overline{xy}), (\overline{yx})) = \frac{1-\mu(xy)}{\mu(xy)} \sum_{\substack{\gamma\in\Gamma_2 \\ y_\gamma = y}} \mathcal{F}(x, \gamma x)\ ,$$

and then

$$\begin{aligned} deg(x) \min_{y\sim x} \left\{ \frac{1-\mu(xy)}{\mu(xy)} \right\} \sum_{\gamma\in\Gamma_2} \mathcal{F}(x, \gamma x) &\leq \sum_{y\sim x} \overline{\mathcal{G}}((\overline{xy}), (\overline{yx})) \\ &\leq deg(x) \max_{y\sim x} \left\{ \frac{1-\mu(xy)}{\mu(xy)} \right\} \sum_{\gamma\in\Gamma_2} \mathcal{F}(x, \gamma x)\ . \end{aligned}$$

This proves the required relation. □

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