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A stabilization law for two semi-infinite interacting strings of characters

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Abstract. We consider a Markov chain that describes the evolution of two interacting strings of symbols. The transitions probalitities of this Markov chain depend only on the rightmost symbols of both strings. The main goal of the present paper is to prove a limit theorem (stabilization law): the distribution of the rightmost symbols converges to some limit correlation function.

Keywords: Markov chain, stabilization law, string of character.

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1 Introduction

A finite string is just a sequence of symbols from a finite alphabet $S = \{1, 2, ..., r\}$. We consider Markov chains with the state space equal to the set of pairs of strings

$$(\bar{x}, \bar{y}), \bar{x} = (x_1, \dots, x_n), \bar{y} = (y_1, \dots, y_m)$$

where x_i, y_i take values from the alphabet S. One-step transition consists in substituting x_n with some word γ and y_m with some word δ. Each of γ and δ can have 0, 1 or 2 symbols. One-step transition probabilities depend only on x_n, γ, y_m, δ.

Some transience and ergodicity conditions for this Markov chain were given in [6].

Note that if $r = 1$ then this reduces to well-known theory of random walks in $\mathbf{Z}_+^2$. The case $r > 1$ is much more complicated.

We consider also semi-infinite strings which are infinite sequences $\alpha = \dots x_{n-1}, x_n$ with values in the same alphabet. Evolution of two interacting

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semi-infinite strings is defined similarly by the same transition probabilities as for finite strings. This Markov process will be called *induced chain*. The induced chain is an uncountable Markov chain, and it describes the behavior of our process far from the origin.

It is natural to introduce invariant measures for the induced chains, and with invariant measures we associate drift vectors that give longtime behavior of the process under consideration. The main problem is that the induced chain may have continuum of invariant measures, so there may exist continuum of drift vectors. We will say that an invariant measure, say with 2-dimensional drift vector $v = (v_1, v_2)$, has the type (σ_1, σ_2), if $\sigma_1 = sign(v_1)$, $\sigma_2 = sign(v_2)$. It was shown [6] that some induced chains may have invariant measures of various types.

The main goal of the present paper is to prove some limit theorems (stabilization laws) for induced chains. Namely, let $n_i(t)$ be the length (or coordinate of the rightmost symbol) of i-th string at time t. Then, under an appropriate choice of the initial distribution, one can show the following:

- as $t \to \infty$, the distribution of symbols inside strings tends to some invariant measure μ.
- as $t \to \infty$
$$\frac{n_i(t)}{t} \to v_i(\mu),$$
where $v_i(\mu)$ are components of the drift vector corresponding to the invariant measure μ.

To obtain the stabilization law we will suppose that for the induced chain all invariant measures have the same type $(-, +)$. In this case it is possible to construct a local Lyapunov function for the induced chain.

Some stabilization laws for one string were given in [2, 3, 5], A stabilization law for two strings, when there is the unique invariant measure of type $(+, +)$, was proved in [5].

The paper is organized as follows. In Section 2 we give definitions and formulate the main result. There we prove also some auxiliary results. In Section 3 we prove the main result.

2 Definitions and Results

Finite strings. Fix a finite set (an alphabet) $S = \{1, 2, \dots r\}$. A finite string is a finite sequence of symbols from S:

$$\alpha = x_1 x_2 \dots x_n, \quad x_i \in S.$$

We denote by $|\alpha|$ the length of the sequence α, and by $\emptyset$ the empty string of length 0. Let $\mathcal{A}$ be the set of all finite strings, including the empty one. The concatenation of two strings $\alpha = x_1x_2 \dots x_n$ and $\beta = y_1y_2 \dots y_n$ is defined by $\alpha\beta = x_1x_2 \dots x_{n+m}$, where $x_{n+1} = y_1, \dots, x_{n+m} = y_n$.

Let $\mathcal{A}^2 = \mathcal{A} \times \mathcal{A}$. An element $\alpha \in \mathcal{A}^2$ is a pair of finite strings

$$\alpha = (\alpha_1, \alpha_2), \quad \alpha_i \in \mathcal{A}.$$

The concatenation of two pairs of finite strings $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ is defined by $\alpha\beta = (\alpha_1\beta_1, \alpha_2\beta_2)$. For any pair $\alpha \in \mathcal{A}^2$ denote by

$$|\alpha| = (|\alpha_1|, |\alpha_2|)$$

the vector of lengths of strings α_1, α_2. We will write $|\alpha| \le \mathbf{c} = (c_1, c_2)$, if $|\alpha_i| \le c_i, i = 1, 2$, where $c_1, c_2 \in \mathbf{Z}^1_+$.

Semi-infinite strings. A semi-infinite string is an infinite sequence $\alpha = \dots y_{n-1}y_n$ of symbols from the alphabet with a specified enumeration. The set of all semi-infinite strings is denoted by $\mathcal{A}_\infty$. The concatenation $\rho\gamma$ of a semi-infinite string $\rho = \dots y_{n-1}y_n \in \mathcal{A}_\infty$ and a finite string $\gamma = x_1x_2 \dots x_m \in \mathcal{A}$ is defined by $\rho\gamma = \dots y_{n-1}y_ny_{n+1} \dots y_{n+m}$, where $y_{n+k} = x_k,\ k = 1, \dots, m$.

Let $\mathcal{A}^2_\infty = \mathcal{A}_\infty \times \mathcal{A}_\infty$. The concatenation $\rho\gamma$ of the pair of semi-infinite strings $\rho = (\rho_1, \rho_2) \in \mathcal{A}^2_\infty$ and the pair of finite strings $\gamma = (\gamma_1, \gamma_2) \in \mathcal{A}^2$ is given by $\rho\gamma = (\rho_1\gamma_1, \rho_2\gamma_2)$.

Define now a discrete time homogeneous Markov chain $\mathcal{L}_\infty$ on the set $\mathcal{A}^2_\infty$. Let $\xi(t) = (\xi_1(t), \xi_2(t))$ denote the state of the chain at time t. Fix some $d \in \mathbf{N}$, and let $\mathbf{d} = (d, d)$. Assume that the transition probabilities $\mathsf{P}\{\xi(t+1) = \beta|\xi(t) = \alpha\} \ne 0$ only if $\beta = \rho\theta$, $\alpha = \rho\gamma$, where $\rho \in \mathcal{A}^2_\infty$, and $\gamma, \theta \in \mathcal{A}^2$ such that $|\gamma| = \mathbf{d}$, $|\theta| \le 2\mathbf{d}$. Assume also that the transition probabilities $\mathsf{P}\{\xi(t+1) = \rho\theta|\xi(t) = \rho\gamma\}$ do not depend on ρ but only on γ, θ. By definition, we put

$$q(\gamma, \theta) = \mathsf{P}\{\xi(t+1) = \rho\theta|\xi(t) = \rho\gamma\}. \tag{1}$$

Invariant measures. Let $\mathbf{Z}_- = \{\dots - 2, -1, 0\}$. By $\mathcal{B}_\infty = S^{\mathbf{Z}_-}$ we denote the infinite product space $\prod_{i=-\infty}^0 S_i$, where $S_i = S$ for all i, endowed with the product topology. An element $\eta \in \mathcal{B}_\infty$ is a (left) semi-infinite sequence $\eta = \dots x_{-1}x_0$, $x_i \in S$. One can consider a semi-infinite string $\alpha = \dots y_{n-1}y_n \in \mathcal{A}_\infty$ as a pair $(n(\alpha), \eta(\alpha))$, where $n(\alpha) \in \mathbf{Z}$ is the position of the right most symbol of the string (or the position of the particle) and $\eta(\alpha)$ is an infinite sequence (the environment on the left of the particle), i.e. a function $\eta(\alpha) : \mathbf{Z}_- \to S$ such that

$$\eta(\alpha) = \dots x_{-1}x_0 \in \mathcal{B}_\infty \text{ where } x_i = y_{i+n}, n = n(\alpha).$$

Let $\mathcal{B}_\infty^2 = \mathcal{B}_\infty \times \mathcal{B}_\infty$. The process $\xi(t) = (\xi_1(t), \xi_2(t))$ can be represented as $\xi(t) = (n(t), \eta(t))$, where

$$n(t) = (n_1(t), n_2(t)) = (n(\xi_1(t)), n(\xi_2(t)) \in \mathbf{Z}^2$$

is the vector of the coordinates of the right most symbols at time t, and

$$\eta(t) = (\eta_1(t), \eta_2(t)) = (\eta(\xi_1(t)), \eta(\xi_2(t))) \in \mathcal{B}_\infty^2$$

is the state of the environment on the left of the particle. Note that the component $\eta(t)$ is Markov chain also.

Let d_0 be any metric on S. Denote by $\mathcal{P}$ the set of all probability measures on $\mathcal{B}_\infty^2$, with the weak topology. $\mathcal{B}_\infty$ is a compact metric space, equipped with the metric $d(\xi, \eta)$ on $\mathcal{B}_\infty$:

$$d(\xi, \eta) = \sum_{i=-\infty}^{0} 2^{-|i|} d_0(x_i, y_i),$$

So, $\mathcal{P}$ is a compact space in the weak topology.

For $\alpha \in \mathcal{A}_\infty(\mathcal{B}_\infty)$ we denote by $[\alpha]_n$ the rightmost substring of length n, i.e. $[\alpha]_n = \gamma$, where $|\gamma| = n$, means that $\alpha = \rho\gamma$ for some $\rho \in \mathcal{A}_\infty$. With the Markov chain $\mathcal{L}_\infty$ we associate the following correlation functions:

$$p_t(\gamma|\rho) = p_t(\gamma_1, \gamma_2|\rho) = \mathsf{P}\{[\xi_1(t)]_{|\gamma_1|} = \gamma_1, [\xi_2(t)]_{|\gamma_2|} = \gamma_2 | \xi(0) = \rho\},$$

where $\gamma \in \mathcal{A}^2$, and $\rho \in \mathcal{A}_\infty^2$ is the initial state of the chain. Note that

$$p_t(\gamma|\rho) = \mathsf{P}\{[\eta_1(t)]_{|\gamma_1|} = \gamma_1, [\eta_2(t)]_{|\gamma_2|} = \gamma_2 \mid \eta(0) = \eta(\rho)\}.$$

The definition of the correlation functions does not depend on $n(0)$ and we will assume that $n(0) = (0, 0)$ unless otherwise stated, i.e. this definition depends only on the component $\eta(t)$, which is a Markov process on $\mathcal{B}_\infty^2$; and when we say about distribution of $\xi(t)$ we mean the distribution of $\eta(t)$. If the initial state of the chain has some distribution $\nu \in \mathcal{P}$ we will write $p_t(\gamma|\nu)$.

For each t correlation functions $p_t(\gamma|\nu)$ uniquely define the measure $\mu(t) \in \mathcal{P}$, where $\mu(0) = \nu$. We will say that $\mu(t)$ is the distribution of the chain $\mathcal{L}_\infty$ at time t.

Definition 1. *Measure $\mu \in \mathcal{P}$ is called invariant for the chain $\mathcal{L}_\infty$ if $\mu(0) = \mu$ implies $\mu(t) = \mu$ for all t.*

Denote by $\mathcal{M}$ the set of all invariant measures for chain $\mathcal{L}_\infty$. Obviously $\mathcal{M}$ is a convex subset of $\mathcal{P}$, closed in the weak topology. So $\mathcal{M}$ is compact, since $\mathcal{P}$ is compact. By Krein-Milman's theorem $\mathcal{M}$ is the closed convex hull of $\mathcal{M}_e$, where $\mathcal{M}_e$ is the set of all extreme points for $\mathcal{M}$.

The following proposition from [6] is a simple consequence of compactness of $\mathcal{P}$.

Proposition 1. *$\mathcal{M}$ is nonempty.*

Let $p_\mu(\gamma)$, $\gamma \in \mathcal{A}^2$, denote the correlation functions corresponding to measure $\mu \in \mathcal{M}$. To each invariant measure μ we associate the drift vector $v(\mu)$

$$v(\mu) = (v_1(\mu), v_2(\mu)) = \sum_{\gamma,\theta} (|\theta| - \mathbf{d}) q(\gamma, \theta) p_\mu(\gamma), \tag{2}$$

where the summation is over all γ, θ such that $|\gamma| = \mathbf{d}$ and $|\theta| \leq 2\mathbf{d}$. With each invariant measure μ we associate also the vector of signs

$$sign(\mu) = (sign(v_1(\mu)), sign(v_2(\mu))),$$

where $sign(v_i(\mu)) = \{-, 0, +\}$. There are 9 of such vectors, and we will speak about invariant measures of the corresponding type. An invariant measure μ is called (σ_1, σ_2)-measure, or measure of type (σ_1, σ_2), if $sign(\mu) = (\sigma_1, \sigma_2)$, where $\sigma_i = \{-, 0, +\}$.

The idea to implement invariant measures to characterise the large time behaviour of strings is based on the following lemma.

Lemma 1. *Suppose that we start with some invariant measure $\mu \in \mathcal{M}_e$. Then*

$$\frac{n(t)}{t} \to v(\mu),$$

P_μ almost surely.

Proof. The proof is straightforward and follows from Birkhoff's ergodic theorem. □

Invariant $(-,+)$**- measure.** In this paper we assume that all invariant measures have the same type $(-,+)$. Introduce the following Lyapunov conditions: there exist $\varepsilon > 0$ and a bounded function $k = k(\rho)$, $0 < k(\rho) \le C$ such that for all initial configurations ρ $\rho = (\rho_1, \rho_2) \in \mathcal{B}^2_\infty$, $n = (n_1, n_2) \in \mathbf{Z}^2$,

$$\mathsf{E}(n_1(k(\rho)) - n_1(0) \mid \xi(0) = (n, \rho)) < -\varepsilon, \tag{3}$$
$$\mathsf{E}(n_2(k(\rho)) - n_2(0) \mid \xi(0) = (n, \rho)) > \varepsilon. \tag{4}$$

The following result was proved in [6].

Theorem 1. *The following statements are equivalent:*

(i) the set $\mathcal{M}$ consists only of invariant measures of type $(-,+)$;

(ii) Lyapunov conditions (3) and (4) hold.

Stabilization law. Below we will assume that $d = 1$, i.e. the transition probabilities $q(\gamma_1, \gamma_2; \theta_1, \theta_2)$ depend only on the rightmost symbols, i.e. $|\gamma_1| = |\gamma_2| = 1$ and $|\theta_i| \le 2$. Assume also that the initial distribution of the Markov chain $\xi(t)$ has the form:

$$\mu_0 = \nu \times \delta_{\rho_2}, \tag{5}$$

where $\nu = \otimes_{i=0}^{-\infty} \nu_i$ is the Bernoulli measure, i.e. ν_i are i.i. distributions on S and δ_{ρ_2} is the Dirac measure, which assigns the mass 1 to $\rho_2 \in \mathcal{B}_\infty$.

Theorem 2. *Suppose that Lyapunov conditions (3) and (4) hold, and the initial distribution μ_0 is given by (5). Then*

(i) for all $\gamma = (\gamma_1, \gamma_2) \in \mathcal{A}^2$

$$\lim_{t\to\infty} p_t(\gamma|\mu_0) = p_\mu(\gamma), \tag{6}$$

where $p_\mu(\gamma)$ are the correlation functions of some invariant measure $\mu \in \mathcal{M}$. Moreover, the convergence in (6) is exponentially fast, i.e. there is some $\chi > 0$, such that

$$\left| p_t(\gamma|\mu_0) - p_\mu(\gamma) \right| < C(\gamma) e^{-\chi t}, \tag{7}$$

for some $C(\gamma)$ depending on γ.

(ii) Let $v_i(\mu)$ be given by (2). Then

$$\frac{n_i(t)}{t} \longrightarrow v_i(\mu) \text{ a.s. when } t \to \infty, \tag{8}$$

where $v_1(\mu) < 0$, $v_2(\mu) > 0$.

Formula for correlation functions of the invariant measure μ. We give a formula for the correlation functions of the measure μ. Let us introduce the following notation.

Define a strictly increasing sequence of random moments $\tau^{(n)}$, by

$$\tau^{(n)} = \min\{t : n_1(t) = n_1(0) - n\},\ \tau^{(0)} = 0 \tag{9}$$

It follows from Lyapunov condition (3) that $\mathsf{P}\{\tau^{(n)} < \infty\} = 1$ for all n. We will say that $\tau^{(n)}$ is the n-th renewal moment. It is evident that for all n the marginal distribution of the first string at the moment $\tau^{(n)}$ coincides with Bernoulli measure ν, and that $\mu^{(1)}(\tau^{(n)}) = \mu^{(1)}(0)$.

Let $\beta,\ \beta_1,\ \beta_2 \in \mathcal{A} \setminus \{\emptyset\}$ and

$$\begin{aligned}
B_\beta &= \{\rho \in \mathcal{B}_\infty :\ [\rho]_{|\beta|} = \beta\},\\
B_{(\beta_1,\beta_2)} &= B_{\beta_1} \times B_{\beta_2}\\
&= \{\rho = (\rho_1, \rho_2) \in \mathcal{B}_\infty^2 :\ [\rho_1]_{|\beta_1|} = \beta_1,\ [\rho_2]_{|\beta_2|} = \beta_2\}.
\end{aligned}$$

Consider the following events:

$$\begin{aligned}
G[t;\beta] &= \{\eta_2(t) \in B_\beta, n_2(t) = n_2(0) + |\beta| - 1\\
&\qquad \text{and for all } k \in (0,t]\ n_2(k) \ge n_2(0)\},\\
F[t;n] &= \{\tau^{(n)} < t, \tau^{(n+1)} \ge t \text{ and for any } k \le n\\
&\qquad \text{there exists } s_k \in (\tau^{(k)}, t] \text{ such that } n_2(s_k) < n_2(\tau^{(k)})\}.
\end{aligned}$$

For all finite strings $\beta,\ \beta_1,\ \beta_2 \in \mathcal{A} \setminus \{\emptyset\}$ and for all symbols $b \in S$ define the quantities:

$$v(\beta, t) = \sum_{n=1}^{t} \mathsf{P}\{\tau^{(n)} = t, \eta_2(t) \in B_\beta \mid \mu_0\}, \tag{10}$$

$$w(b;\beta_1,\beta_2,t) = \sum_{n=1}^{t} \mathsf{P}\{\eta_1(t) \in B_{\beta_1}, G[t;\beta_2], F[t;n] \mid \nu \times \delta_{\rho b}\} \tag{11}$$

By definition $v(\beta, t)$ depends on the initial distribution $\mu_0 = \nu \times \delta_{\rho_2}$, but $w(b;\beta_1,\beta_2,t)$ depends only on ν and b. For sake of simplicity we will not indicate this dependence.

We will show in Section 3.1 (Lemma 4 and Corollary 1) that for all nonempty words $\beta \in \mathcal{A}$ there exists a positive limit

$$\lim_{t\to\infty} v(\beta, t) = v(\beta),$$

and

$$w(b, \gamma_1, \gamma_2) = \sum_{t=1}^{\infty} \sum_{\alpha \in \mathcal{A}} w(b; \gamma_1, \alpha\gamma_2, t) < \infty,$$
$$\tilde{w}(b, \gamma_1, \gamma_2) = \sum_{t=1}^{\infty} w(b; \gamma_1, \gamma_2, t) < \infty$$

Then the following representation for the correlation functions holds.

Theorem 3. *The correlation functions of the invariant measure μ are given by*

$$p_\mu(\gamma) = \sum_{b \in S} v(b) w(b; \gamma_1, \gamma_2) + \sum_{b \in S} \sum_{\substack{\gamma', \gamma'' \in \mathcal{A} \setminus \{\emptyset\}: \\ \gamma_2 = \gamma'\gamma''}} v(\gamma' b) \tilde{w}(b; \gamma_1, \gamma''). \quad (12)$$

3 Proofs

3.1 Proofs of Theorem 2 and Theorem 3

The proof of main results is based on the following lemmas.

Lemma 2. *Let μ_0 be the initial distribution given by (5) and let us fix $\gamma = (\gamma_1, \gamma_2) \in \mathcal{A}^2$. Then*

$$\begin{aligned} p_t(\gamma | \mu_0) = & \sum_{b \in S} \sum_{t_1 + t_2 = t} v(b, t_1) \sum_{\beta \in \mathcal{A}} w(b; \gamma_1, \beta\gamma_2, t_2) \\ & + \sum_{b \in S} \sum_{t_1 + t_2 = t} \sum_{\substack{\gamma', \gamma'' \in \mathcal{A} \setminus \{\emptyset\}: \\ \gamma_2 = \gamma'\gamma''}} v(\gamma' b, t_1) w(b; \gamma_1, \gamma'', t_2) + r(\gamma, t), \end{aligned} \quad (13)$$

where

$$r(\gamma, t) = \sum_{n=0}^{t} \mathsf{P}\{\eta(t) \in B_\gamma \text{ and } F[t; n] \mid \mu_0\}$$

We will prove this lemma in Section 3.2.

Fix some nonempty finite word β. Let

$$\begin{aligned} T_\beta = \min\{\tau^{(n)} & \text{ such that } \eta_2(\tau^{(n)}) \in B_\beta \\ & \text{and for all } t > \tau^{(n)}\ n_2(t) > n_2(\tau^{(n)})\}, \end{aligned} \quad (14)$$

We will say that T_β is a β - *renewal moment*.

Lemma 3. *There exist constants C, δ, such that*

$$\mathsf{P}\{T_\beta \geq t \mid \delta_\rho\} < Ce^{-\delta t}, \tag{15}$$

for every initial state $\rho = (\rho_1, \rho_2)$.

We will prove it in Section 3.3.

Corollary 1. *There exist constants C_1, δ_1 such that*

$$r(\gamma, t) < C_1 e^{-\delta_1 t},$$
$$w(b; \beta_1, \beta_2, t) < C_1 e^{-\delta_1 t},$$

for all $\gamma = (\gamma_1, \gamma_2)$, $\gamma_i, \beta_i \in \mathcal{A} \setminus \{\emptyset\}$, $b \in S$.

The corollary immediately follows from the evident inequalities

$$r(\gamma, t) < \mathsf{P}\{\min_{b \in S} T_b \geq t\},$$
$$w(b; \beta_1, \beta_2, t) < \mathsf{P}\{\min_{b \in S} T_b \geq t\}.$$

Note that the constants in the corollary do not depend on γ_i, β_i and b.

Lemma 4. *Suppose that $\mu_0 = \nu \times \delta_{\rho_2}$ be the initial distribution (see (5)) and $v(\beta, t)$ is given by (10). Then for all nonempty finite $\beta \in \mathcal{A}$ there exists*

$$\lim_{t \to \infty} v(\beta, t) = v(\beta) > 0,$$

where $v(\beta)$ does not depend on ρ_2. Moreover, the convergence is exponentially fast:

$$|v(\beta, t) - v(\beta)| \leq C_\beta e^{-\delta t},$$

for some $\delta, C_\beta > 0$.

We will prove Lemma 4 in Section 3.4.

For all $\gamma_1,\ \gamma_2 \in \mathcal{A} \setminus \{\emptyset\}$ and $b \in S$ introduce

$$w(b, \gamma_1, \gamma_2) = \sum_{t=1}^{\infty} \sum_{\alpha \in \mathcal{A}} w(b; \gamma_1, \alpha\gamma_2, t),$$
$$\tilde{w}(b, \gamma_1, \gamma_2) = \sum_{t=1}^{\infty} w(b; \gamma_1, \gamma_2, t).$$

Thanks to Corollary 1 the above series are convergent. Hence, Lemma 4 gives that

$$\lim_{t\to\infty} p_t(\gamma|\mu_0) = \sum_{b\in S} v(b)w(b,\gamma_1,\gamma_2) + \sum_{b\in S} \sum_{\substack{\gamma',\gamma''\in\mathcal{A}\setminus\{\emptyset\}:\\ \gamma_2=\gamma'\gamma''}} v(\gamma' b)\tilde{w}(b,\gamma_1,\gamma'').$$

and the convergence of correlation functions is exponentially fast. This proves Theorem 2 and Theorem 3. □

3.2 Proof of Lemma 2

Let us fix some $\rho = (\rho_1, \rho_2) \in \mathcal{B}^2_\infty$ and consider the following set of trajectories of the process $\mathcal{L}_\infty$:

$$\Omega_\rho(t) = \{\Gamma = g_0 g_1 \dots g_t : g_0 = [(0,0), \rho]\},$$

where $g_k = [n(k), \eta(k)] = [(n_1(k), n_2(k)), (\eta_1(k), \eta_2(k))]$ is the state of the process at the moment k, and $|\Gamma|$ is the length of trajectory Γ, $|\Gamma| = t + 1$. Let us fix some pair of finite words $\gamma = (\gamma_1, \gamma_2) \in \mathcal{A}^2$. Define subset $\Omega_\rho(\gamma, t) \subset \Omega_\rho(t)$:

$$\Omega_\rho(\gamma, t) = \{\Gamma \in \Omega_\rho(t) : \eta(t) \in B^2_\gamma\}.$$

The correlation functions $p_t(\gamma|\delta_\rho)$ can be given by the formula

$$p_t(\gamma|\delta_\rho) = \sum_{\Gamma\in\Omega_\rho(\gamma,t)} \mathcal{I}(\Gamma), \tag{16}$$

where $\mathcal{I}(\Gamma)$ is the weight of the trajectory Γ:

$$\mathcal{I}(\Gamma) = \prod_{i=1}^{t} \mathsf{P}\{g_{i-1} \to g_i\}.$$

Fix a symbol $b \in S$. With each trajectory $\Gamma \in \Omega_\rho(\gamma, t)$ one can associate a sequence of moments $T_i(\Gamma, b),\ i = 1, \dots, n(\Gamma, b)$ such that

(i) for all $k < T_i$ $n_1(k) > n_1(T_i)$, i.e. T_i is a renewal moment; if $n_1(0) - n_1(T_i) = n$, then we will write $T_i = \tau^{(n)}$;

(ii) $\eta_2(T_i) \in B_b$;

(iii) for all $k \in (T_i, t]$ $n_2(k) > n_2(T_i)$.

If for a fixed Γ it is not possible to define T_1 satisfied $(i) - (iii)$ then we put $n(\Gamma, b) = 1$ and $T_1(\Gamma, b) = 0$. In any case $0 < T_i(\Gamma, b) < t$. Let

$$\sigma(\Gamma) := \max_{b\in S} \max_{1\leq i\leq n(\Gamma,b)} \{T_i(\Gamma, b)\}.$$

Divide the set $\Omega_\rho(\gamma, t)$ into two nonintersecting subsets

$$\Omega_\rho(\gamma, t) = \Omega^1_\rho(\gamma, t) \bigcup \Omega^2_\rho(\gamma, t).$$

where

$$\begin{aligned} \Omega^1_\rho(\gamma, t) &= \{\Gamma : \sigma(\Gamma) \neq 0\}, \\ \Omega^2_\rho(\gamma, t) &= \{\Gamma : \sigma(\Gamma) = 0\}, \end{aligned}$$

and split sum (16) in the following way

$$\sum_{\Gamma\in\Omega_\rho(\gamma,t)} \mathcal{I}(\Gamma) = \sum_{\Gamma\in\Omega^1_\rho(\gamma,t)} \mathcal{I}(\Gamma) + \sum_{\Gamma\in\Omega^2_\rho(\gamma,t)} \mathcal{I}(\Gamma). \tag{17}$$

Then we have

$$\begin{aligned} p_t(\gamma|\mu_0) &= \int_{\mathcal{B}^2_\infty} p_t(\gamma|\delta_\rho) d\mu_0(\rho) \qquad (18) \\ &= \int_{\mathcal{B}^2_\infty} \sum_{\Gamma\in\Omega^1_\rho(\gamma,t)} \mathcal{I}(\Gamma) d\mu_0(\rho) + \int_{\mathcal{B}^2_\infty} \sum_{\Gamma\in\Omega^2_\rho(\gamma,t)} \mathcal{I}(\Gamma) d\mu_0(\rho). \end{aligned}$$

The set $\Omega^2_\rho(\gamma, t)$ consists of trajectories for which event $F[t, n]$ occurs for some $n \geq 0$. It follows from the fact that

$$\sum_{\Gamma\in\Omega^2_\rho(\gamma,t)} \mathcal{I}(\Gamma) = \sum_{n=0}^{t} \mathsf{P}\{\eta(t) \in B^2_\gamma,\, F[t; n] \mid \delta_\rho\},$$

that

$$\int_{\mathcal{B}^2_\infty} \sum_{\Gamma\in\Omega^2_\rho(\gamma,t)} \mathcal{I}(\Gamma) d\mu_0(\rho) = r(\gamma, t).$$

Consider the first sum in the right-hand side of (17). We divide the set of trajectories $\Omega^1_\rho(\gamma, t)$ into two nonintersecting subsets $\Omega^{1,1}_\rho(\gamma, t)$ and $\Omega^{1,2}_\rho(\gamma, t)$, where

$$\begin{aligned} \Omega^{1,1}_\rho(\gamma, t) &= \{\Gamma \in \Omega^1_\rho(\gamma, t) : \; n_2(\sigma(\Gamma)) \leq n_2(t) - |\gamma_2|\}, \\ \Omega^{1,2}_\rho(\gamma, t) &= \{\Gamma \in \Omega^1_\rho(\gamma, t) : \; n_2(\sigma(\Gamma)) > n_2(t) - |\gamma_2|\}. \end{aligned}$$

The sum of weights of all trajectories from $\Omega_{\rho}^{1,1}(\gamma, t)$ can be rewritten as follows

$$\sum_{\Gamma\in\Omega_{\rho}^{1,1}(\gamma,t)} \mathcal{I}(\Gamma) = \sum_{b\in S}\sum_{t_1+t_2=t}\sum_{n=1}^{t}\sum_{\substack{\Gamma:\sigma(\Gamma)=T_{n(\Gamma,b)}\\ =\tau^{(n)}=t_1}} \mathcal{I}(\Gamma), \tag{19}$$

and

$$\sum_{\Gamma:\sigma(\Gamma)=T_{n(\Gamma,b)}=\tau^{(n)}=t_1} \mathcal{I}(\Gamma) \tag{20}$$

$$= \sum_{\Gamma^1\in\Gamma^1([\rho_1]_n,b,t_1)}\sum_{m=1}^{t_2}\sum_{\beta\in\mathcal{A}}\sum_{\Gamma^2\in\Gamma^2(b,[\theta_n\rho_1]_m;\gamma_1,\beta\gamma_2,t_2)} \mathcal{I}(\Gamma^1)\mathcal{I}(\Gamma^2).$$

Here $\Gamma^1([\rho_1]_n, \beta, t)$ is the set of trajectories Γ with length $|\Gamma| = t + 1$ such that

(i) $\eta(0) = (\rho_1, \rho_2)$, $n(0) = (0, 0)$;

(ii) for all $k < t$ $n_1(k) > n_1(t) = -n$, i.e. $\tau^{(n)} = t$;

(iii) $\eta_2(t) \in B_\beta$;

the set $\Gamma^2(b, [\rho_1]_m; \alpha_1, \alpha_2, t)$ consists of trajectories Γ with length $|\Gamma| = t + 1$ such that

(i) $\eta_1(0) = \rho_1$, $\eta_2(0) \in B_b$ and $n(0) = (0, 0)$;

(ii) for all $k \in (0, t]$ $n_2(k) > 0$, $\eta(t) \in B^2_{(\alpha_1,\alpha_2)}$ and $n_2(t) = |\alpha_2| - 1$, i.e. the event $G[t; \alpha_2] \cap (\eta_1(t) \in B_{\alpha_1})$ occurs;

(iii) $m = -\min_{0\le k\le t} n_1(k)$;

(iv) the event $F[t; n]$ occurs for some $n \le m$.

Note that the sum of weights over all trajectories from $\Gamma^1([\rho_1]_n, \beta, t)$ does not depend on $\theta_n\rho_1$. Let

$$I_1([\rho_1]_n, \beta, t) = \sum_{\Gamma\in\Gamma^1([\rho_1]_n,\beta,t)} \mathcal{I}(\Gamma). \tag{21}$$

Note also, that for any ρ_2 such that $\eta_2(0) = \rho_2 b$ the sum of all weights over trajectories from $\Gamma^2(b, [\rho_1]_n; \alpha_1, \alpha_2, t)$ does not depend on ρ_2 and $\theta_n\rho_1$. Let

$$I_2(b, [\rho_1]_n; \alpha_1, \alpha_2, t) = \sum_{\Gamma\in\Gamma^2(b,[\rho_1]_n;\alpha_1,\alpha_2,t)} \mathcal{I}(\Gamma). \tag{22}$$

Together with $v(\beta,t)$ and $w(b;\alpha_1,\alpha_2,t)$ for all $\rho=(\rho_1,\rho_2)\in\mathcal{B}^2_\infty$ we define $v_\rho(\beta,t)$, $w_{\rho_1}(b;\alpha_1,\alpha_2,t)$:

$$\begin{aligned} v_\rho(\beta,t) &= \sum_{n=1}^{t} \mathsf{P}\{\tau^{(n)}=t,\eta_2(t)\in B_\beta \mid \delta_\rho\},\\ w_{\rho_1}(b;\beta_1,\beta_2,t) &= \sum_{n=1}^{t} \mathsf{P}\{\eta_1(t)\in B_{\beta_1}, G[t;\beta_2], F[t;n] \mid \delta_{(\rho_1,\rho_2 b)}\}.\end{aligned}$$

Using formulas (21) and (22) we can represent $v_\rho(\beta,t)$ and $w_{\rho_1}(b;\alpha_1,\alpha_2,t)$ in the following way:

$$v_\rho(\beta,t) = \sum_{n=1}^{t} I_1([\rho_1]_n,\beta,t), \tag{23}$$

$$w_{\rho_1}(b;\alpha_1,\alpha_2,t) = \sum_{n=1}^{t} I_2(b,[\rho_1]_n;\alpha_1,\alpha_2,t). \tag{24}$$

So, we have

$$\begin{aligned} v(\beta,t) &= \int_{\mathcal{B}_\infty} v_\rho(\beta,t)d\nu(\rho_1)\\ &= \sum_{n=1}^{t}\sum_{\alpha\in\mathcal{A}:|\alpha|=n} p_\nu(\alpha)I_1(\alpha,\beta,t) \qquad (25)\\ w(b;\alpha_1,\alpha_2,t) &= \int_{\mathcal{B}_\infty} w_{\rho_1}(b;\alpha_1,\alpha_2,t)d\nu(\rho_1)\\ &= \sum_{n=1}^{t}\sum_{\alpha\in\mathcal{A}:|\alpha|=n} p_\nu(\alpha)I_2(b,\alpha;\alpha_1,\alpha_2,t) \qquad (26)\end{aligned}$$

Integrating the first term in the right-hand side of (17) over initial measure μ_0 we get

$$\int_{\mathcal{B}^2_\infty}\sum_{\Gamma\in\Omega^1_\rho(\gamma,t)} \mathcal{I}(\Gamma)d\mu_0(\rho) \tag{27}$$

$$= \int_{\mathcal{B}^2_\infty}\sum_{\Gamma\in\Omega^{1,1}_\rho(\gamma,t)} \mathcal{I}(\Gamma)d\mu_0(\rho) + \int_{\mathcal{B}^2_\infty}\sum_{\Gamma\in\Omega^{1,2}_\rho(\gamma,t)} \mathcal{I}(\Gamma)d\mu_0(\rho).$$

Consider the first term in the right-hand side of the above formula. By formulas (20), (23), (24) we find

$$\int_{\mathcal{B}_\infty^2} \sum_{\Gamma\in\Omega_\rho^{1,1}(\gamma,t)} \mathcal{I}(\Gamma) d\mu_0(\rho) = \int_{\mathcal{B}_\infty} \sum_{\Gamma\in\Omega_\rho^{1,1}(\gamma,t)} \mathcal{I}(\Gamma) d\nu(\rho_1)$$

$$= \int_{\mathcal{B}_\infty} \sum_{b\in S} \sum_{t_1+t_2=t} \sum_{n=1}^{t_1} I_1([\rho_1]_n, b, t_1) \times$$

$$\times \sum_{m=1}^{t_2} \sum_{\beta\in\mathcal{A}} I_2(b, [\theta_n\rho_1]_m; \gamma_1, \beta\gamma_2, t_2) d\nu(\rho_1)$$

Using formulas (25), (26) we get

$$\int_{\mathcal{B}_\infty^2} \sum_{\Gamma\in\Omega_\rho^{1,1}(\gamma,t)} \mathcal{I}(\Gamma) d\mu_0(\rho) = \sum_{b\in S} \sum_{t_1+t_2=t} v(b, t_1) \sum_{\beta\in\mathcal{A}} w(b; \gamma_1, \beta\gamma_2, t_2).$$

The right-hand side of the above display is equal to the first term in formula (13).

Rewritting in the same way (see (19) and (20)) the sum of weigth of all trajectories from $\Omega_\rho^{1,2}(\gamma, t)$ we have

$$\sum_{\Gamma:\sigma(\Gamma)=T_{n(\Gamma,b)}=\tau^{(n)}=t_1} \mathcal{I}(\Gamma) \tag{28}$$

$$= \sum_{\substack{\gamma',\gamma''\in\mathcal{A}\setminus\{\emptyset\}:\\ \gamma_2=\gamma'\gamma''}} \sum_{\Gamma^1\in\Gamma^1([\rho_1]_n,\gamma b,t_1)} \sum_{m=1}^{t_2} \sum_{\Gamma^2\in\Gamma^2(b,[\theta_n\rho_1]_m;\gamma_1,\gamma'',t_2)} \mathcal{I}(\Gamma^1)\mathcal{I}(\Gamma^2)$$

Using formulas (28), (23), (24) (25), (26), we find

$$\int_{\mathcal{B}_\infty^2} \sum_{\Gamma\in\Omega_\rho^{1,2}(\gamma,t)} \mathcal{I}(\Gamma) d\mu_0(\rho)$$

$$= \sum_{b\in S} \sum_{t_1+t_2=t} \sum_{\substack{\gamma',\gamma''\in\mathcal{A}\setminus\{\emptyset\}:\\ \gamma_2=\gamma'\gamma''}} v(b, t_1) w(b; \gamma_1, \beta\gamma_2, t_2).$$

The right-hand side of the above display is equal to the second term in formula (13). The lemma is proved. □

3.3 Proof of Lemma 3

Let $\mathsf{P}_{n,\rho}\{\cdot\} := \mathsf{P}\{\cdot \mid \eta(0) = \rho,\ n(0) = n\}$. If $n(0) = (0, 0)$, we will write $\mathsf{P}_\rho\{\cdot\}$.

Proposition 2. *There exists* $\varepsilon \in (0, 1)$ *such that*

$$\mathsf{P}_{n,\rho}\{\text{for all } t > 0, n_2(t) > n_2(0)\} \geq \varepsilon > 0, \tag{29}$$

for all $\rho \in \mathcal{B}^2_\infty,\ n \in \mathbf{Z}^2$.

Proof. In order to prove the proposition it is enough to show that there exist $N, \varepsilon > 0$ such that for any $\rho \in \mathcal{B}^2_\infty, n \in \mathbf{Z}^2$

$$\mathsf{P}_{n,\rho}\{\text{for all } t > 0, n_2(t) > n_2(0) - N\} \geq \varepsilon > 0. \tag{30}$$

This probability does not depend on n and without loss of generality we can assume that $n_1(0) = n_2(0) = 0$.

It follows from Lyapunov conditions (3), (4) that there exist posititive δ, C such that for any ρ

$$\mathsf{P}_\rho\{n_1(t) \geq 0\} < Ce^{-\delta t}, \tag{31}$$
$$\mathsf{P}_\rho\{n_2(t) \leq 0\} < Ce^{-\delta t}. \tag{32}$$

Let

$$B_m = \bigcap_{t=m}^{\infty}\{n_2(t) > 0\}.$$

Then

$$\mathsf{P}_\rho\{B_m\} = 1 - \mathsf{P}_\rho\{\bigcup_{t=m}^{\infty}\{n_2(t) \leq 0\}\} > 1 - \sum_{t=m}^{\infty}\mathsf{P}_\rho\{n_2(t) \leq 0\}$$

and by (32)

$$\sum_{t=m}^{\infty}\mathsf{P}_\rho\{n_2(t) \leq 0\} < C_1 e^{-\delta m}.$$

for some constant $C_1 > 0$. Hence, there exist γ, m_0, such that for any $m > m_0$

$$\mathsf{P}_\rho\{B_m\} > \gamma > 0. \tag{33}$$

Thus, we obtain (30). Proposition 2 is proved. □

Now we construct an infinite sequence of nonintersecting $A_i \subset \Omega_\rho$ where Ω_ρ is the set of all trajectories of $\mathcal{L}_\infty$ provided that it starts at point ρ. We do it by induction. Let

$$t_\beta^{(1)} := \min\{t \geq 0 : \text{ (there exists } n : \ \tau^{(n)} = t) \cap (\eta_2(t) \in B_\beta)\}.$$

Put

$$A_1 := \{\omega \in \Omega_\rho : \text{ for all } t > t_\beta^{(1)} \ n_2(t) > n_2(t_\beta^{(1)})\}.$$

With each trajectory $\omega \in \Omega_\rho \setminus A_1$ we associate the moments

$$\begin{aligned} t_1(\omega) &:= \min\{t > 0 : \ n_2(t + t_\beta^{(1)}) < n_2(t_\beta^{(1)})\}, \\ t_\beta^{(2)}(\omega) &:= \min\{t > 0 : \text{ (for all } k < \tau_\beta^{(1)} + t_1 + t \\ &\qquad n_1(k) > n_1(t_\beta^{(1)} + t_1 + t)) \cap (\eta_2(t_\beta^{(1)} + t_1 + t) \in B_\beta)\}. \end{aligned}$$

Using these moments we define

$$\begin{aligned} A_2 := \{\omega \in \Omega_\rho \setminus A_1 \text{: for all } t > t_\beta^{(1)} + t_1 + t_\beta^{(2)} \\ n_2(t) > n_2(t_\beta^{(1)} + t_1 + t_\beta^{(2)})\}, \end{aligned}$$

and so on. Suppose now that we have constructed the subsets

$$A_1, A_2, \ldots, A_n.$$

For all $\omega \in \Omega_\rho \setminus (\cup_{i=1}^n A_i)$ put by definition

$$\begin{aligned} t_n(\omega) &:= \min\{t > 0 : \\ &\qquad n_2(t + t_\beta^{(1)} + \cdots + t_\beta^{(n)}) < n_2(t_\beta^{(1)} + \cdots + t_\beta^{(n)})\}, \\ t_\beta^{(n+1)}(\omega) &:= \min\{t > 0 : \text{ for all } k < t_\beta^{(1)} + t_1 + \cdots + t_\beta^{(n)} + t \\ &\qquad n_1(k) > n_1(t_\beta^{(1)} + t_1 + \cdots + t_\beta^{(n)} + t_n + t), \\ &\qquad \text{and } \eta_2(t_\beta^{(1)} + t_1 + \cdots + t_\beta^{(n)} + t_n + t) \in B_\beta\}, \end{aligned}$$

and introduce $A_{n+1} \subset \Omega_\rho \setminus (\cup_{i=1}^n A_i)$ as follows

$$\begin{aligned} A_{n+1} := \{\omega \in \Omega_\rho \setminus (\cup_{i=1}^n A_i) \text{ such that} \\ \text{for all } t > t_\beta^{(1)} + t_1 + \cdots + t_\beta^{(n)} + t_n + t_\beta^{(n+1)} \\ n_2(t) > n_2(t_\beta^{(1)} + t_1 + \cdots + t_\beta^{(n)} + t_n + t_\beta^{(n+1)})\} \end{aligned}$$

It follows from the above construction that for all $\omega \in A_n$

$$T_\beta(\omega) = t_\beta^{(1)} + t_1 + t_\beta^{(2)} + \cdots + t_\beta^{(n)} + t_n + t_\beta^{(n+1)}$$

where T_β is *β-renewal moment*. By (29) we have

$$\mathsf{P}\{A_n\} < (1-\varepsilon)^{n-1}.$$

In order to prove the exponential estimation for β - renewal moment we need the following proposition.

Proposition 3. *There exist positive C, δ such that*

$$\mathsf{P}_{n,\rho}\{t_1 > t\} < Ce^{-\delta t}, \tag{34}$$

$$\mathsf{P}_{n,\rho}\{t_\beta^{(1)} > t\} < Ce^{-\delta t}, \tag{35}$$

where constants C, δ do not depend on ρ and n.

Proof. Inequality (34) follows from (32). To prove (35) it is sufficient to show that for λ small enough

$$\mathsf{E}_\rho(e^{\lambda t_\beta^{(1)}}) < c(\lambda), \tag{36}$$

where $c(\lambda) \to 1$ as $\lambda \to 0$. Let

$$M_n = \{\text{for all } i < n \ \eta_2(\tau^{(i)}) \notin B_\beta,\ \eta_2(\tau^{(n)}) \in B_\beta\}.$$

For $\omega \in M_n$, $t_\beta^{(1)}(\omega) = \tau^{(n)}(\omega)$ and we have

$$\mathsf{E}_\rho\, e^{\lambda t_\beta^{(1)}} = \sum_{n=1}^{\infty} \mathsf{E}_\rho(e^{\lambda t_\beta^{(1)}} I_{M_n}) = \sum_{n=1}^{\infty} \mathsf{E}_\rho(e^{\lambda \tau^{(n)}} I_{M_n}). \tag{37}$$

We need to prove the convergence of the above series. Note that there is $\epsilon > 0$ such that for any n,

$$\mathsf{P}_\rho\{\eta_2(\tau^{(n)}) \in B_\beta\} > \epsilon,$$

uniformly in $\rho \in \mathcal{B}^2_\infty$. Hence,

$$\mathsf{P}_\rho\{M_n\} < (1-\epsilon)^{n-1}. \tag{38}$$

It follows from (31) that

$$\mathsf{P}_\rho\{\tau^{(1)} > t\} < Ce^{-\delta t},$$

so, for λ small enough

$$\mathsf{E}_\rho\, e^{\lambda \tau^{(1)}} \leq c_1(\lambda),$$

where $c_1(\lambda) \to 1$, as $\lambda \to 0$. By (38) we have that for some $q \in (0, 1)$, $\lambda > 0$

$$\mathsf{E}_\rho\, e^{\lambda\tau^{(1)}} I_{M_1} < q, \quad \mathsf{E}_\rho\, e^{\lambda\tau^{(1)}} I_{\overline{M}_1} < q.$$

Let $\rho_n = \eta(\tau_n)$. Then one can write

$$\begin{aligned}\mathsf{E}_\rho(e^{\lambda\tau^{(n)}} I_{M_n}) &= \mathsf{E}_\rho(e^{\lambda\sum_{i=1}^{n}(\tau^{(i)}-\tau^{(i-1)})} I_{M_n})\\ &= \mathsf{E}_\rho(\mathsf{E}_\rho(e^{\lambda(\tau^{(n)}-\tau^{(n-1)})} e^{\lambda\sum_{i=1}^{n-1}(\tau^{(i)}-\tau^{(i-1)})} I_{M_n} \mid \rho_{n-1}))\\ &< q\, \mathsf{E}_\rho(e^{\lambda\sum_{i=1}^{n-1}(\tau^{(i)}-\tau^{(i-1)})} I_{M_n}),\end{aligned}$$

which yields immediately

$$\mathsf{E}_\rho(e^{\lambda\tau^{(n)}} I_{M_n}) < q^n.$$

So, series (37) is convergent for sufficiently small λ . Proposition 3 is proved. □

Corollary 2. *For any i*

$$\mathsf{P}_\rho\{t_i > t\} < Ce^{-\delta t} \text{ and } \mathsf{P}\{t_\beta^{(i)} > t\} < Ce^{-\delta t}.$$

To complete the proof of the lemma we need the following fact.

Proposition 4. *There exists $\lambda > 0$ such that*

$$\mathsf{E}_\rho\, e^{\lambda T_\beta} < c_2(\lambda), \tag{39}$$

where the function $c_2(\lambda)$ does not depend on ρ.

Proof. Define

$$\begin{aligned}\tau(t) &= \max\{\tau^{(n)} : \tau^{(n)} \leq t\},\\ b(t) &= n_1(t) - n_1(\tau(t))\end{aligned}$$

and let $\phi(\rho, \lambda) = \mathsf{E}_\rho\, e^{\lambda T_\beta}$. This exponential moment can be represented as

$$\mathsf{E}_\rho\, e^{\lambda T_\beta} = \sum_{n=1}^{\infty}(\mathsf{E}_\rho\, e^{\lambda T_\beta} I_{A_n}).$$

Consider functions $\phi_n(\rho, \lambda) = \mathsf{E}_\rho(e^{\lambda T_\beta} I_{A_n})$. Note that

$$\phi_n(\rho, 0) = \mathsf{P}_\rho\{A_n\} < (1 - \varepsilon)^{n-1},$$

where ε is from Proposition 2. It is sufficient to prove that there exist $q \in (0,1)$, $\lambda_0 > 0$ such that for any positive $\lambda < \lambda_0$

$$\phi_n(\rho, \lambda) < q^{n-1} c(\lambda), \tag{40}$$

uniformly in $\rho \in \mathcal{B}^2_\infty$. Suppose that the initial state is $\rho' = (\rho_1, \rho_2\beta)$. In this case $t^{(1)}_\beta = 0$. Let us estimate $\phi_2(\rho', \lambda)$. Note that

$$\phi_2(\rho', \lambda) < \mathsf{E}_{\rho'}(e^{\lambda t_1 + \lambda t^{(2)}_\beta} I_{\Omega \setminus A_1}).$$

We have

$$\begin{aligned}
&\mathsf{E}_{\rho'}(e^{\lambda t_1 + \lambda t^{(2)}_\beta} I_{\Omega \setminus A_1}) \\
&= \sum_{s_1, s_2} \sum_{\rho} \sum_{n} e^{\lambda s_1 + \lambda s_2} \mathsf{P}_{\rho'}\{t_1 = s_1, \eta(s_1) = \rho, b(s_1) = n, t^{(2)}_\beta = s_2\} \\
&= \sum_{s_1, s_2} \sum_{\rho} \sum_{n} e^{\lambda s_1} \mathsf{P}_{\rho'}\{t_1 = s_1, \eta(s_1) = \rho, b(s_1) = n\} \\
&\qquad \times e^{\lambda s_2} \mathsf{P}_{\rho'}\{t^{(2)}_\beta = s_2 \mid t_1 = s_1, \eta(s_1) = \rho, b(s_1) = n\} \\
&= \sum_{s_1} \sum_{\rho} \sum_{n} e^{\lambda s_1} \mathsf{P}_{\rho'}\{t_1 = s_1, \eta(s_1) = \rho, b(s_1) = n\} \\
&\qquad \times \sum_{s_2} e^{\lambda s_2} \mathsf{P}_{\rho'}\{t^{(2)}_\beta = s_2 \mid \eta(s_1) = \rho, b(s_1) = n\},
\end{aligned}$$

where

$$\sum_{s_2} e^{\lambda s_2} \mathsf{P}_{\rho'}\{t^{(2)}_\beta = s_2 \mid \eta(s_1) = \rho, b(s_1) = n\} \le c_1^n(\lambda) c(\lambda).$$

Here $c_1(\lambda)$, $c(\lambda)$ are from the proof of Proposition 3. Thus,

$$\begin{aligned}
&\mathsf{E}_{\rho'}(e^{\lambda t_1 + \lambda t^{(2)}_\beta} I_{\Omega \setminus A_1}) \\
&\le \sum_{s} \sum_{n} e^{\lambda s} \mathsf{P}_{\rho'}\{t_1 = s, b(s) = n\} c_1^n(\lambda) c(\lambda).
\end{aligned} \tag{41}$$

By Cauchy-Schwartz inequality

$$\mathsf{P}_{\rho'}\{t_1 = s, b(s) = n\} \le \sqrt{\mathsf{P}_{\rho'}\{t_1 = s\}\, \mathsf{P}_{\rho'}\{b(s) = n\}}. \tag{42}$$

By Proposition 3 we have

$$\mathsf{P}_{\rho'}\{b(t_1) \geq n\} < \mathsf{P}_{\rho'}\{t_1 \geq n\} < C^{-\delta n}. \tag{43}$$

Combining (42) and (43) with Proposition 3, we get that there exists $\lambda > 0$ such that series (41) converges. Since

$$\sup_{\rho'} |\phi_2(\rho', \lambda) - \phi_2(\rho', 0)| \to 0, \text{ when } \lambda \to 0,$$

there exist λ_0 and $q \in (0, 1)$ such that

$$\mathsf{E}_{\rho'}(e^{\lambda_0 T_\beta} I_{\Omega \setminus A_1}) < q. \tag{44}$$

Now we can estimate $\phi_n(\rho, \lambda)$. Indeed,

$$\phi_n(\rho, \lambda) < \mathsf{E}_\rho(e^{\lambda T_\beta} I_{\Omega \setminus \cup_{i=1}^{n-1} A_i}).$$

Let $\rho_n = \eta(t_\beta^{(1)} + t_1 + \cdots + \lambda t_\beta^{(n)})$. For any $\lambda \leq \lambda_0$ we have

$$\begin{aligned}
&\mathsf{E}_\rho(e^{\lambda T_\beta} I_{\Omega \setminus \cup_{i=1}^{n-1} A_i}) = \mathsf{E}_\rho(e^{\lambda(t_\beta^{(1)} + t_1 + \cdots + t_\beta^{(n)})} I_{\Omega \setminus \cup_{i=1}^{n-1} A_i}) \\
&= \mathsf{E}_\rho(\mathsf{E}_\rho(e^{\lambda(t_\beta^{(1)} + t_1 + \cdots + t_\beta^{(n)})} I_{\Omega \setminus \cup_{i=1}^{n-1} A_i} \mid \rho_{n-1})) \\
&= \mathsf{E}_\rho(e^{\lambda(t_\beta^{(1)} + t_1 + \cdots + t_\beta^{(n-1)})} I_{\Omega \setminus \cup_{i=1}^{n-1} A_i} \mathsf{E}_\rho(e^{\lambda t_{n-1} + \lambda t_\beta^{(n)}} \mid \rho_{n-1})) \\
&< q\, \mathsf{E}_\rho(e^{\lambda(t_\beta^{(1)} + t_1 + \cdots + t_\beta^{(n-1)})} I_{\Omega \setminus \cup_{i=1}^{n-1} A_i}) \\
&< q\, \mathsf{E}_\rho(e^{\lambda(t_\beta^{(1)} + t_1 + \cdots + t_\beta^{(n-1)})} I_{\Omega \setminus \cup_{i=1}^{n-2} A_i})
\end{aligned}$$

which yields immediately

$$\mathsf{E}_\rho(e^{\lambda T_\beta} I_{\Omega \setminus \cup_{i=1}^{n-1} A_i}) < q^{n-1}\, \mathsf{E}_\rho\, e^{\lambda t_\beta^{(1)}} \overset{(36)}{\leq} q^{n-1} c(\lambda).$$

So, the formula (40) holds. The Proposition 4 is proved. □

3.4 Proof of Lemma 4

Define an increasing sequence of random moments:

$$\begin{aligned}
\tau_\beta^{(1)} &= \min\{\tau^{(k)} :\ \eta_2(\tau^{(k)}) \in B_\beta\}, \\
\tau_\beta^{(n)} &= \min\{\tau^{(k)} > \tau_\beta^{(n-1)} :\ \eta_2(\tau^{(k)}) \in B_\beta\}.
\end{aligned}$$

Let $n(0) = (0, 0)$ and

$$F_\beta[t; n] \;=\; \{\tau_\beta^{(n)} < t,\, \tau_\beta^{(n+1)} = t \text{ and for any } k \le n \text{ there exists } s_k \in (\tau^{(k)}, t] \text{ such that } n_2(s_k) < n_2(\tau^{(k)})\}.$$

Consider the following probability:

$$f_\beta(t) \;=\; \sum_{n=0}^{t} \mathsf{P}\{F_\beta[t; n] \text{ and for all } k \in (0, t]\; n_2(k) > n_2(0) \mid \nu \times \delta_{\rho\beta}\}.$$

Proposition 5. *For all* $\beta \in \mathcal{A} \setminus \{\emptyset\}$

$$v(\beta, t) = \sum_{t_1+t_2=t} v(\beta, t_1) f_\beta(t_2) + r(\beta, t), \tag{45}$$

where

$$r(\beta, t) = \sum_{n=0}^{t} \mathsf{P}\{F_\beta[t; n] \mid \mu_0\}. \tag{46}$$

Proof. Let $\rho \in \mathcal{B}_\infty^2$ and let $\Upsilon_\rho(\beta, t)$ be the following set of trajectories.

$$\Upsilon_\rho(\beta, t) \;=\; \{\Gamma = g_0 g_1 \dots g_t \text{ such that } \eta(0) = \rho,\, \eta_2(t) \in B_\beta \text{ and } n_1(t) < n_1(k) \text{ for all } k < t\}.$$

So

$$v_\rho(\beta, t) = \sum_{\Gamma \in \Upsilon_\rho(\beta,t)} \mathcal{I}(\Gamma). \tag{47}$$

For any $\Gamma \in \Upsilon_\rho(\beta, t)$ define the moment $\sigma(\Gamma)$ as the maximal time T such that:

(i) for all $k < T$ $n_1(k) > n_1(T)$;

(ii) for all $k \in (T, t]$ $n_2(k) \ge n_2(T)$;

(iii) $\eta_2(T) \in B_\beta$.

If there is no T with the above properties put $\sigma(\Gamma) = 0$. Split the sum (47)

$$\sum_{\Gamma \in \Upsilon^1_\rho(\beta,t)} \mathcal{I}(\Gamma) + \sum_{\Gamma \in \Upsilon^2_\rho(\beta,t)} \mathcal{I}(\Gamma), \tag{48}$$

where

$$\Upsilon^1_\rho(\beta, t) = \{\Gamma : \sigma(\Gamma) \neq 0\} \text{ and } \Upsilon^2_\rho(\beta, t) = \{\Gamma : \sigma(\Gamma) = 0\}.$$

Consider the second sum in (48). The set $\Upsilon^2_\rho(\beta, t)$ consists of all trajectories for which event $F_\beta[t; n]$, $n \geq 0$ occurs. So we have

$$\sum_{\Gamma \in \Upsilon^2_\rho(\beta,t)} \mathcal{I}(\Gamma) = \sum_{n=0}^{t} \mathsf{P}\{F_\beta[t; n] \mid \delta_\rho\}.$$

and, by integrating over initial distribution μ_0, we come to

$$r(\beta, t) = \int \sum_{\Gamma \in \Upsilon^2_\rho(\beta,t)} \mathcal{I}(\Gamma) d\mu_0(\rho). \tag{49}$$

Remark 1. It follows from the definition of T_β (see (14)) that

$$r(\beta, t) < \mathsf{P}\{T_\beta \geq t \mid \mu_0\}.$$

By Lemma 3 $r(\beta, t)$ converges to zero for all nonempty finite words β as $t \to \infty$.

Denote by $\Gamma_\rho([\rho_1]_n, \beta, t)$, $\rho = (\rho_1, \rho_2) \in \mathcal{B}^2_\infty$, $\beta \in \mathcal{A}$, $n \in \mathbf{N}$ the set of all trajectories satisfying the following properties:

(i) $\eta(0) = (\rho_1, \rho_2\beta)$;

(ii) $\eta_1(t) = \theta_n \rho_1$, $\eta_2(t) \in B_\beta$;

(iii) $n_1(k) > n_1(t)$ for all $k < t$;

(iv) $n_2(k) > n_2(0)$ for all $k \in (0, t]$;

(v) for all $t' < t$, for which (ii), (iii) hold, there exists k' such that $t' + k' < t$ and $n_2(t' + k') < n_2(t')$; in another words, at the moment $t' + k'$ the word β is destroyed.

Note that the sum of weights over trajectories from the set $\Gamma_\rho([\rho_1]_n, \beta, t)$, does not depend on ρ_2. Let

$$f_{[\rho_1]_n,\beta,t} = \sum_{\Gamma\in\Gamma_\rho([\rho_1]_n,\beta,t)} \mathcal{I}(\Gamma) \tag{50}$$

and

$$f_{\rho_1,\beta}(t) = \sum_{n=1}^{t} f_{[\rho_1]_n,\beta}(t). \tag{51}$$

Then one can write

$$f_\beta(t) = \int_{\mathcal{B}_\infty} f_{\rho_1,\beta}(t)d\nu(\rho_1). \tag{52}$$

For the first sum in (48) we have

$$\begin{aligned}
\sum_{\Gamma\in\Upsilon_\rho^1(\beta,t)} \mathcal{I}(\Gamma) &= \sum_{t_1=1}^{t}\sum_{n=1}^{t_1} \sum_{\Gamma:\sigma(\Gamma)=\tau^{(n)}=t_1} \mathcal{I}(\Gamma) \\
&= \sum_{t_1+t_2=t}\sum_{n=1}^{t_1} \sum_{\substack{\Gamma\in\Upsilon_\rho(\beta,t_1):\\ n_1(0)-n_1(t_1)=n}} \mathcal{I}(\Gamma) f_{\theta_n\rho_1,\beta}(t_2) \\
&= \sum_{t_1+t_2=t}\sum_{n=1}^{t_1} I_\Upsilon([\rho_1]_n, t_1) f_{\theta_n\rho_1,\beta}(t_2),
\end{aligned} \tag{53}$$

where

$$I_\Upsilon([\rho_1]_n, t) = \sum_{\substack{\Gamma\in\Upsilon_\rho(\beta,t_1):\\ n_1(0)-n_1(t_1)=n}} \mathcal{I}(\Gamma).$$

Using formulas (48), (49), (53) and integrating over initial distribution μ_0, we get

$$\begin{aligned}
v(\beta,t) &= \int_{\mathcal{B}_\infty^2} \sum_{\Gamma\in\Upsilon_\rho(\beta,t)} \mathcal{I}(\Gamma)d\mu_0(\rho) = \int_{\mathcal{B}_\infty} \sum_{\Gamma\in\Upsilon_\rho(\beta,t)} \mathcal{I}(\Gamma)d\nu(\rho_1) \\
&= \int_{\mathcal{B}_\infty} \sum_{t_1+t_2=t}\sum_{n=1}^{t_1} I_\Upsilon([\rho_1]_n, t_1) f_{\theta_n\rho_1,\beta}(t_2)d\nu(\rho_1) + r(\beta,t) \\
&= \sum_{t_1+t_2=t}\sum_{n=1}^{t_1}\sum_{\alpha:|\alpha|=n} p_\nu(\alpha) I_\Upsilon(\alpha,t_1) \int_{\mathcal{B}_\infty} f_{\theta_n\rho_1,\beta}(t_2)d\nu(\theta_n\rho_1) \\
&\quad + r(\beta,t) = \sum_{t_1+t_2=t} v(\beta,t_1) f_\beta(t_2) + r(\beta,t).
\end{aligned}$$

This completes the proof of (45). □

Proposition 6. *For any* $\beta \in \mathcal{A} \setminus \{\emptyset\}$

$$\sum_{t=1}^{\infty} f_\beta(t) = 1.$$

Proof. Consider measures μ such that

$$\mu\{\rho = (\rho_1, \rho_2) : \ \rho_2 \in B_\beta\} = 1. \tag{54}$$

Denote

$$p(\mu^{(1)}, \beta) = \mathsf{P}\{\text{for all } t, \ n_2(t) > n_2(0) \mid \mu\}. \tag{55}$$

It is obvious that this probability depends only on the marginal measure $\mu^{(1)}$ of the first string and on the fixed rightmost finite word β which belongs to the second string. For $\mu^{(1)} = \delta_{\rho_1}$, $\rho_1 \in \mathcal{B}_\infty$, we will write $p(\rho_1, \beta)$ instead of $p(\mu^{(1)}, \beta)$. Under condition (54) we have

$$p(\mu^{(1)}, \beta) = \int_{\mathcal{B}^2_\infty} p(\rho_1, \beta) d\mu(\rho).$$

Let $\eta(0) = (\rho_1, \rho_2\beta)$, $\rho = (\rho_1, \rho_2) \in \mathcal{B}^2_\infty$ and

$$A(\rho_1, \rho_2\beta) = \{\text{for all } t \ n_2(t) > n_2(0)\}.$$

Now we define function $\sigma(\Gamma)$ on the set $A(\rho_1, \rho_2\beta)$. For any trajectory $\Gamma = g_0 g_1 \dots g_t \dots \in A(\rho_1, \rho_2\beta)$ put $\sigma(\Gamma)$ be equal to the minimal moment T satisfying the conditions:

(i) for all $t > T$ $n_2(t) > n_2(T)$;

(ii) for all $t < T$ $n_1(t) > n_1(T)$;

(iii) $\eta_2(T) \in B_\beta$.

If for a fixed trajectory Γ it is not possible to define the moment T satisfied $(i)-(ii)$ then we put $\sigma(\Gamma) = 0$. Divide the set $A(\rho_1, \rho_2\beta)$ into two nonintersecting sets:

$$A(\rho_1, \rho_2\beta) = A_1(\rho_1, \rho_2\beta) \cup A_2(\rho_1, \rho_2\beta),$$

where

$$A_1(\rho_1, \rho_2\beta) = \{\Gamma : \ \sigma(\Gamma) \neq 0\},$$

$$A_2(\rho_1, \rho_2\beta) = \{\Gamma : \ \sigma(\Gamma) = 0\}.$$

Now we will prove that for any $\beta \in \mathcal{A} \setminus \{\emptyset\}$

$$\mathsf{P}\{\Gamma : \Gamma \in A_2(\rho_1, \rho_2\beta)\} = 0. \tag{56}$$

Indeed, we have

$$\begin{aligned}
&\mathsf{P}\{\Gamma : \Gamma \in A_2(\rho_1, \rho_2\beta)\} \\
&\leq \lim_{t\to\infty} \sum_{n=0}^{t} \mathsf{P}\{\tau_\beta^{(n)} < t, \tau_\beta^{(n+1)} \geq t \text{ and for any } k \leq n \\
&\qquad\qquad \text{there exists } s_k \in (\tau^{(k)}, t] \text{ such that } n_2(s_k) < n_2(\tau^{(k)})\} \\
&= \lim_{t\to\infty} \mathsf{P}\{T_\beta \geq t \mid \delta_{(\rho_1,\rho_2\beta)}\}.
\end{aligned}$$

Thus, (56) follows from Lemma 3.

Consider now the set $A_1(\rho_1, \rho_2\beta)$. By definition of $\sigma(\cdot)$ we get

$$\begin{aligned}
&\mathsf{P}\{A(\rho_1, \rho_2\beta)\} = \mathsf{P}\{A_1(\rho_1, \rho_2\beta)\} \\
&= \sum_{t=1}^{\infty} \int_{\Gamma:\ \sigma(\Gamma)=t} \mathsf{P}(d\Gamma) = \sum_{t=1}^{\infty}\sum_{n=1}^{t} \int_{\Gamma:\ \sigma(\Gamma)=\tau^{(n)}=t} \mathsf{P}(d\Gamma).
\end{aligned} \tag{57}$$

Let the initial measure be $\mu = \nu \times \delta_{\rho_2\beta}$. Consider the probability of the event $\{\text{for all } t,\ n_2(t) > n_2(0)\}$ provided that the initial measure is μ. Using (57) we come to

$$\begin{aligned}
p(\nu, \beta) &= \int_{\mathcal{B}_\infty} \mathsf{P}\{A_1(\rho_1, \rho_2\beta)\} d\nu(\rho_1) \\
&= \sum_{t=1}^{\infty}\sum_{n=1}^{t} \int_{\mathcal{B}_\infty} f_{[\rho_1]_n,\beta}(t) p(\theta_n\rho_1, \beta) d\nu(\rho_1) \\
&= \sum_{t=1}^{\infty}\sum_{n=1}^{t} \sum_{\alpha:|\alpha|=n} p_\nu(\alpha) f_{\alpha,\beta}(t) \int_{\mathcal{B}_\infty} p(\theta_n\rho_1, \beta) d\nu(\theta_n\rho_1) \\
&= p(\nu, \beta) \sum_{t=1}^{\infty}\sum_{n=1}^{t} \sum_{\alpha:|\alpha|=n} p_\nu(\alpha) f_{\alpha,\beta}(t) = \sum_{t=1}^{\infty} f_\beta(t) p(\nu, \beta).
\end{aligned}$$

Here the last equality holds because

$$\begin{aligned}
f_\beta(t) &= \int_{\mathcal{B}_\infty} f_{\rho_1,\beta}(t) d\nu(\rho_1) = \int_{\mathcal{B}_\infty} \sum_{n=1}^{t} f_{[\rho_1]_n,\beta}(t) d\nu(\rho_1) \\
&= \sum_{n=1}^{t} \int_{\mathcal{B}_\infty} f_{[\rho_1]_n,\beta}(t) d\nu(\rho_1) = \sum_{n=1}^{t} \sum_{\alpha:|\alpha|=n} p_\nu(\alpha) f_{\alpha,\beta}(t).
\end{aligned}$$

Hence,

$$p(\nu, \beta) = \sum_{t=1}^{\infty} f_{\beta}(t) p(\nu, \beta).$$

Proposition 6 is proved. □

Proposition 7. *Let condition (54) hold for the initial measure μ. Then*

$$f_{\beta}(t) < \mathsf{P}\{T_{\beta} \geq t \mid \mu\}.$$

Proof. Let

$$C_t = \{\text{there exists } t_0 \leq t \text{ such that } n_2(t_0) < n_2(0)\}.$$

Then

$$\begin{aligned} f_{\beta}(t) &\leq \mathsf{P}\{T_{\beta} > t,\ \overline{C_t} \mid \mu\} \\ &< \mathsf{P}\{T_{\beta} > t,\ \overline{C_t} \mid \mu\} + \mathsf{P}\{T_{\beta} > t,\ C_t \mid \mu\} = \mathsf{P}\{T_{\beta} > t \mid \mu\}. \end{aligned}$$

The proposition is proved. □

By Propositions 6, 7 and Lemma 3

$$\sum_{t=1}^{\infty} t f_{\beta}(t) < \infty.$$

Note that equation (45) is a *renewal equation*. By *the renewal theorem* we obtain that there exists the limit

$$\lim_{t \to \infty} v(\beta, t) = v(\beta) \tag{58}$$

and, moreover,

$$v(\beta) = \frac{\sum_{t=1}^{\infty} r(\beta, t)}{\sum_{t=1}^{\infty} t f_{\beta}(t)}.$$

Since we have exponential estimates for $f_{\beta}(t)$ and $r(\beta, t)$, standard arguments (see, for example, Chapter 7 of [1]) show that the convergence in (58) is exponentially fast.

Proposition 8. *Let $\mu_0 = \nu \times \delta_{\rho_2}$, where $\rho_2 \in \mathcal{B}_{\infty}$. Then*

$$p(\nu, \beta) \sum_{t=1}^{\infty} r(\beta, t) = 1, \tag{59}$$

for any ρ_2, where $p(\nu, \beta)$ is defined by (55).

Proof. Let the initial distribution be $\delta_\rho,\ \rho = (\rho_1, \rho_2)$. The distribution of the β-renewal moment T_β can be represented as follows

$$\mathsf{P}_\rho\{T_\beta = t\} = \sum_{n=1}^{t} \sum_{\substack{\Gamma \in \Upsilon_\rho^2(\beta,t): \\ n_1(0)-n_1(t)=n}} \mathcal{I}(\Gamma) p(\theta_n \rho_1, \beta).$$

Let

$$I_2([\rho_1]_n, \beta, t) = \sum_{\substack{\Gamma \in \Upsilon_\rho^2(\beta,t): \\ n_1(0)-n_1(t)=n}} \mathcal{I}(\Gamma).$$

Integrating over ν, we get

$$\begin{aligned}
\mathsf{P}\{T_\beta = t \mid \mu_0\} &= \int_{\mathcal{B}_\infty} \mathsf{P}_\rho\{T_\beta^{(1)} = t\} d\nu(\rho^{(1)}) \\
&= \sum_{n=1}^{t} \int_{\mathcal{B}_\infty} I_2([\rho^{(1)}]_n, \beta, t) p(\theta_n \rho^{(1)}, \beta) d\nu(\rho^{(1)}) \\
&= \sum_{n=1}^{t} \sum_{\alpha:|\alpha|=n} p_\nu(\alpha) I_2(\alpha, \beta, t) \int_{\mathcal{B}_\infty} p(\theta_n \rho^{(1)}, \beta) d\nu(\theta_n \rho^{(1)}) \\
&= \sum_{n=1}^{t} \sum_{\alpha:|\alpha|=n} p_\nu(\alpha) I_2(\alpha, \beta, t) p(\nu, \beta).
\end{aligned} \tag{60}$$

Since

$$\begin{aligned}
r(\beta, t) &= \int_{\mathcal{B}_\infty} \sum_{n=1}^{t} \sum_{\substack{\Gamma \in \Upsilon_\rho^2(\beta,t): \\ n_1(0)-n_1(t)=n}} \mathcal{I}(\Gamma) d\nu(\rho_1) \\
&= \sum_{n=1}^{t} \sum_{\alpha:|\alpha|=n} I_2(\alpha, \beta, t),
\end{aligned}$$

the sum (60) can be rewritten as

$$\sum_{n=1}^{t} \sum_{\alpha:|\alpha|=n} p_\nu(\alpha) I_2(\alpha, \beta, t) p(\nu, \beta) = r(\beta, t) p(\nu, \beta).$$

So

$$\mathsf{P}\{T_\beta = t \mid \mu_0\} = p(\nu, \beta) r(\beta, t). \tag{61}$$

Since

$$\sum_{t=1}^{\infty} \mathsf{P}\{T_\beta = t \mid \mu_0\} = 1,$$

using formula (61), we come to (59).

Corollary 3. $v(\beta)$ *does not depend on the state of the second string at the initial moment.*

The Lemma 4 is proved. □

As we noted before Theorems 2, 3 follows from Lemmas 2 – 4.

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