

# Weak convergence to the fractional Brownian sheet in Besov spaces

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**Abstract.** In this paper we study the problem of the approximation in law of the fractional Brownian sheet in the topology of the anisotropic Besov spaces. We prove the convergence in law of two families of processes to the fractional Brownian sheet: the first family is constructed from a Poisson process in the plane and the second family is defined by the partial sums of two sequences of real independent fractional brownian motions.

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**Mathematical subject classification:** 60F05, 60G15.

## 1 Introduction

Let  $T = [0, 1]$  the unit interval and  $(W_t^\alpha)_{t \in T}$  a fractional Brownian motion of Hurst parameter  $\alpha \in (0, 1)$  on some probability space  $(\Omega, \mathcal{F}, P)$ . That is,  $W^\alpha$  is a centered Gaussian process, starting from zero and its covariance is given by

$$R(t, s) = E(B_t^\alpha B_s^\alpha) = \frac{1}{2} (t^{2\alpha} + s^{2\alpha} - |t - s|^{2\alpha}).$$

Recall that the fractional Brownian motion of Hurst parameter  $\alpha \in (0, 1)$  admits a Wiener integral representation with respect to  $W$  of the form

$$W_t^\alpha = \int_0^t K_\alpha(t, s) dW_s \quad (1)$$

where  $K_\alpha$  is the kernel defined on the set  $\{0 < s < t\}$  and it is given by

$$K_\alpha(t, s) = d_\alpha (t - s)^{\alpha - \frac{1}{2}} + d_\alpha \left( \frac{1}{2} - \alpha \right) \int_s^t (u - s)^{\alpha - \frac{3}{2}} \left( 1 - \left( \frac{s}{u} \right)^{\frac{1}{2} - \alpha} \right) du, \quad (2)$$

with  $d_\alpha$  the following normalizing constant

$$d_\alpha = \left( \frac{2\alpha \Gamma(\frac{3}{2} - \alpha)}{\Gamma(\alpha + \frac{1}{2})\Gamma(2 - 2\alpha)} \right)^{\frac{1}{2}}.$$

Let now  $(W_{u,v})_{(u,v) \in T^2}$  a Brownian sheet. The fractional Brownian sheet can be also defined by a Wiener integral with respect to the Brownian sheet (see [4])

$$W_{s,t}^{\alpha,\beta} = \int_0^s \int_0^t K_\alpha(s, u) K_\beta(t, v) dW_{u,v}. \quad (3)$$

where  $\alpha, \beta \in (0, 1)$  and the kernels  $K_\alpha, K_\beta$  are defined above. Note that this process is a two-parameters centered Gaussian process, starting from  $(0, 0)$ , and its covariance is given by

$$E \left( W_{s,t}^{\alpha,\beta} W_{s',t'}^{\alpha,\beta} \right) = \frac{1}{2} (s'^{2\alpha} + s^{2\alpha} - |s' - s|^{2\alpha}) \frac{1}{2} (t'^{2\beta} + t^{2\beta} - |t' - t|^{2\beta}), \quad (4)$$

and it coincides in law with the process introduced in [10] or [1].

The aim of this work is to study the weak convergence of some continuous processes in the anisotropic Besov spaces to the fractional Brownian sheet. We will consider two types of approximations. First, we let

$$y_\varepsilon(s, t) = \int_0^t \int_0^s \frac{1}{\varepsilon^2} \sqrt{xy} (-1)^{N(\frac{x}{\varepsilon}, \frac{y}{\varepsilon})} dx dy, \quad \varepsilon > 0,$$

where  $\{N(x, y), (x, y) \in \mathbb{R}_+^2\}$  is a standard Poisson process in the plane. It was proved in [3] (using the result of Stroock [11] for the one-dimensional case) that this process converges in law in the space of continuous functions on  $[0, 1]^2$ , denoted by  $C([0, 1]^2)$ , as  $\varepsilon$  tends to 0, to the ordinary Brownian sheet. Using this result and the representation (3) a natural way to obtain an approximation in law of the fractional Brownian sheet is to put

$$X_\varepsilon(s, t) = \int_0^t \int_0^s K_\alpha(s, u) K_\beta(t, v) \frac{1}{\varepsilon^2} \sqrt{uv} (-1)^{N(\frac{u}{\varepsilon}, \frac{v}{\varepsilon})} du dv.$$

The weak convergence of the family of processes  $X_\varepsilon$  to the fractional Brownian sheet  $W^{\alpha,\beta}$  in the space of continuous functions has been showed in [4]. In our first result we prove that the sequence  $X_\varepsilon$  converges also to  $W^{\alpha,\beta}$  in the stronger topology of the anisotropic Besov space  $lip_p^*((\alpha, \beta), b)$ . See the next Section

for the definition of Besov spaces. To simplify the notation, we put  $n = \frac{1}{\varepsilon^2}$  and we will consider the family of processes

$$X^n(s, t) = n \int_0^s \int_0^t K_\alpha(s, x) K_\beta(t, y) \sqrt{xy} (-1)^{N_n(x, y)} dx dy. \quad (5)$$

In the last expression  $N_n(x, y) := N(x\sqrt{n}, y\sqrt{n})$ . Observe that  $N_n$  is a Poisson process with intensity  $n$ .

On the other hand, it has been proved in [5] that if  $(B^n)_n, (C^n)_n$  are two families of independent one dimensional Brownian motions then the process

$$W^n(s, t) = \frac{1}{\sqrt{n}} \sum_{j=1}^n B_s^j C_t^j$$

converges weakly to the Brownian sheet in the two dimensional Besov space  $lip_p^*((\frac{1}{2}, \frac{1}{2}), b)$  for any  $p > \frac{2}{b}$ . Our second approximation result is an extension of the result of [5]. That is, if we put

$$Z^n(s, t) = \frac{1}{\sqrt{n}} \sum_{j=1}^n B_s^{j, \alpha} C_t^{j, \beta} \quad (6)$$

where  $(B^{n, \alpha})_n, (C^{n, \beta})_n$  are two families of independent one dimensional fractional Brownian motions, then the process  $Z^n$  converges in law, as  $n \rightarrow \infty$ , to the fractional Brownian sheet  $W^{\alpha, \beta}$  in the space  $lip_p^*((\alpha, \beta), b)$ .

For the weak convergence in Besov spaces to the one dimensional fractional Brownian motion we refer to [9].

## 2 Anisotropic Besov spaces

We recall in this section some basic elements on the two-dimensional Besov spaces. We refer to [10] for a complete presentation on this subject.

Let  $T^2 = [0, 1]^2$  and  $D = \{1, 2\}$ . For a function  $f : T^2 \rightarrow \mathbb{R}$ ,  $i \in D$ ,  $h \in \mathbb{R}$  and  $e_i = (\delta_{i1}, \delta_{i2})$  unit vectors, let

$$\Delta_{h,i} = \begin{cases} f(z + he_i) - f(z), & \text{if } z, z + he_i \in T^2 \\ 0, & \text{otherwise} \end{cases}$$

denote the progressive difference of  $f$  in the direction  $e_i$ . Let

$$\Delta_{(h_1, h_2)} f = \Delta_{h_1, 1} \circ \Delta_{h_2, 2} f, \quad \text{for } (h_1, h_2) \in \mathbb{R}^2.$$

If  $A = \{i\}$  is a subset of  $D$  with one element, then we put  $\Delta_{(h_1, h_2), A} f = \Delta_{h_i, i} f$  and  $\Delta_{(h_1, h_2), A} f = f$  if  $A = \Phi$ .

Now, for  $f \in L^p(T^2)$  if  $1 \leq p < \infty$  or  $f \in C(T^2)$  (the space of continuous functions on  $T^2$ ) if  $p = \infty$ , we define its  $L^p$ -modulus of continuity by

$$\omega_{p,A}(f, (t_1, t_2)) = \sup_{0 \leq h_1 \leq t_1, 0 \leq h_2 \leq t_2} \|\Delta_{(h_1, h_2), A} f\|_p \text{ for } (t_1, t_2) \in \mathbb{R}_+^2.$$

For  $b$  real and  $\bar{a} = (a_1, a_2)$ ,  $a_1, a_2 > 0$  consider the real valued application on  $T^2$  given by

$$\omega_{\bar{a}, b}((t_1, t_2), A) = \prod_{i \in A} t_i^{a_i} \left( 1 + \sum_{i \in A} \log \frac{1}{t_i} \right)^b$$

for any  $A \subset D$  with  $\omega_{\bar{a}, b}((t_1, t_2), \Phi) = 1$ .

**Definition 2.1.** Let  $\bar{a} = (a_1, a_2)$ ,  $a_1, a_2 > 0$ ,  $b \in \mathbb{R}$  and  $1 \leq p \leq \infty$ . The anisotropic Besov space  $Lip_p(\bar{a}, b)$  is defined by

$$Lip_p(\bar{a}, b) = \left\{ f \in L^p(T^2); \sum_{A \subset D} \sup_{t_1, t_2 > 0} \frac{\omega_{p,A}(f, (t_1, t_2))}{\omega_{\bar{a}, b}((t_1, t_2), A)} < \infty \right\}$$

and this space is endowed with the norm

$$\|f\|_p^{\bar{a}, b} = \sum_{A \subset D} \sup_{t_1, t_2 > 0} \frac{\omega_{p,A}(f, (t_1, t_2))}{\omega_{\bar{a}, b}((t_1, t_2), A)}.$$

In this way  $Lip_p(\bar{a}, b)$  becomes a non-separable Banach space.

We also introduce the subspace  $lip_p^*(\bar{a}, b)$  of  $Lip_p(\bar{a}, b)$  by

$$lip_p^*(\bar{a}, b) = \left\{ f \in L^p(T^2); \forall \Phi \neq A \subset D, \lim_{\delta_A(t_1, t_2) \rightarrow 0} \frac{\omega_{p,A}(f, (t_1, t_2))}{\omega_{\bar{a}, b}((t_1, t_2), A)} = 0 \right\}$$

where  $\delta_A(t_1, t_2) = \min\{t_i, i \in A\}$ .

The following result on Besov spaces and fractional Brownian motion has been proved in [10].

**Theorem 2.1.** For any  $2 < p < \infty$ , it holds

$$P(W^{\alpha, \beta} \in Lip_p((\alpha, \beta), 0)) = 1 \text{ and } P(W^{\alpha, \beta} \in lip_p^*((\alpha, \beta), 0)) = 0.$$

For similar results in the one parameter case we refer to [7].

### 3 Weak convergence to the fractional Brownian sheet in Besov spaces

We will need the following tightness criterion in Besov spaces given by [5].

**Lemma 3.1.** *Let  $(U^n(s, t))_{(s,t) \in [0,1]^2}$  be a sequence of continuous adapted processes such that there exists  $a = (a_1, a_2)$ ,  $a_1, a_2 > 0$  and for every  $p \geq 1$  it exists a constant  $C_p > 0$  such that, for  $s, s', t, t' \in [0, 1]$ ,  $s \leq s', t \leq t'$  it holds*

$$E \left| \Delta_{s'-s, t'-t} U^n(s, t) \right|^p \leq C_p |s' - s|^{a_1 p} |t' - t|^{a_2 p} \quad (7)$$

and

$$\begin{aligned} E \left| \Delta_{s'-s, 1} U^n(s, t) \right|^p &\leq C_p |s' - s|^{a_1 p}, \\ E \left| \Delta_{t'-t, 2} U^n(s, t) \right|^p &\leq C_p |t' - t|^{a_2 p}. \end{aligned} \quad (8)$$

Then  $U^n$  is tight in the space  $lip_p^*(a, b)$  for any  $p > \max_i \frac{1}{a_i} \vee \frac{2}{b}$ .

We prove now the tightness of the approximating families  $X^n$  and  $Z^n$  given by (5) and (6).

#### Lemma 3.2.

- 1) *Let  $(X^n(s, t))_{s,t \in [0,1]^2}$  be the family of processes given by (5). Then  $X^n$  is tight in the space  $lip_p^*((\alpha, \beta), b)$  for any  $p > \frac{2}{b} \vee \frac{1}{\alpha} \vee \frac{1}{\beta}$ .*
- 2) *Let  $(Z^n(s, t))_{s,t \in [0,1]^2}$  be the family of processes given by (6). Then  $Z^n$  is tight in the space  $lip_p^*((\alpha, \beta), b)$  for any  $p > \frac{2}{b} \vee \frac{1}{\alpha} \vee \frac{1}{\beta}$ .*

**Proof. Tightness of  $X^n$ :** Note first that, since the Besov norms are increasing in  $p$  it suffices to prove the result for  $p$  even. We will show that

$$\sup_n E \left[ \Delta_{s,t} X^n(s', t') \right]^p \leq C_p (s' - s)^{p\alpha} (t' - t)^{p\beta}$$

for any  $p$  even. By Lemma 4.1 of [4], it suffices to check this inequality for  $s' \geq s > 0, t' \geq t > 0, s' - s < s$  and  $t' - t < t$ .

We will extend the kernels  $K_\alpha$  and  $K_\beta$  over all  $(0, 1]$  by putting

$$\tilde{K}_\alpha(s, x) = \begin{cases} K_\alpha(s, x) & \text{if } s > x \\ 0 & \text{if } s \leq x \end{cases}$$

and for the sake of simplicity we will denote also by  $K_\alpha$  this extension. We introduce also the following notations

$$K_{\alpha,\beta}(s, t, x, y) = K_\alpha(s, x) K_\beta(t, y),$$

and

$$\Delta_{s,t} K_{\alpha,\beta}(s', t', x, y) = (K_{\alpha}(s', x) - K_{\alpha}(s, x))(K_{\beta}(t', y) - K_{\beta}(t, y)).$$

We have that, with the notations introduced above,

$$\begin{aligned} E [\Delta_{s,t} X_n(s', t')]^p &= n^p E \left[ \int_{[0,1]^2} \Delta_{s,t} K_{\alpha,\beta}(s', t', x, y) \sqrt{xy} (-1)^{N_n(x,y)} dx dy \right]^p \\ &= n^p E \left[ \int_{[0,1]^{2p}} \prod_{i=1}^p (\Delta_{s,t} K_{\alpha,\beta}(s', t', x_i, y_i) \sqrt{x_i y_i} (-1)^{N_n(x_i, y_i)}) dx_1 \cdots dy_p \right]. \end{aligned}$$

We obtain now a bound for the expectation of the random part of the last expression. First of all, we have that

$$(-1)^{\sum_{i=1}^p N_n(x_i, y_i)} = (-1)^{\sum_{i=1}^p \Delta_{0,0} N_n(x_i, y_i)},$$

and that

$$\begin{aligned} \sum_{i=1}^p \Delta_{0,0} N_n(x_i, y_i) &= \sum_{i=1}^p \Delta_{s,t} N_n(x_i, y_i) + \sum_{i=1}^p \Delta_{s,0} N_n(x_i, t) \\ &\quad + \sum_{i=1}^p \Delta_{0,t} N_n(s, y_i) + p N_n(s, t), \end{aligned}$$

and hence

$$\begin{aligned} (-1)^{\sum_{i=1}^p \Delta_{0,0} N_n(x_i, y_i)} &= \\ (-1)^{\sum_{i=1}^p \Delta_{s,t} N_n(x_i, y_i)} (-1)^{\sum_{i=1}^p \Delta_{s,0} N_n(x_i, t)} (-1)^{\sum_{i=1}^p \Delta_{0,t} N_n(s, y_i)}. \end{aligned}$$

Since for all  $x_i, y_i, x_j, y_k$  the three intervals  $((s, t), (x_i, y_i)]$ ,  $((s, 0), (x_j, t))$  and  $((0, t), (s, y_k))$  are disjoint sets, the three factors of the last product are independent random variables. Moreover, if we majorize the expectation of the first factor by 1 we will obtain

$$\begin{aligned} E \left( (-1)^{\sum_{i=1}^p \Delta_{0,0} N_n(x_i, y_i)} \right) &\leq \exp\{-2nt[(x_{(p)} - x_{(p-1)}) + \cdots + (x_{(2)} - x_{(1)})]\} \\ &\quad \times \exp\{-2ns[(y_{(p)} - y_{(p-1)}) + \cdots + (y_{(2)} - y_{(1)})]\}, \end{aligned}$$

where  $x_{(1)}, \dots, x_{(p)}$  are the variables  $x_1, \dots, x_p$  ordered in increasing order.

Using now the fact that  $2t > t'$  and  $2s > s'$ , the last expression can be bounded by

$$\begin{aligned} & \exp\{-nt'[(x_{(p)} - x_{(p-1)}) + \cdots + (x_{(2)} - x_{(1)})]\} \\ & \times \exp\{-ns'[(y_{(p)} - y_{(p-1)}) + \cdots + (y_{(2)} - y_{(1)})]\} \\ & \leq \exp\{-n[(x_{(p)} - x_{(p-1)})y_{(p-1)} + \cdots + (x_{(2)} - x_{(1)})y_{(1)}]\} \\ & \times \exp\{-n[(y_{(p)} - y_{(p-1)})x_{(p-1)} + \cdots + (y_{(2)} - y_{(1)})x_{(1)}]\}. \end{aligned}$$

Therefore,

$$\begin{aligned} & E[\Delta_{s,t}X_n(s', t')]^p \\ & \leq (p!)^2 n^p \int_{[0,1]^{2p}} \left( \prod_{i=1}^p (\Delta_{s,t}K_{\alpha,\beta}(s', t', x_i, y_i) \sqrt{x_i y_i}) \right. \\ & \quad \times \exp\{-n[(x_p - x_{p-1})y_{p-1} + \cdots + (x_2 - x_1)y_1]\} \\ & \quad \times \exp\{-n[(y_p - y_{p-1})x_{p-1} + \cdots + (y_2 - y_1)x_1]\} \\ & \quad \times 1_{\{x_1 \leq \cdots \leq x_p\}} 1_{\{y_1 \leq \cdots \leq y_p\}} \Big) dx_1 \cdots dy_p \\ & \leq C_p \left( n^2 \int_{[0,1]^2} \left( \Delta_{s,t}K_{\alpha,\beta}(s', t', x_1, y_1) \Delta_{s,t}K_{\alpha,\beta}(s', t', x_2, y_2) \sqrt{x_1 x_2 y_1 y_2} \right. \right. \\ & \quad \times \exp[-n(x_2 - x_1)y_1 - n(y_2 - y_1)x_1] \Big) 1_{\{x_1 \leq x_2\}} 1_{\{y_1 \leq y_2\}} dx_1 \cdots dy_2 \Big)^{\frac{p}{2}}. \end{aligned} \quad (9)$$

We now divide the region of integration in two parts:  $A = \{x_1 \leq x_2 \leq 2x_1, y_1 \leq y_2 \leq 2y_1\}$  and  $A^c$ .

The integral of expression (9) over the region  $A$  can be majorized by

$$\begin{aligned} & C_p n^2 \int_{[0,1]^4} \left( \Delta_{s,t}K_{\alpha,\beta}(s', t', x_1, y_1) \Delta_{s,t}K_{\alpha,\beta}(s', t', x_2, y_2) x_1 y_1 \right. \\ & \quad \times \exp\{-2n[(x_2 - x_1)y_1 + (y_2 - y_1)x_1]\} 1_{\{x_1 \leq x_2, y_1 \leq y_2\}} \Big) dx_1 \cdots dy_2 \\ & \leq C_p n^2 \int_{[0,1]^4} \left( K_{\alpha}(s', x_1) - K_{\alpha}(s, x_1) \right)^2 \left( K_{\beta}(t', y_2) - K_{\beta}(t, y_2) \right)^2 x_1 y_1 \\ & \quad \times \exp\{-2n[(x_2 - x_1)y_1 + (y_2 - y_1)x_1]\} 1_{\{x_1 \leq x_2, y_1 \leq y_2\}} \Big) dx_1 \cdots dy_2 \\ & + C_p n^2 \int_{[0,1]^4} \left( K_{\alpha}(s', x_2) - K_{\alpha}(s, x_2) \right)^2 \left( K_{\beta}(t', y_1) - K_{\beta}(t, y_1) \right)^2 x_1 y_1 \\ & \quad \times \exp\{-2n[(x_2 - x_1)y_1 + (y_2 - y_1)x_1]\} 1_{\{x_1 \leq x_2, y_1 \leq y_2\}} \Big) dx_1 \cdots dy_2. \end{aligned}$$

These last two summands can be treated in a similar way. In the first one we first integrate with respect to  $x_2$  and then with respect to  $y_1$ , and in the second

we integrate with respect  $x_1$  and  $y_2$ . In both cases we obtain the bound

$$\begin{aligned} & C_p \left( \int_{[0,1]^2} (K_\alpha(s', x) - K_\alpha(s, x))^2 (K_\beta(t', y) - K_\beta(t, y))^2 dx dy \right)^{\frac{p}{2}} \\ &= C_p \left( E \left( \Delta_{s,t} W_{s',t'}^{\alpha,\beta} \right)^2 \right)^{\frac{p}{2}} = C_p (s' - s)^{\alpha p} (t' - t)^{\beta p}. \end{aligned}$$

We consider now the integral given in (9) over the region  $A^c$ . This region is the union of  $B_1 = \{x_1 \leq x_2, y_1 \leq y_2, x_2 > 2x_1\}$  and  $B_2 = \{x_1 \leq x_2, y_1 \leq y_2, y_2 > 2y_1\}$ . We will deal with the integral over  $B_2$ , the other one is analogous. In this case we obtain the following inequalities:

$$\begin{aligned} 2(y_2 - y_1)x_1 + 2(x_2 - x_1)y_1 &\geq (y_2 - y_1)x_1 + x_1y_1 + 2(x_2 - x_1)y_1 \\ &\geq \frac{1}{2}(y_2 - y_1)x_1 + x_1y_1 + \frac{1}{2}(x_2 - x_1)y_1 = \frac{1}{2}(y_2x_1 + y_1x_2). \end{aligned}$$

Thus, the integral given in expression (9) over the region  $B_2$  can be bounded by

$$\begin{aligned} & C_p n^2 \int_{[0,1]^4} \prod_{i=1}^2 (\Delta_{s,t} K_{\alpha,\beta}(s', t', x_i, y_i) \sqrt{x_i y_i}) \\ & \quad \times \exp\left[-\frac{n}{2}(y_2x_1 + x_2y_1)\right] 1_{\{x_1 \leq x_2, y_1 \leq y_2\}} dx_1 \cdots dy_2 \\ & \leq C_p n^2 \left[ \int_{[0,1]^4} \left( K_\alpha(s', x_1) - K_\alpha(s, x_1) \right)^2 \left( K_\beta(t', y_1) - K_\beta(t, y_1) \right)^2 x_1 y_1 \right. \\ & \quad \times \exp\left[-\frac{n}{2}(y_2x_1 + x_2y_1)\right] 1_{\{x_1 \leq x_2, y_1 \leq y_2\}} dx_1 \cdots dy_2 \\ & \quad + \int_{[0,1]^4} \left( K_\alpha(s', x_2) - K_\alpha(s, x_2) \right)^2 \left( K_\beta(t', y_2) - K_\beta(t, y_2) \right)^2 x_2 y_2 \\ & \quad \times \exp\left[-\frac{n}{2}(y_2x_1 + x_2y_1)\right] 1_{\{x_1 \leq x_2, y_1 \leq y_2\}} dx_1 \cdots dy_2 \Big]. \end{aligned}$$

By integrating in the first summand of the last expression with respect to  $x_2$  and  $y_2$  and in the second summand with respect to  $x_1$  and  $y_1$ , we obtain that the last expression is bounded by

$$\begin{aligned} & C_p \left( \int_{[0,1]^2} (K_\alpha(s', x) - K_\alpha(s, x))^2 (K_\beta(t', y) - K_\beta(t, y))^2 dx dy \right)^{\frac{p}{2}} \\ &= C_p \left( E \left( \Delta_{s,t} W_{s',t'}^{\alpha,\beta} \right)^2 \right)^{\frac{p}{2}} = C_p (s' - s)^{\alpha p} (t' - t)^{\beta p}. \end{aligned}$$



Using the same calculus we can prove that conditions (8) of Lemma 1 are also verified. This finishes the proof of tightness of  $X^n$ .

**Tightness of  $Z^n$ :** Concerning 2), observe that the rectangular increments of the process  $Z^n$  can be written as

$$\Delta_{s,t} Z_{s',t'}^n = \frac{1}{\sqrt{n}} \sum_{j=1}^n \left( B_{s'}^{j,\alpha} - B_s^{j,\alpha} \right) \left( C_{t'}^{j,\beta} - C_t^{j,\beta} \right)$$

and by decomposing  $B_{s'}^{j,\alpha} - B_s^{j,\alpha} = \frac{B_{s'}^{j,\alpha} - B_s^{j,\alpha}}{\sqrt{E(B_{s'}^{j,\alpha} - B_s^{j,\alpha})^2}} |s' - s|^\alpha$  and similarly for  $C_{t'}^{j,\beta} - C_t^{j,\beta}$ , we get, with  $(\zeta_i, \eta_i)_i$  a double sequence of i.i.d. random variables with standard normal distribution,

$$\begin{aligned} & E \left| \Delta_{s'-s, t'-t} Z_{s,t}^n \right|^p \\ & \leq (s' - s)^{p\alpha} (t' - t)^{p\beta} E \left[ \frac{1}{\sqrt{n}} \sum_{j=1}^n \zeta_i \eta_i \right]^p \\ & \leq C_p (s' - s)^{p\alpha} (t' - t)^{p\beta} E \left[ \frac{1}{n} \sum_{i=1}^n \zeta_i^2 \right]^{\frac{p}{2}} \\ & \leq C_p (s' - s)^{p\alpha} (t' - t)^{p\beta} \frac{1}{n} \sum_{i=1}^n E |\zeta_i|^p \leq C_p (s' - s)^{p\alpha} (t' - t)^{p\beta} \end{aligned}$$

and similarly (8) can be checked. □

We can state now our main result.

**Theorem 3.3.** *Let  $X^n$  and  $Z^n$  be the families defined by (5) and (6). Then the following weak convergences hold*

$$X^n \rightarrow W^{\alpha,\beta} \text{ in } lip_p^* ((\alpha, \beta), b) \text{ as } n \rightarrow \infty$$

and

$$Z^n \rightarrow W^{\alpha,\beta} \text{ in } lip_p^* ((\alpha, \beta), b) \text{ as } n \rightarrow \infty .$$

**Proof.** We refer to [4] for the proof of the convergence of the finite dimensional distributions of  $X^n$  to those of  $W^{\alpha,\beta}$ . Concerning the family  $Z^n$ , let  $N$  integer,  $a_1, \dots, a_N \in \mathbb{R}$  and  $(s_1, t_1), \dots, (s_N, t_N) \in [0, 1]^2$ . We must see that the random variables

$$\sum_{k=1}^N a_k Z_n(s_k, t_k) = \frac{1}{\sqrt{n}} \sum_{k=1}^N a_k \times \sum_{j=1}^n \left( \int_0^{s_k} K_\alpha(s_k, u_k) dW_{u_k}^j \right) \left( \int_0^{t_k} K_\beta(t_k, v_k) dW_{v_k}^j \right)$$

converge in law, as  $n$  tends to infinity to

$$\sum_{k=1}^N a_k W_{s_k, t_k}^{\alpha, \beta} = \sum_{k=1}^N \int_0^{s_k} \int_0^{t_k} K_\alpha(s_k, u_k) K_\beta(t_k, v_k) dW_{u_k, v_k}.$$

To prove this convergence, we will prove the convergence of the associated characteristic functions. In the sequel we will consider the extension of the kernel  $K_\alpha(t, s)$  on  $[0, 1]$  by putting zero if  $s \geq t$ . For the sake of simplicity we will denote this extension also by  $K_\alpha(t, s)$ .

Let  $(\gamma^{k,l})_l$  be a sequence of elementary functions converging, for every  $s_k$ , to  $K_\alpha(s_k, \cdot)$  in  $L^2(T)$  as  $l$  goes to  $\infty$  and for every  $t_k$ , let  $(\rho^{k,l})_l$  a sequence of elementary functions converging in  $L^2(T)$ , as  $l \rightarrow \infty$ , to  $K_\beta(t_k, \cdot)$ .

Let us denote by

$$Z_{k,l}^n = \frac{1}{\sqrt{n}} \sum_{j=1}^n \left( \int_0^{s_k} \gamma^{k,l}(u_k) dW_{u_k}^j \right) \left( \int_0^{t_k} \rho^{k,l}(v_k) dW_{v_k}^j \right)$$

and

$$Z_{k,l} = \int_0^{s_k} \int_0^{t_k} K_\alpha(s_k, u) K_\beta(t_k, v) dW_{u,v}.$$

For every  $\lambda$  real, we have the following bound

$$\begin{aligned} & \left| E \left( e^{i\lambda \sum_{k=1}^N a_k Z^n(s_k, t_k)} \right) - E \left( e^{i\lambda \sum_{k=1}^N a_k W_{s_k, t_k}^{\alpha, \beta}} \right) \right| \\ & \leq \left| E \left( e^{i\lambda \sum_{k=1}^N a_k Z^n(s_k, t_k)} \right) - E \left( e^{i\lambda \sum_{k=1}^N a_k Z_{k,l}^n} \right) \right| \\ & \quad + \left| E \left( e^{i\lambda \sum_{k=1}^N a_k Z_{k,l}^n} \right) - E \left( e^{i\lambda \sum_{k=1}^N a_k Z_{k,l}} \right) \right| \\ & \quad + \left| E \left( e^{i\lambda \sum_{k=1}^N a_k Z_{k,l}} \right) - E \left( e^{i\lambda \sum_{k=1}^N a_k W_{s_k, t_k}^{\alpha, \beta}} \right) \right| = I_1 + I_2 + I_3 \end{aligned}$$

By the mean value theorem we can bound the term  $I_1$  by

$$\max_k E |Z^n(s_k, t_k) - Z_{k,l}^n|$$

and moreover

$$\begin{aligned} & E |Z^n(s_k, t_k) - Z_{k,l}^n| \\ & \leq \frac{1}{\sqrt{n}} E \left| \sum_{j=1}^n \left( \int_0^1 K_\alpha(s_k, u_k) dW_{u_k}^j \right) \left( \int_0^1 K_\beta(t_k, v_k) dW_{v_k}^j \right) \right. \\ & \quad \left. - \left( \int_0^1 \gamma^{k,l}(u_k) dW_{u_k}^j \right) \left( \int_0^1 \rho^{k,l}(v_k) dW_{v_k}^j \right) \right| \\ & \leq \frac{1}{\sqrt{n}} E \sum_{j=1}^n \left| \int_0^1 (K_\alpha(s_k, u) - \gamma^{k,l}(u)) dW^j(u) \right| \left| \int_0^1 K_\beta(t_k, v) dW_v^j \right| \\ & \quad + \frac{1}{\sqrt{n}} E \sum_{j=1}^n \left| \int_0^1 \gamma^{k,l}(u) dW_u^j \right| \left| \int_0^1 (K_\beta(t_k, v) - \rho^{k,l}(v)) dW_v^j \right| \\ & \leq \frac{1}{\sqrt{n}} \left( E \left( \sum_{j=1}^n \left| \int_0^1 (K_\alpha(s_k, u) - \gamma^{k,l}(u)) dW^j(u) \right| \left| \int_0^1 K_\beta(t_k, v) dW_v^j \right| \right) \right)^{\frac{1}{2}} \\ & \quad + \frac{1}{\sqrt{n}} \left( E \left( \sum_{j=1}^n \left| \int_0^1 \gamma^{k,l}(u) dW^j(u) \right| \left| \int_0^1 (K_\beta(t_k, v) - \rho^{k,l}(v)) dW_v^j \right| \right) \right)^{\frac{1}{2}}. \end{aligned}$$

Using the independence of increments, we can majorize the last expression by

$$\begin{aligned} & \left( \int_0^1 (K_\alpha(s_k, u) - \gamma^{k,l}(u))^2 du \right) \left( \int_0^1 K_\beta^2(t_k, v) dv \right) \\ & + \left( \int_0^1 (\gamma^{k,l}(u))^2 du \right) \left( \int_0^1 (K_\beta(t_k, v) - \rho^{k,l}(v))^2 dv \right). \end{aligned}$$

Now, since  $\gamma^{k,l}$  and  $\rho^{k,l}$  are elementary functions, the convergence of  $I_2$  to 0 as  $n \rightarrow \infty$  follows from the convergence of  $W^n$  to  $W^{\alpha,\beta}$ . Finally, concerning the last term  $I_3$ , we have

$$\begin{aligned} & \left| E \left( e^{i\lambda \sum_{k=1}^N a_k Z_{k,l}} \right) - E \left( e^{i\lambda \sum_{k=1}^N a_k W_{s_k,t_k}^{\alpha,\beta}} \right) \right| \leq C \max_k E |Z_{k,l} - W^{\alpha,\beta}(s_k, t_k)| \\ & = C \max_j \{ \bar{E} \left| \int_{[0,1]^2} [\gamma^{kl}(x) \rho^{kl}(y) - K_\alpha(s_k, x) K_\beta(t_k, y)] dW_{x,y} \right| \} \\ & \leq C \max_j \|\gamma^{kl} \otimes \rho^{kl} - K_\alpha(s_k, \cdot) \otimes K_\beta(t_k, \cdot)\|_{L^2([0,1]^2)}. \end{aligned}$$

This last norm in  $L^2([0, 1]^2)$  tends to zero, as  $l \rightarrow \infty$ , independently of  $n$ . This fact concludes the proof.  $\square$

## References

- [1] A. Ayache, S. Leger and M. Pontier, *Drap brownien fractionnaire*. To appear in Potential Anal. (2001).
- [2] X. Bardina, *The Complex Brownian Motion as a Weak Limit of Processes constructed from a Poisson Process*. Progress in Probability, Vol.48 Birkhauser 149–158 *Weak convergence to the multiple Stratonovich integrals*. Stochastic Process. Appl., **90**(2) (2001): 277–300.
- [3] X. Bardina and M. Jolis, *Weak approximation of the Brownian sheet from the Poisson process in the plane*. Bernoulli, **6**(4) (2000): 653–665.
- [4] X. Bardina, M. Jolis and C.A. Tudor, *Weak convergence to the fractional Brownian sheet*. Preprint. (2001).
- [5] B. Boufoussi and M. Dozzi, *On the local time of multiparameter symmetric stable processes. Regularity and limit theorem in Besov spaces*. Preprint. (2002).
- [6] Z. Ciesielski and A. Kamont, *Levy's fractional Brownian random field and function spaces*. Acta Sci. Math. (Szeged) **60**(1-2) (1995): 99–118.
- [7] Z. Ciesielski, G. Keryacharian and B. Roynette, *Quelques espaces fonctionnels associés à des processus gaussiens*. Studia Mathematica, **107**(2) (1993): 171–204.
- [8] R. Delgado and M. Jolis, *Weak approximation for a class of Gaussian processes*. J. Appl. Probab., **37**(2) (2000): 400–407.
- [9] H. Hult, *Weak convergence to fractional Brownian motion in Besov spaces*. Preprint. (2001).
- [10] A. Kamont, *On the fractional anisotropic Wiener field*. Probab. Math. Statist., **16**(1) (1996): 85–98.
- [11] D. Stroock, *Topics in Stochastic Differential Equations*. Springer Verlag. (1982).

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