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When does the Hessian determinant vanish identically ?

(On Gordan and Noether's Proof of Hesse's Claim)

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Abstract. In 1851, Hesse claimed that the Hessian determinant of a homogeneous polynomial f vanishes identically if and only if the projective hypersurface $V(f)$ is a cone. We follow the lines of the 1876 paper of Gordan and Noether to give a proof of Hesse's claim for curves and surfaces. For higher dimensional hypersurfaces, the claim is wrong in general. We review the construction of polynomials with vanishing Hessian determinant but $V(f)$ not being a cone. For three dimensional hypersurfaces the latter gives, again, the complete answer to the question asked in the title.

Keywords: Hessian determinant, cone.

Mathematical subject classification: 14N15, 14J70.

1 Introduction

Let $f \in \mathbb{K}[x_0, \dots, x_n]$, $n \geq 1$, be a homogeneous polynomial over an algebraically closed field of characteristic zero. Its *Hessian determinant* is the determinant of the matrix of second derivatives,

$$H_f(x_0, \dots, x_n) := \det \begin{pmatrix} \frac{\partial^2 f}{\partial x_0 \partial x_0} & \cdots & \frac{\partial^2 f}{\partial x_0 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_0} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}.$$

In his original 1851 paper [1], Hesse claims that

> *the equality $H_f \equiv 0$ holds iff, after a suitable homogeneous coordinate change, the polynomial depends on at most n variables.*

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In other words, he claims that the Hessian determinant vanishes identically if and only if the hypersurface $V(f) := \{f = 0\} \subset \mathbb{P}^n$ is a *cone*. While the "if" implication is trivial, the "only if" statement is not trivial at all. Actually, it turned out very soon that Hesse's proof is wrong[1] and it seems to be Weierstraß who first expressed his doubts on the correctness of Hesse's claim[2]. Finally, in 1876, Gordan and Noether [2] showed that the general "only if" statement holds true exactly for $n \leq 3$. For higher dimensions, the "only if" statement is true as long as we restrict ourselves to quadratic forms f, while for degree ≥ 3, there is an explicit construction method for (series of) counterexamples. The most basic is probably the following.

Example 1.1. *Let* $f = x_0^2x_2 + x_0x_1x_3 + x_1^2x_4 + x_0^3 + x_1^3 \in \mathbb{K}[x_0, \ldots, x_4]$. *Then* $H_f \equiv 0$, *while the variety* $V(f)$ *is not a cone.*

In Section 2 of this note we follow the lines of Gordan and Noether to give a proof of Hesse's claim in case $n \leq 3$:

Theorem 1.2. *Let* $f \in \mathbb{K}[x_0, \ldots, x_n]$, $n \in \{1, 2, 3\}$, *be a homogeneous polynomial over an algebraically closed field of characteristic zero.*[3] *Then the following are equivalent:*

(a) $H_f \equiv 0$.

(b) *The partial derivatives* $\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}$ *are linearly dependent (over* $\mathbb{K}$*).*

(c) $V(f)$ *is a cone, that is, after a suitable homogeneous coordinate change,* f *depends on at most* n *variables.*

More precisely, the present proof is obtained by collecting the key observations in [2] needed for the case $n \leq 3$, by reorganizing them and by modernizing their proof.

Finally, in Section 3, we review the general construction of polynomials with $H_f \equiv 0$ but $V(f)$ not being a cone. For $n = 4$ the latter gives, again, the complete answer to the question asked in the title.

[1] Hesse implicitly assumes that the matrix of second derivatives has corank 1, thus, the adjoint matrix has rank 1, but then (without any justification) claims that the rows of the adjoint matrix are proportional to a non-constant vector. The latter statement is crucial for the conclusion, since it yields that a non-trivial $\mathbb{K}$-linear combination of the first derivatives vanishes.

[2] See the footnote in Gordan and Noether's paper [2]

[3] In positive characteristic, the situation is quite different [3]. Here, in general, only the implications $(b) \Rightarrow (a)$ and $(c) \Rightarrow (a)$ are true.

2 Proof of Theorem 1.2.

The implications $(b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$ being trivial, it suffices to show $(a) \Rightarrow (b)$.

Step 1. We prove that (a) implies

(a') *There exists a homogeneous polynomial* $\pi \in \mathbb{K}[y_0, \dots, y_n]$ *such that*

$$\pi\left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right) = 0\,.$$

Indeed, consider the rational map

$$\phi = \left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right) : \mathbb{A}^{n+1}_{\mathbb{K}} \longrightarrow \mathbb{A}^{n+1}_{\mathbb{K}}\,.$$

Then $H_f \equiv 0$ implies that ϕ cannot be dominant. Indeed, if ϕ would be dominant, there would be a non-empty Zariski open set $U \subset \mathbb{A}^{n+1}_{\mathbb{K}}$ such that for each $\boldsymbol{p} \in U$ the induced map on the Zariski tangent spaces $T_{\boldsymbol{p}}(\phi) : T_{\boldsymbol{p}}(\mathbb{A}^{n+1}_{\mathbb{K}}) \to T_{\phi(\boldsymbol{p})}(\mathbb{A}^{n+1}_{\mathbb{K}})$ is surjective, hence an isomorphism (see, e.g., [4, III.10.4 and III.10.5]). But the latter would mean that the determinant of the Jacobian matrix of ϕ (that is, H_f) does not vanish at $\boldsymbol{p}$, in contradiction to the assumption $H_f \equiv 0$.

Since ϕ is not dominant, the closure of the image of ϕ is contained in some hypersurface. Hence, there exists some

$$\pi \in \mathbb{K}[y_0, \dots, y_n] \quad \text{such that} \quad \pi\left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right) = 0.$$

Since the partials are all homogeneous of the same degree, we can assume π to be homogeneous, too.

Step 2. We prove that (a') implies (b).

Let $\pi \in \mathbb{K}[y_0, \dots, y_n]$ be homogeneous, of minimal degree $d \geq 1$, such that

$$\pi\left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right) = 0\,. \tag{2.1}$$

We introduce the following notations:

- $\pi_i := \dfrac{\partial \pi}{\partial y_i}\left(\dfrac{\partial f}{\partial x_0}, \dots, \dfrac{\partial f}{\partial x_n}\right) \in \mathbb{K}[x_0, \dots, x_n]\,;$

- $\rho := \gcd(\pi_0, \ldots, \pi_n) \in \mathbb{K}[x_0, \ldots, x_n]$;
- $h_i := \dfrac{\pi_i}{\rho} \in \mathbb{K}[x_0, \ldots, x_n]$.

Note that $h_0, \ldots, h_n$ are homogeneous polynomials of the same degree $e \geq 0$. Moreover, the minimality of d implies that π is irreducible, that we have equivalences

$$h_i \neq 0 \iff \pi_i \neq 0 \iff \frac{\partial \pi}{\partial y_i} \neq 0 \tag{2.2}$$

for $i = 0, \ldots, n$, and that

$$\dim_{\mathbb{K}}\big(\langle h_0, \ldots, h_n\rangle_{\mathbb{K}}\big) = \dim_{\mathbb{K}}\big(\langle \pi_0, \ldots, \pi_n\rangle_{\mathbb{K}}\big) = \dim_{\mathbb{K}}\left(\left\langle \frac{\partial \pi}{\partial y_0}, \ldots, \frac{\partial \pi}{\partial y_n}\right\rangle_{\mathbb{K}}\right), \tag{2.3}$$

where $\langle h_0, \ldots, h_n\rangle_{\mathbb{K}}$ denotes the $\mathbb{K}$-vector space spanned by $h_0, \ldots, h_n$.

The latter equality can be seen as follows: each relation $\sum_i \alpha_i \pi_i = 0, \alpha_i \in \mathbb{K}$, corresponds to a polynomial

$$\pi' := \sum_i \alpha_i \frac{\partial \pi}{\partial y_i} \in \mathbb{K}[y_0, \ldots, y_n]$$

of degree at most $d-1$ satisfying $\pi'\big(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\big) = 0$. Hence, by minimality of d, $\sum_i \alpha_i \frac{\partial \pi}{\partial y_i} = 0$.

We start by considering a special case (which will turn out to be the only possible case for $n \leq 3$):

Case 2a. Let $\dim_{\mathbb{K}}\big(\langle h_0, \ldots, h_n\rangle_{\mathbb{K}}\big) \leq 2$. Then, by (2.3), we may assume (w.l.o.g.) that each $\frac{\partial \pi}{\partial y_i}$, $i \geq 2$, can be written as a $\mathbb{K}$-linear combination of $\frac{\partial \pi}{\partial y_0}$ and $\frac{\partial \pi}{\partial y_1}$:

$$\frac{\partial \pi}{\partial y_i} = \alpha_i \frac{\partial \pi}{\partial y_0} + \beta_i \frac{\partial \pi}{\partial y_1}, \quad \alpha_i, \beta_i \in \mathbb{K}.$$

Choosing the new coordinates $y'_0 := y_0 + \sum_{i\geq 2} \alpha_i y_i$, $y'_1 := y_1 + \sum_{i \geq 2} \beta_i y_i$, $y'_i := y_i$ for $i \geq 2$, the chain rule gives, for $i \geq 2$,

$$\begin{aligned}\frac{\partial \pi}{\partial y'_i}(y_0, \ldots, y_n) &= \frac{\partial \pi}{\partial y'_i}\left(y'_0 - \sum_{i\geq 2} \alpha_i y'_i,\ y'_1 - \sum_{i \geq 2} \beta_i y'_i,\ y'_2, \ldots, y'_n\right)\\ &= \frac{\partial \pi}{\partial y_i}(y_0, \ldots, y_n) - \alpha_i \frac{\partial \pi}{\partial y_0}(y_0, \ldots, y_n) - \beta_i \frac{\partial \pi}{\partial y_1}(y_0, \ldots, y_n)\\ &= 0.\end{aligned}$$

In other words, after a homogeneous coordinate change, π depends on at most two variables. But each homogeneous polynomial in two variables splits (over any algebraically closed field) into linear factors. Hence, π gives a linear relation between the first derivatives, that is, condition (b) is satisfied.

Case 2b. Assume that $\dim_{\mathbb{K}}\langle h_0, \ldots, h_n\rangle_{\mathbb{K}} \geq 3$; in particular, either $h_i = 0$ or $h_i \notin \mathbb{K}$.

- Since π is homogeneous of degree d, Euler's Lemma implies

$$0 = d \cdot \pi\left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right) = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{\partial \pi}{\partial y_i}\left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right),$$

  hence

$$\sum_{i=0}^{n} \frac{\partial f}{\partial x_i} \cdot h_i = 0\,. \tag{2.4}$$

- Differentiating (2.1) with respect to x_j and applying the chain rule gives

$$0 = \frac{\partial}{\partial x_j}\left(\pi\left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right)\right) = \sum_{i=0}^{n} \frac{\partial \pi}{\partial y_i}\left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right) \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}\,.$$

We derive that

$$\sum_{i=0}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} \cdot h_i = 0\,, \tag{2.5}$$

for all $j = 0, \ldots, n$. Since π_k is a polynomial expression in the partials $\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}$, the latter implies, for all $k = 0, \ldots, n$,

$$\sum_{i=0}^{n} \frac{\partial \pi_k}{\partial x_i} \cdot h_i = \sum_{i=0}^{n}\left(\sum_{j=0}^{n} \frac{\partial \pi_k}{\partial\left(\frac{\partial f}{\partial x_j}\right)} \cdot \frac{\partial^2 f}{\partial x_j \partial x_i}\right) \cdot h_i = 0\,. \tag{2.6}$$

Now, we come to the main point of Gordan and Noether's proof.

- Let $g \in \mathbb{K}[x_0, \ldots, x_n]$. Then

$$\sum_{i=0}^{n} \frac{\partial g}{\partial x_i} \cdot h_i = 0 \iff \forall\, \lambda \in \mathbb{K}\colon\ g(x_0 + \lambda h_0, \ldots, x_n + \lambda h_n) = g(x_0, \ldots, x_n)\,. \tag{2.7}$$

Moreover, if $g = g_1 g_2$ with $g_1, g_2 \in \mathbb{K}[x_0, \ldots, x_n]$, and if g satisfies the equivalent conditions (2.7) then the respective conditions are also satisfied for each of the factors g_1, g_2.

To see the latter, consider $g_i(\boldsymbol{x} + \lambda \boldsymbol{h})$, respectively their product, as elements of $(\mathbb{K}[\boldsymbol{x}])[\lambda]$. Since the product is a constant (that is, in $\mathbb{K}[\boldsymbol{x}]$), and since $\mathbb{K}[\boldsymbol{x}]$ is an integral domain, the factors have to be constant, too.

It remains to see the equivalence (2.7), or, more precisely, to see the implication "$\Rightarrow$" in (2.7) (the other implication is an immediate consequence of the chain rule). One possible proof (as given in [2]) is to apply Taylor's Formula and to consider successively the coefficients for λ^k, $k = 1, \ldots, \deg(g)$:

$$g(\boldsymbol{x} + \lambda \boldsymbol{h}) - g(\boldsymbol{x}) = \sum_{k=1}^{\deg(g)} \lambda^k \cdot \Phi_k \,,$$

where

$$\Phi_k := \sum_{i_1, \ldots, i_k} \frac{\partial^k g}{\partial x_{i_1} \ldots \partial x_{i_k}} \cdot \frac{h_{i_1} \cdots h_{i_k}}{k!} \,.$$

The statement follows by induction on k. Indeed, $\Phi_1 = 0$ by hypothesis. Let $k \geq 2$ and suppose $\Phi_{k-1} = 0$. Then $\Phi_k = 0$, because

$$\begin{aligned}
\Phi_k &= \frac{1}{k} \cdot \sum_{i=0}^{n} h_i \cdot \left(\sum_{i_1, \ldots, i_{k-1}} \frac{\partial}{\partial x_i} \left(\frac{\partial^{k-1} g}{\partial x_{i_1} \ldots \partial x_{i_{k-1}}} \right) \cdot \frac{h_{i_1} \cdots h_{i_{k-1}}}{(k-1)!} \right) \\
&\overset{(2.6)}{=} \frac{1}{k \cdot \rho^{k-1}} \cdot \sum_{i=0}^{n} h_i \cdot \left(\sum_{i_1, \ldots, i_{k-1}} \frac{\partial}{\partial x_i} \left(\frac{\partial^{k-1} g}{\partial x_{i_1} \ldots \partial x_{i_{k-1}}} \right) \cdot \frac{\pi_{i_1} \cdots \pi_{i_{k-1}}}{(k-1)!} \right. \\
&\qquad\qquad \left. + \sum_{i_1, \ldots, i_{k-1}} \frac{\partial^{k-1} g}{\partial x_{i_1} \ldots \partial x_{i_{k-1}}} \cdot \frac{\partial}{\partial x_i} \left(\frac{\pi_{i_1} \cdots \pi_{i_{k-1}}}{(k-1)!} \right) \right) \\
&= \frac{1}{k \cdot \rho^{k-1}} \cdot \sum_{i=0}^{n} h_i \cdot \frac{\partial (\Phi_{k-1} \cdot \rho^{k-1})}{\partial x_i} \,.
\end{aligned}$$

• Due to (2.6), the equivalent conditions (2.7) hold for π_k, $k = 0, \ldots, n$, hence also for each factor of π_k. In particular, we obtain

$$\sum_{i=0}^{n} \frac{\partial h_k}{\partial x_i} \cdot h_i = 0 \quad \text{and} \quad \forall \, \lambda \in \mathbb{K} : \; h_k(\boldsymbol{x} + \lambda \boldsymbol{h}) = h_k(\boldsymbol{x}) \,. \tag{2.8}$$

Since, by our assumptions, each $h_k \neq 0$ is homogeneous of degree $e \geq 1$, comparing the coefficients for λ^e gives

$$h_0(h_0, \dots, h_n) = \dots = h_n(h_0, \dots, h_n) = 0 \in \mathbb{K}[x_0, \dots, x_n]. \tag{2.9}$$

- Finally, consider the rational map

$$\psi : \mathbb{P}^n_{\mathbb{K}} \longrightarrow \mathbb{P}^n_{\mathbb{K}}, \quad (x_0 : \dots : x_n) \longmapsto \big(h_0(\boldsymbol{x}) : \dots : h_n(\boldsymbol{x})\big). \tag{2.10}$$

By (2.9), we know that

$$\mathrm{Im}(\psi) \subset V(h_0, \dots, h_n) \subset \mathbb{P}^n_{\mathbb{K}}.$$

Since the h_i are homogeneous of the same degree $e \geq 1$ with $\gcd(h_0, \dots, h_n) = 1$, and since at least two of the h_i are non-zero, the latter implies

$$1 \leq \dim\big(\overline{\mathrm{Im}(\psi)}\big) \leq \dim\big(V(h_0, \dots, h_n)\big) \leq n-2 \tag{2.11}$$

(dim denoting the dimension as projective variety).

In particular, a priori, we get $n \geq 3$. In the following, we shall show that, indeed,

the case under consideration can only occur for $n \geq 4$.

The Case $\mathbf{n = 3}$. By the above, the closure of the image of ψ could only be an irreducible, 1-dimensional projective subvariety of $\mathbb{P}^3$. We shall show that it necessarily would be a *linear* subspace $\mathbb{P}^1 \subset \mathbb{P}^3$; in other words, we would have

$$\dim_{\mathbb{K}}\big(\langle h_0, \dots, h_3\rangle_{\mathbb{K}}\big) \leq 2, \tag{2.12}$$

contradicting our assumptions.

Since the claim is obviously true if $e = \deg h_i = 1$, we can assume that $e \geq 2$. Let $\mathrm{LinHull}(\mathrm{Im}(\psi)) \subset \mathbb{P}^3$ be the linear subspace spanned by $\mathrm{Im}(\psi)$. We have to show that

$$\dim\big(\mathrm{LinHull}(\mathrm{Im}(\psi))\big) = 1. \tag{2.13}$$

First, due to the dimension formula, each (non-empty) fibre of ψ has dimension 2. Hence, for each $\boldsymbol{\xi} \in \mathrm{Im}(\psi)$, there exists a reduced polynomial $\chi^{\boldsymbol{\xi}} \in \mathbb{K}[x_0, \dots, x_3]$ such that $\psi^{-1}(\boldsymbol{\xi}) = V(\chi^{\boldsymbol{\xi}})$.

Assume that the image of ψ is not a linear subspace of $\mathbb{P}^3$. Then, for any two points $\boldsymbol{\xi} = (\xi_0 : \ldots : \xi_3)$, $\boldsymbol{\xi}' = (\xi'_0 : \ldots : \xi'_3) \in \operatorname{Im}(\psi)$, there exists a linear subspace $V(\alpha_0 y_0 + \ldots + \alpha_3 y_3)$ such that

$$\boldsymbol{\xi}, \boldsymbol{\xi}' \in \operatorname{Im}(\psi) \cap V\big(\textstyle\sum_i \alpha_i y_i\big) \quad \text{and} \quad \operatorname{Im}(\psi) \not\subset V\big(\textstyle\sum_i \alpha_i y_i\big)\,.$$

But then the polynomial $\sum_i \alpha_i h_i \in \mathbb{K}[x_0, \ldots, x_3]$ vanishes along each of the (finitely many) fibres over points in $\operatorname{Im}(\psi) \cap V(\sum_i \alpha_i y_i)$, in particular, we obtain that $\chi^{\boldsymbol{\xi}}, \chi^{\boldsymbol{\xi}'}$ both divide $\sum_i \alpha_i h_i$.

Now, by (2.8), we know that

$$\sum_{i=0}^{3} \frac{\partial\big(\sum_j \alpha_j h_j\big)}{\partial x_i} \cdot h_i = 0\,.$$

The latter and (2.7) (which applies to each factor of $\sum_j \alpha_j h_j$) imply that

$$\sum_{i=0}^{3} \frac{\partial \chi^{\boldsymbol{\xi}}}{\partial x_i} \cdot h_i = 0\,.$$

In particular,

$$\forall\, \boldsymbol{p} \in \psi^{-1}(\boldsymbol{\xi}') : \quad \sum_{i=0}^{3} \xi'_i \cdot \frac{\partial \chi^{\boldsymbol{\xi}}}{\partial x_i}(\boldsymbol{p}) = 0\,,$$

which implies that

$$\chi^{\boldsymbol{\xi}'} \ \text{divides} \ \sum_{i=0}^{3} \xi'_i \cdot \frac{\partial \chi^{\boldsymbol{\xi}}}{\partial x_i}\,. \tag{2.14}$$

Exchanging the role of $\boldsymbol{\xi}, \boldsymbol{\xi}'$, we get the analogous statement

$$\chi^{\boldsymbol{\xi}} \ \text{divides} \ \sum_{i=0}^{3} \xi'_i \cdot \frac{\partial \chi^{\boldsymbol{\xi}'}}{\partial x_i}\,. \tag{2.15}$$

For degree reasons, (2.14) and (2.15) can only hold if one (hence, each) of the polynomials on the right-hand side is zero, that is,

$$\sum_{i=0}^{3} \xi'_i \cdot \frac{\partial \chi^{\boldsymbol{\xi}}}{\partial x_i} = 0\,. \tag{2.16}$$

As before (cf. (2.7)), the latter gives

$$\forall\,\lambda \in \mathbb{K}: \quad \chi^{\xi}(x_0 + \lambda\xi'_0, \ldots, x_3 + \lambda\xi'_3) = \chi^{\xi}(x_0, \ldots, x_3)\,.$$

We conclude that, for each $\boldsymbol{p} \in \psi^{-1}(\boldsymbol{\xi})$, the (2-dimensional) fibre $\psi^{-1}(\boldsymbol{\xi})$ contains the linear space spanned by $\boldsymbol{p}$ and $\mathrm{Im}(\psi)$; more precisely,

$$\mathrm{LinHull}\big(\boldsymbol{p}, \mathrm{Im}(\psi)\big) \setminus V(h_0, \ldots, h_3) \subset \psi^{-1}(\boldsymbol{\xi})\,.$$

In particular, either $\mathrm{LinHull}\big(\mathrm{Im}(\psi)\big)$ has dimension at most 1, or it has dimension 2 and contains $\boldsymbol{p}$. Since, for dimension reasons, the latter is impossible to hold for each point $\boldsymbol{p} \in \mathbb{P}^3 \setminus V(h_0, \ldots, h_3)$, we conclude (2.13).

3 Higher Dimensions

Gordan and Noether aim at giving the complete answer (for $n \geq 4$) to the question asked in the title, that is, they try to describe all polynomials $f \in \mathbb{K}[x_0, \ldots, x_n]$, with vanishing Hessian determinant. However, they succeed to give the complete answer only for $n = 4$, while for higher dimensions they give a method for constructing series of polynomials f satisfying $H_f \equiv 0$ and $V(f)$ not being a cone (which answers the question under certain additional implicit assumptions). In the following, we give an outline of the approach applied and of the results obtained.

General Approach. The first step is to describe the (homogeneous) polynomials f with $H_f \equiv 0$ and $V(f)$ not being a cone as solutions of systems of partial differential equations:

Lemma 3.1. *Let $f \in \mathbb{K}[x_0, \ldots, x_n]$ be a homogeneous polynomial such that $V(f)$ is not a cone. Then $H_f \equiv 0$ iff f is a solution of some system of PDE,*

$$\sum_{i=0}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{\partial h_i}{\partial x_k} = 0\,, \qquad k = 0, \ldots, n\,, \tag{3.1}$$

where $h_0, \ldots, h_n \in \mathbb{K}[x_0, \ldots, x_n]$ are either zero or homogeneous of the same degree $e \geq 2$, at least one of the h_i being non-zero.

Proof. First, note that the equalities (3.1) are equivalent to the union of (2.4) and (2.5). Indeed, differentiating (2.4) with respect to x_k and taking into account (2.5) gives (3.1). Vice versa, multiplying the equalities (3.1) by x_k and summing up, Euler's lemma gives (2.4), hence, again by differentiating, (2.5).

But (2.5) implies $H_f \equiv 0$ as at least one of the h_i is non-zero. On the other hand, if $H_f \equiv 0$ then we can choose $h_0, \dots, h_n$ as in the above Step 2, obtaining, in particular, the equalities (2.4) and (2.5). □

Now, for special cases, we can write down the general analytic solution of the system (3.1). For instance, if $h_0 = \cdots = h_s = 0$ and if $h_{s+1}, \dots, h_n$ depend only on $x_0, \dots, x_s$ for some $1 \le s \le n-3$. Then the general solution of (3.1) is given by

$$f = \varphi\big(Q^{(1)}, \dots, Q^{(n-s-\mu-1)},\ x_0,\ x_1, \dots, x_s\big)\,, \tag{3.2}$$

with φ an analytic function, and

$$Q^{(i)} := \det \begin{pmatrix} x_{s+1} & x_{s+2} & \cdots & x_n \\ \frac{\partial h_{s+1}}{\partial A_0} & \frac{\partial h_{s+2}}{\partial A_0} & \cdots & \frac{\partial h_n}{\partial A_0} \\ \vdots & \vdots & & \vdots \\ \frac{\partial h_{s+1}}{\partial A_\mu} & \frac{\partial h_{s+2}}{\partial A_\mu} & \cdots & \frac{\partial h_n}{\partial A_\mu} \\ P^{(i)}_{1,1} & P^{(i)}_{1,2} & \cdots & P^{(i)}_{1,n-s} \\ \vdots & \vdots & & \vdots \\ P^{(i)}_{n-s-\mu-2,1} & P^{(i)}_{n-s-\mu-2,2} & \cdots & P^{(i)}_{n-s-\mu-2,n-s} \end{pmatrix}, \tag{3.3}$$

where

- μ is the dimension of the image of the rational map $\psi : \mathbb{P}^n_{\mathbb{K}} \to \mathbb{P}^n_{\mathbb{K}}$ defined by $h_0, \dots, h_n$ (see (2.10)),
- $A_0, \dots, A_\mu$ are analytic functions in $x_0, \dots, x_s$ such that each h_j can be written as $H_j(A_0, \dots, A_\mu)$ for some H_j,
- $P^{(i)}_{\alpha,1}, \dots, P^{(i)}_{\alpha,n-s}$ are generically chosen analytic functions in $x_0, \dots, x_s$.

Note that for $\mu = 1$, that is, in the case that the closure of the image of ψ is a projective curve (studied in some detail in Section 2), the above assumptions on $h_0, \dots, h_n$ are no restriction. It turns out that in this case, up to a homogeneous coordinate change, the polynomials h_i constructed in Section 2 satisfy $h_0 = \cdots = h_s = 0$ and $h_{s+1}, \dots, h_n \in \mathbb{K}[x_0, \dots, x_s]$ for some $1 \le s \le n-3$. The latter observation allows to deduce the complete answer to the Hesse problem for $n = 4$.

The Case n = 4. In this case, the homogenenous polynomials f satisfying $H_f \equiv 0$ are either cones or (up to homogeneous coordinate change) of the form

$$f = \varphi\big(x_2 P_2 + x_3 P_3 + x_4 P_4,\ x_0,\ x_1\big)\,, \quad \varphi \in K[y_1, \ldots, y_3]\,,$$

with $P_2, \ldots, P_4 \in \mathbb{K}[x_0, x_1]$ linearly independent over $\mathbb{K}$ and homogeneous of the same degree.

Indeed, the latter satisfy $H_f \equiv 0$, since (for dimension reasons) there is a non-trivial algebraic relation between the polars

$$\frac{\partial f}{\partial x_j} = P_j(x_0, x_1) \cdot \frac{\partial \varphi}{\partial y_1}\big(x_2 P_2 + x_3 P_3 + x_4 P_4,\ x_0,\ x_1\big)\,, \quad j = 2, 3, 4\,.$$

Moreover, of course, they include all homogeneous polynomial solutions of the type (3.2) for $\mu = 1$. Hence, the remaining point is to show that for $V(f)$ not a cone the image of the rational map ψ can only be one-dimensional (a priori, it could also have dimension 2).

Gordan and Noether's proof of the latter statement can be sketched as follows: if ψ has a two-dimensional image then the variety $V(h_0, \ldots, h_n)$ can be written as union of (infinitely many) lines. Indeed, this can be deduced from (2.8), when considering the closure of the fibres of ψ in $\mathbb{P}^4_{\mathbb{K}}$: a point $\boldsymbol{q} \in V(h_0, \ldots, h_n)$ is contained in the closure of the fibre $\psi^{-1}(\boldsymbol{p})$ iff the line connecting $\boldsymbol{p}$ and $\boldsymbol{q}$ is contained in it.

Now, $\mathrm{Im}(\psi) \subset V(h_0, \ldots, h_n)$ and both are two-dimensional. Hence, we derive that the closure of the image of ψ is a union of (infinitely many) lines L, too.

The closure of each preimage $\psi^{-1}(L)$ (which is a hypersurface in $\mathbb{P}^4_{\mathbb{K}}$) is easily seen to contain the image of ψ. Then the above reasoning allows to conclude that, a priori, there are only two possible situations:

(1) $\overline{\mathrm{Im}(\psi)}$ consists of concurrent lines meeting in a point $\boldsymbol{p} \in \mathrm{Im}(\psi)$, or

(2) each point $\boldsymbol{q} \in \mathrm{Im}(\psi)$ is the intersection point of two different lines in $\overline{\mathrm{Im}(\psi)}$.

To complete the proof, Gordan and Noether show that in both situations one necessarily obtains $\mathbb{K}$-linear relations between the partials of f, contradicting the assumption that $V(f)$ is not a cone.

The Case n ≥ 5. The following general construction leads to series of polynomials f satisfying $H_f \equiv 0$, whose general elements are not cones.

Let $1 \leq s \leq n-3$ and $1 \leq \mu \leq s$. Choose $A_0, \ldots, A_\mu \in \mathbb{K}[x_0, \ldots, x_s]$ algebraically independent homogeneous polynomials of the same degree $d \geq 1$. Moreover, choose homogeneous polynomials $H_{s+1}, \ldots, H_n \in \mathbb{K}[y_0, \ldots, y_\mu]$ of the same degree such that

$$h_k := H_k\big(A_0, \ldots, A_\mu\big) \in \mathbb{K}[x_0, \ldots, x_s], \qquad k = s+1, \ldots, n,$$

are homogeneous of the same degree $e \geq 2$ and such that the image of the rational map $\mathbb{P}^s_{\mathbb{K}} \to \mathbb{P}^{n-s-1}_{\mathbb{K}}$, $(x_0 : \ldots : x_s) \mapsto \big(h_{s+1}(\boldsymbol{x}) : \ldots : h_n(\boldsymbol{x})\big)$, is μ-dimensional (which is satisfied for H_k generically chosen).

Finally, let the polynomials $Q^{(1)}, \ldots, Q^{(n-s-\mu-1)} \in \mathbb{K}[x_0, \ldots, x_n]$ be defined as in (3.3), with $P^{(i)}_{\alpha,1}, \ldots, P^{(i)}_{\alpha,n-s} \in \mathbb{K}[x_0, \ldots, x_s]$ homogeneous of the same degree. Then, for each homogeneous $\varphi \in \mathbb{K}[y_1, \ldots, y_{n-\mu}]$,

$$f := \varphi\big(Q^{(1)}, \ldots, Q^{(n-s-\mu-1)}, x_0, x_1, \ldots, x_s\big),$$

is a homogeneous polynomial solving the system of PDE (3.1), hence, satisfying $H_f \equiv 0$, and, for generic choices, $V(f)$ is not a cone.

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References

[1] Hesse, M.O.: *Über die Bedingung, unter welcher eine homogene ganze Function von n unabhängigen Variabeln durch lineäre Substitutionen von n andern unabhängigen Variabeln auf eine homogene Function sich zurückführen läßt, die eine Variable weniger enthält.* Journal für reine und angew. Math. **42** (1851), 117–124.

[2] Gordan, P.; Noether, M.: *Über die algebraischen Formen deren Hesse'sche Determinante identisch verschwindet.* Math. Ann. **10** (1876), 547–568.

[3] Permutti, R.: *Sul teorema di Hesse per forme sopra un campo a caratteristica qualunque.* Matematiche **18** (1964), 116–128.

[4] Hartshorne, R.: *Algebraic Geometry.* Springer (1977).

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