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Remarks on osculating linear spaces to projective varieties

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Abstract. Let $X \subset \mathbf{P}^N$ be an integral n-dimensional variety and $m(X, P, i)$ (resp. $m(X, i)$), $1 \leq i \leq N - n + 1$, the Hermite invariants of X measuring the osculating behaviour of X at P (resp. at its general point). Here we prove $m(X, x) + m(X, y) \leq m(X, x + y)$ and $m(X, P, x) + m(X, y) \leq m(X, P, x + y)$ for all integers x, y such that $x + y \leq N - n + 1$, the case $n = 1$ being known (M. Homma, A. Garcia and E. Esteves).

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1 Introduction

We work over an algebraically closed field $\mathbf{K}$. Let $X \subset \mathbf{P}^N$ be an integral non-degenerate variety. Set $n := \dim(X)$. For any $Q \in X$ set $m(X, Q, 0) = 0$ and $m(X, Q, 1) = 1$. Fix an integer i with $2 \leq i \leq N - n + 1$; if there is a linear space V with $\dim(V) = i - 1$, $Q \in V$ and such that $V \cap X$ contains a curve C with $Q \in C$, set $m(X, Q, i) = +\infty$; if there is no such linear space, let $m(X, Q, i)$ be the supremum of the length of the connected component supported by Q of all schemes $X \cap V$, where V is a linear space with $\dim(V) = i - 1$ and $Q \in V$. For all integers i with $0 \leq i \leq N - n + 1$, let $m(X, i)$ be the infimum of all $m(X, Q, i)$ with $Q \in X$. Thus $m(X, 0) = 0$ and $m(X, 1) = 1$. If $n = 1$ the integers $m(X, i)$, $0 \leq i \leq N$, are called the Hermite invariants of the curve X (see [6] or [3]). If $\mathrm{char}(\mathbf{K}) = 0$ and $\dim(X) = 1$ we have $m(X, i) = i$ for every i; for curves in positive characteristic, see [3], Remark 1.5. For the osculating behaviour of surfaces in characteristic zero, see [7]. We always make

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the convention that $+\infty + +\infty = +\infty$ and $a + +\infty = +\infty$ for all integers $a \geq 0$.

Remark 1. Let $X \subset \mathbf{P}^N$ be an integral n-dimensional non-degenerate variety and $Q \in X$. We have $m(X, x) \leq m(X, y)$ and $m(X, Q, x) \leq m(X, Q, y)$ if $x \leq y \leq N - n + 1$. If $m(X, x) \neq +\infty$ (resp. $m(X, Q, x) \neq +\infty$) and $x < y$, then $m(X, x) < m(X, y)$ (resp. $m(X, Q, x) < m(X, Q, y)$). Fix an integer i with $2 \leq i \leq N - n + 1$ and assume that there is no linear space V with $\dim(V) = i - 1$ and such that $V \cap X$ contains an irreducible curve C with $Q \in C$. Since the Grassmannian $G(i, N + 1)$ of all $(i - 1)$-dimensional linear subspaces of $\mathbf{P}^N$ is algebraic, the value of $m(X, i)$ is computed by a maximum, not just a supremum, and in particular $m(X, i) < +\infty$.

Here are our results.

Theorem 1. *Let $X \subset \mathbf{P}^N$ be an integral n-dimensional variety and x, y integers with $x \geq 0$, $y \geq 0$ and $x + y \leq N - n + 1$. Then $m(X, x) + m(X, y) \leq m(X, x + y)$.*

Theorem 2. *Let $X \subset \mathbf{P}^N$ be an integral n-dimensional variety, $P \in X$, and x, y integers with $x \geq 0$, $y \geq 0$ and $x+y \leq N-n+1$. Then $m(X, P, x)+m(X, y) \leq m(X, P, x + y)$.*

Theorem 3. *Let $X \subset \mathbf{P}^N$ be an integral n-dimensional variety, $n \geq 2$, and C an integral curve contained in X. Fix $P \in C$ and a general $Q \in C$. Let x, y be integers with $x \geq 0$, $y \geq 0$ and $x + y \leq N - n + 1$. Then $m(X, P, x) + m(X, Q, y) \leq m(X, P, x + y)$.*

In the case of curves Theorem 1 was proved using combinatorial techniques by M. Homma ([3], Th. 1). His proof was simplified by A. Garcia still using combinatorial techniques ([2]). E. Esteves gave a geometric proof of a more general inequality. In the same year M. Homma gave a very short proof of his original inequality and another proof of Esteves ' inequality ([4]).

Remark 2. Let $X \subset \mathbf{P}^{g-1}$ be the canonical model of a smooth curve of genus g. Assume that the Hermite invariants of X are not classical in the sense of [6]. Such curves do exists in positive characteristics ([8], [5] or [3]). Theorem 1 gives a relation for the Hermite sequence of any Weierstrass point of X.

2 The proofs

Proof of Theorem 1. It is sufficient to prove the case $y \geq x$. The result is obvious if $x = 0$ or $m(X, x) = +\infty$. The second part of Remark 1 gives that either $m(X, y) = +\infty$ or $m(X, y) < m(X, y+1)$, proving the case $x = 1$. Hence we may assume $x \geq 2$. Fix a general pair $(P, Q) \in X \times X$ and linear subspaces V and W with $\dim(V) = x - 1$, $P \in V$, $X \cap V$ containing a zero-dimensional scheme $A(P)$ of length $m(X, x)$ with $A(P)_{red} = \{P\}$, $\dim(W) = y-1$, $Q \in W$ and $X \cap W$ containing a zero-dimensional scheme $Z(Q)$ of length $m(X, y)$ with $Z(Q)_{red} = \{Q\}$ (use the last part of Remark 1). The linear span $\langle V \cup W\rangle$ of $V \cup W$ has dimension at most $x + y - 1$. We choose a linear space M with $\dim(M) = x + y - 1$ and $V \cup W \subseteq M$. There is a flat family of pairs $\{Q_t, W_t\}_{t\in T}$ such that T is an integral curve, $o \in T$, $Q_o = Q$, $W_o = W$, for every $t \in T$, $Q_t \in X$, W_t is a linear subspace of $\mathbf{P}^N$ with $\dim(W_t) = y - 1$, $W_t \cap X$ contains a zero-dimensional scheme $Z(Q_t)$ such that $Z(Q_t)_{red} = \{Q_t\}$ and $\text{length}(Z(Q_t)) \geq m(X, y)$, $Z(Q_o) = Z(Q)$, and there is $a \in T$ with $Q_a = P$. Indeed, since $m(X, Q, y) \neq +\infty$, we may find such a flat family with $m(X, Q_t, y) = m(X, y)$ and $\text{length}(Z(Q_t)) = m(X, y)$ for general $t \in T$. By the properness of the Grassmannian $G(x + y, N + 1)$ of all $(x + y - 1)$-dimensional linear subspaces of $\mathbf{P}^N$, we may construct (taking if necessay a finite covering of T) a flat family $\{M_t\}_{t\in T}$ of $(x + y - 1)$-dimensional linear subspaces of $\mathbf{P}^N$ with $M_o = M$ and $W_t \cup V \subseteq M_t$ for every t. In particular $P \in M_a$. By the properness of the Hilbert scheme $\text{Hilb}(X)$ of X, the scheme $M_a \cap X$ contains a zero-dimensional subscheme of length $m(X, x) + m(X, y)$ with P as support; here we use $Q_t \neq Q_a$ for general $t \in T$ and hence $Z(Q_t) \cap A(P) = \emptyset$ and $\text{length}(Z(Q_t) \cup A(P)) = \text{length}(Z(Q_t)) + \text{length}(A(P))$ for general $t \in T$. Thus $m(X, P, x + y) \geq m(X, x) + m(X, y)$. Since P is general, we have $m(X, x + y) = m(X, P, x + y)$, concluding the proof. □

Proofs of Theorems 2 and 3. Just copy verbatim the proof of Theorem 1 with P fixed and not general. For the proof of Theorem 3 take a flat family $\{Q_t, W_t\}_{t\in T}$ with $Q_t \in C$ for every t. □

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