BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 35(1), 127-150

Determination of the top Baum-Bott number via classical intersection theory

Marcio G. Soares

Abstract. We determine geometrically the number $\int_{\mathbf{V}^m} c_m(\mathrm{T}\mathbf{V}^m - i^*\mathrm{T}_{\mathcal{F}})$, associated to a smooth m-dimensional projective variety $\mathbf{V}^m$, invariant by a one-dimensional holomorphic foliation $\mathcal{F}$ of $\mathbb{P}^n_{\mathbb{C}}$, using polar divisors associated to the foliation and Bézout's theorem.

Keywords: holomorphic foliation, characteristic numbers.

Mathematical subject classification: 32S65.

1 Introduction

Baum-Bott's theorem is a deep generalization, in the complex realm, of both Poincaré-Hopf and Gauss-Bonnet theorems. It relates the evaluation of Chern classes (curvatures) to a sum of residues (indices) of a holomorphic foliation along a singular locus.

The aim of this work is to determine the characteristic number $\int_{\mathbf{V}^m} c_m(\mathrm{T}\mathbf{V}^m - i^*\mathrm{T}_{\mathcal{F}})$, associated to a holomorphic one-dimensional foliation $\mathcal{F}$ of $\mathbb{P}^n_{\mathbb{C}}$ and to a smooth algebraic m-dimensional variety $i : \mathbf{V}^m \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ invariant by the foliation, where $\mathrm{T}\mathbf{V}^m$ and $\mathrm{T}_{\mathcal{F}}$ are the tangent bundles of the variety and of the foliation, respectively, $i^*\mathrm{T}_{\mathcal{F}}$ is the pull-back, via the imbedding i, of $\mathrm{T}_{\mathcal{F}}$ to $\mathbf{V}^m$ and $c_m(\mathrm{T}\mathbf{V}^m - i^*\mathrm{T}_{\mathcal{F}})$ is the top Chern class of the virtual bundle $\mathrm{T}\mathbf{V}^m - i^*\mathrm{T}_{\mathcal{F}}$, which coincides with the class $c_m(\mathrm{T}\mathbf{V}^m \otimes i^*\mathrm{T}^*_{\mathcal{F}})$ since $\mathrm{T}_{\mathcal{F}}$ has rank one. Assuming the singularities of $\mathcal{F}$ along the variety $\mathbf{V}^m$, $sing(\mathcal{F}) \cap \mathbf{V}^m$, to be a finite set of points, this gives a positive integer which we refer to as the top Baum-Bott number of $\mathcal{F}$ relative to $\mathbf{V}^m$. It measures the number of singularities, counting multiplicities, of the foliation along the invariant variety, or the degree of the algebraic 0-cycle $S = sing(\mathcal{F}) \cap \mathbf{V}^m$.

Received 13 June 2002.

This determination is not new since theorem I of [6] exhibits this number through usual properties of Chern classes. In this paper we recover this result by using Bézout's theorem in projective spaces, hence providing an effective and geometrical computation. The motivation for the approach we give here comes from the fact that theorem I of [6] displays this number as a polynomial on the degree of the foliation, whose coefficients are alternating sums of the polar classes of the invariant variety. It is then natural to relate the extrinsic geometry of a variety, invariant by a foliation, to some geometrical object associated to the foliation. This object is what we call the polar (or tangency) divisor of the foliation with respect to a pencil of hyperplanes. Our result states that, given a holomorphic one-dimensional foliation $\mathcal{F}$ of $\mathbb{P}^n_{\mathbb{C}}$ and a smooth irreducible $\mathcal{F}$-invariant variety $i : \mathbf{V}^m \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ then, by choosing conveniently m polar divisors of $\mathcal{F}$, say $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_m$, the number $\int_{\mathbf{V}^m} c_m(\mathrm{T}\mathbf{V}^m - i^*\mathrm{T}_{\mathcal{F}})$ can be obtained by intersecting the $\mathcal{D}_i$'s with $\mathbf{V}^m$ and by taking suitable hyperplane sections. The main result is given in section 6.

2 The top Baum-Bott number, Grothendieck residues and counting degrees

We start by recalling the particular case of Baum-Bott's theorem which is of interest to us. Let $\mathcal{W}$ be a compact complex manifold of dimension m and $\mathcal{F}$ be a one-dimensional holomorphic foliation on $\mathcal{W}$. Assume the singular set $sing(\mathcal{F}) \subset \mathcal{W}$ is a finite set of points. Such a foliation is given by a bundle map

$$\Phi : \mathcal{L}^{-1} \longrightarrow \mathrm{T}\mathcal{W}$$

where $\mathcal{L}$ is a holomorphic line bundle on $\mathcal{W}$ and $sing(\mathcal{F})$ is the analytic subvariety $sing(\mathcal{F}) = \{p : \Phi(p) = 0\}$. $\mathcal{L}^{-1}$ is called the tangent bundle of the foliation and $\mathcal{L}$ is then the cotangent bundle of $\mathcal{F}$. These are denoted by $\mathrm{T}_{\mathcal{F}}$ and $\mathrm{T}^*_{\mathcal{F}}$, respectively. Consider the Chern classes of the virtual bundle $\mathrm{T}\mathcal{W} - \mathrm{T}_{\mathcal{F}}$,

$$c_k(\mathrm{T}\mathcal{W} - \mathrm{T}_{\mathcal{F}}) = c_k(\mathcal{W}) + c_{k-1}(\mathcal{W})c_1(\mathrm{T}^*_{\mathcal{F}}) + \cdots + (c_1(\mathrm{T}^*_{\mathcal{F}}))^k, \;\; 1 \le k \le m$$

and let

$$c^{\alpha}(\mathrm{T}\mathcal{W} - \mathrm{T}_{\mathcal{F}}) = c_1^{\alpha_1}(\mathrm{T}\mathcal{W} - \mathrm{T}_{\mathcal{F}}) \dots c_m^{\alpha_m}(\mathrm{T}\mathcal{W} - \mathrm{T}_{\mathcal{F}})$$

where

$$\alpha = (\alpha_1, \dots, \alpha_m) \qquad \alpha_1 + 2\alpha_2 + \cdots + m\alpha_m = m.$$

By *top Baum-Bott number* we mean the integer

$$\int_{\mathcal{W}} c_m(\mathrm{T}\mathcal{W} - \mathrm{T}_{\mathcal{F}})$$

obtained by taking $\alpha = (0, 0, \dots, 0, 1)$ and where integration is over the fundamental class of $\mathcal{W}$.

We now specialize to projective spaces. Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on $\mathbb{P}^n_{\mathbb{C}}$. Since line bundles on $\mathbb{P}^n_{\mathbb{C}}$ are classified by their Chern classes, such a $\mathcal{F}$ is given by a bundle map

$$\Psi : \mathcal{O}^{\otimes (1-d)} \longrightarrow \mathrm{T}\mathbb{P}^n_{\mathbb{C}}$$

where $\mathcal{O}$ is the hyperplane bundle on $\mathbb{P}^n_{\mathbb{C}}$, $d \in \mathbb{Z}$ and $d \geq 0$. This integer d is called the degree of the foliation and we assume throughout that $d \geq 2$. Also, $\mathcal{O}^{\otimes (1-d)} = \mathrm{T}_{\mathcal{F}}$, the tangent bundle of the foliation and we will write $\mathcal{F}^d$ for a foliation $\mathcal{F}$ of degree d of $\mathbb{P}^n_{\mathbb{C}}$. In general, the singular set $sing(\mathcal{F}^d)$ of $\mathcal{F}^d$ is an analytic set of codimension at least 2. Clearly, $\mathcal{F}^d$ is given locally by a polynomial vector field.

Now, let $\boldsymbol{i} : \mathbf{V}^m \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ be a smooth projective variety of dimension m and assume $\mathbf{V}^m$ is $\mathcal{F}^d$-invariant. Moreover, suppose $S = sing(\mathcal{F}^d) \cap \mathbf{V}^m$ is a finite set. Our aim is to determine

$$\int_{\mathbf{V}^m} c_m(\mathrm{T}\mathbf{V}^m - \boldsymbol{i}^*\mathrm{T}_{\mathcal{F}^d}).$$

In order to evaluate this number we recall the point residues introduced by Grothendieck. Let $f_1, \dots, f_m$ be germs of holomorphic functions at $0 \in \mathbb{C}^m$ whose divisors intersect properly, that is, set theoretically $f^{-1}(0) = \{0\}$, where $f = (f_1, \dots, f_m)$. Consider a germ of holomorphic m-form ω at $0 \in \mathbb{C}^m$ and let Γ be the real (homological) m-cycle defined by $\Gamma = \{z : |f_i(z)| = \epsilon_i\}$, $0 < \epsilon_i$ sufficiently small, $i = 1, \dots, m$. Give Γ the orientation prescribed by declaring positive the m-form $d(\arg f_1) \wedge \cdots \wedge d(\arg f_m)$. The Grothendieck residue symbol at 0 is defined by

$$\mathrm{Res}_0\begin{bmatrix} \omega \\ f_1 \cdots f_m \end{bmatrix} = \left(\frac{1}{2\pi\sqrt{-1}}\right)^m \int_\Gamma \frac{\omega}{f_1 \cdots f_m}.$$

Let $p \in S$ and choose local coordinates around p such that, locally, $\mathbf{V}^m$ is given by $z_{m+1} = \cdots = z_n = 0$. Assume p is sent to $0 \in \mathbb{C}^m$ by this local coordinate chart. The vector field defining $\mathcal{F}^d$ around p is sent, say, to the vector field

$$Y = \sum_{i=1}^{n} Y_i \frac{\partial}{\partial z_i} = \sum_{i=1}^{m} Y_i \frac{\partial}{\partial z_i} + \sum_{i=m+1}^{n} Y_i \frac{\partial}{\partial z_i},$$

with $Y(0) = 0$ and $Y_* = \sum_{i=1}^{m} Y_i \frac{\partial}{\partial z_i}$ being the tangential component to $\mathbf{V}^m$ of Y. Let $(z_1, \ldots, z_m, 0, \ldots, 0) = (z_*, 0)$ and

$$\mathrm{J}Y_*(z_*, 0) = \left(\frac{\partial Y_j}{\partial z_i}\right)(z_*, 0)$$

be the Jacobian matrix of Y_* along $\mathbf{V}^m$ and define $\mathrm{Res}_p(\boldsymbol{i}^*\mathcal{F}^d, c_m)$ by

$$\mathrm{Res}_p(\boldsymbol{i}^*\mathcal{F}^d, c_m) = \mathrm{Res}_0\begin{bmatrix}\det(\mathrm{J}Y_*(z_*, 0))\ dz_1 \wedge \cdots \wedge dz_m \\ Y_1(z_*, 0) \cdots Y_m(z_*, 0)\end{bmatrix}.$$

A particular case of Baum-Bott's theorem (see [1]or [7]) asserts that

Theorem.

$$\int_{\mathbf{V}^m} c_m(\mathrm{T}\mathbf{V}^m - \boldsymbol{i}^*\mathrm{T}_{\mathcal{F}^d}) = \sum_{p\in S} \mathrm{Res}_p(\boldsymbol{i}^*\mathcal{F}^d, c_m). \qquad \square$$

Lemma 1. $\int_{\mathbf{V}^m} c_m(\mathrm{T}\mathbf{V}^m - \boldsymbol{i}^*\mathrm{T}_{\mathcal{F}^d}) = \sum\limits_{p\in S} (\mathbf{V}^m \cdot sing(\mathcal{F}^d))_p$, *the sum of the intersection multiplicities of* $\mathbf{V}^m$ *with* $sing(\mathcal{F}^d)$.

Proof. It is shown in ([3]-chapter 5) or [8] that

$$\mathrm{Res}_0\begin{bmatrix} df_1 \wedge \cdots \wedge df_m \\ f_1 \cdots f_m \end{bmatrix} = \dim_{\mathbb{C}} \mathcal{O}_m/\langle f_1 \cdots f_m\rangle$$

and hence the residue gives the intersection multiplicity $(D_1 \cdots D_m)_0$ at $0 \in \mathbb{C}^m$ of the divisors D_i defined by f_i.

Now,

$$\mathrm{Res}_0\begin{bmatrix}\det(\mathrm{J}Y_*(z_*, 0))\ dz_1 \wedge \cdots \wedge dz_m \\ Y_1(z_*, 0) \cdots Y_m(z_*, 0)\end{bmatrix} = \mathrm{Res}_0\begin{bmatrix} dY_1(z_*, 0) \wedge \cdots \wedge dY_m(z_*, 0) \\ Y_1(z_*, 0) \cdots Y_m(z_*, 0)\end{bmatrix}$$

and since $\mathbf{V}^m$ is smooth, we conclude that $\mathrm{Res}_p(\boldsymbol{i}^*\mathcal{F}^d, c_m) = (\mathbf{V}^m \cdot sing(\mathcal{F}^d))_p$. The lemma follows from Baum-Bott's theorem. $\square$

We shall refer to the number $\sum_{p\in S} (\mathbf{V}^m \cdot sing(\mathcal{F}^d))_p$ as the *degree* of the algebraic zero-cycle S and denote it by $\mathcal{N}(\boldsymbol{i}^*\mathcal{F}^d, \mathbf{V}^m)$. It measures the number of singularities of the foliation along $\mathbf{V}^m$, counting multiplicities.

3 The polar divisor of a foliation

We recall the notion of a polar (or tangency) divisor of a foliation with respect to a pencil of hyperplanes (to my knowledge this idea is already present in the works of Painlevé, [5]). Let $\mathcal{F}^d$ be a one-dimensional holomorphic foliation on $\mathbb{P}^n_{\mathbb{C}}$ of degree $d \geq 2$, with singular set of codimension at least 2. We associate a *polar divisor* to $\mathcal{F}^d$ as follows:

Choose affine coordinates $(z_1, \ldots, z_n)$ such that the hyperplane at infinity, with respect to these, is not $\mathcal{F}^d$-invariant. In these coordinates $\mathcal{F}^d$ is given by a vector field of the form $X = gR + \sum_{i=1}^n Y_i \frac{\partial}{\partial z_i}$ (this is nothing but the local expression of a bundle map $\Psi : \mathcal{O}^{\otimes(1-d)} \to \mathrm{T}\mathbb{P}^n_{\mathbb{C}}$), where $R = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}$, $g(z_1, \ldots, z_n) \not\equiv 0$ is homogeneous of degree d and $Y_i(z_1, \ldots, z_n)$ is a polynomial of degree $\leq d$, $1 \leq i \leq n$. Let H be a hyperplane in $\mathbb{P}^n_{\mathbb{C}}$ which is not invariant by $\mathcal{F}^d$. Then, the set of points in H which are either singular points of $\mathcal{F}^d$ or at which the leaves of $\mathcal{F}^d$ are not transverse to H is an algebraic set, denoted by $\mathcal{T}(H, \mathcal{F})$, of dimension $n-2$ and degree d (observe that $g(z_1, \ldots, z_n) = 0$ is precisely $\mathcal{T}(H_\infty, \mathcal{F})$).

Definition. *Consider a pencil of hyperplanes $\mathcal{H} = \{H_t\}_{t \in \mathbb{P}^1_{\mathbb{C}}}$ with axis $\mathbb{L}^{n-2}$. The polar (or tangency) divisor of $\mathcal{F}^d$ with respect to $\mathcal{H}$ is*

$$\mathcal{D}_{\mathcal{H}} = \bigcup_{t \in \mathbb{P}^1_{\mathbb{C}}} \mathcal{T}(H_t, \mathcal{F}^d).$$

Observe that $\mathcal{D}_{\mathcal{H}}$ is the tangency variety between two foliations of $\mathbb{P}^n_{\mathbb{C}}$; the first one is the codimension one foliation $\mathcal{H}$, which is singular along the axis $\mathbb{L}^{n-2}$, and the other is the one-dimensional foliation $\mathcal{F}^d$.

Lemma 2. *$\mathcal{D}_{\mathcal{H}}$ is either $\mathbb{P}^n_{\mathbb{C}}$ or a (possibly singular) hypersurface of degree $d+1$. Moreover, the set of axes $\mathbb{L}^{n-2}$, of pencils $\mathcal{H}$ for which $\mathcal{D}_{\mathcal{H}}$ is a hypersurface, is Zariski open in the Grassmanian $Gr(n-2; n)$.*

Proof. If there is a hyperplane belonging to the pencil $\mathcal{H}$ which is not $\mathcal{F}^d$-invariant, set it to be the hyperplane at infinity. Then choose coordinates in $\mathbb{P}^n_{\mathbb{C}}$ such that the pencil $\mathcal{H}$ is given, in affine coordinates, by $z_n = c$, $c \in \mathbb{C}$. The vector field inducing $\mathcal{F}^d$ has an expression of the form

$$\mathcal{X} = \sum_{i=1}^n \left[z_i g(z_1, \ldots, z_n) + Y_i(z_1, \ldots, z_n) \right] \frac{\partial}{\partial z_i}$$

where $g \not\equiv 0$. Then $\mathcal{D}_{\mathcal{H}}$ is given by $F_n(z_0, \ldots, z_n) = 0$, the homogenization of the last component of $\mathcal{X}$, $z_n g(z_1, \ldots, z_n) + Y_n(z_1, \ldots, z_n)$, and hence is a hypersurface of degree $d+1$. Another way of showing that it has degree $d+1$ is the following: let p be a point in $\mathbb{L}^{n-2}$, the axis of the pencil. If $p \in sing(\mathcal{F})$ then p is necessarily in $\mathcal{D}_{\mathcal{H}}$, otherwise p is a regular point of $\mathcal{F}^d$. In this case, if $\mathcal{L}$ is the leaf of $\mathcal{F}^d$ through p, then either $\mathrm{T}_p\mathcal{L} \subset \mathbb{L}^{n-2}$ or $\mathrm{T}_p\mathcal{L}$, together with $\mathbb{L}^{n-2}$, determine a hyperplane $H_\alpha \in \mathcal{H}$, and hence we have $p \in \mathcal{T}(H_\alpha, \mathcal{F}^d) \subset \mathcal{D}_{\mathcal{H}}$, so that $\mathbb{L}^{n-2} \subset \mathcal{D}_{\mathcal{H}}$. Now, let $p \in \mathbb{L}^{n-2}$ be a regular point of $\mathcal{F}^d$ and choose a generic line ℓ, transverse to $\mathbb{L}^{n-2}$, passing through p and such that $\mathbb{L}^{n-2}$ and ℓ determine a hyperplane H_β, distinct from H_α. This line ℓ meets $\mathcal{D}_{\mathcal{H}}$ at p and at d further points, counting multiplicities, corresponding to the intersections of ℓ with $\mathcal{T}(H_\beta, \mathcal{F}^d)$. Hence $\mathcal{D}_{\mathcal{H}}$ has degree $d+1$. If all hyperplanes in the pencil $\mathcal{H}$ are $\mathcal{F}^d$-invariant, then $\mathcal{T}(H_t, \mathcal{F}^d) = H_t$ for all $t \in \mathbb{P}^1_{\mathbb{C}}$, so that $\mathcal{D}_{\mathcal{H}} = \mathbb{P}^n_{\mathbb{C}}$. Observe that in this case the singular set of the foliation contains the axis $\mathbb{L}^{n-2}$ of the pencil as a component. Now, choose a hyperplane H in the pencil $\mathcal{H}$ to be the hyperplane at infinity and choose coordinates such that the pencil is given, in the affine space $\mathbb{P}^n_{\mathbb{C}} \setminus H$, by $z_n = c$, $c \in \mathbb{C}$. The foliation is then represented by a vector field of the form

$$\mathcal{X} = \sum_{i=1}^{n} Y_i(z_1, \ldots, z_n) \frac{\partial}{\partial z_i}$$

where $Y_n \equiv 0$. By slightly moving the axis $\mathbb{L}^{n-2}$ we obtain a pencil whose hyperplanes are transverse to the hyperplanes $z_n = c$ and, in this new pencil, there is a hyperplane which is not $\mathcal{F}^d$-invariant. Then apply the reasoning at the begining of the proof of this lemma. □

Example 1. If we consider the two-dimensional Jouanolou's example

$$\mathcal{X} = \left(y^d - x^{d+1}\right)\frac{\partial}{\partial x} + \left(1 - yx^d\right)\frac{\partial}{\partial y}$$

and the pencil $\mathcal{H} = \{(at, bt) : t \in \mathbb{C}\ ,\ (a : b) \in \mathbb{P}^1_{\mathbb{C}}\}$, a straightforward manipulation shows that $\mathcal{D}_{\mathcal{H}}$ is given, in homogeneous coordinates $(X : Y : Z)$ in $\mathbb{P}^2_{\mathbb{C}}$, by the equation $Y^{d+1} - XZ^d = 0$. Now, considering the pencils of vertical lines $x = c$ and of horizontal lines $y = c$ (both based along the line at infinity), we obtain divisors whose equations are just the homogenization of the coordinates of the vector field $\mathcal{X}$ namely, $Y^d Z - X^{d+1} = 0$ and $Z^{d+1} - X^d Y = 0$.

Remark. If H is a hyperplane belonging to the pencil $\mathcal{H}$, then $H \cap \mathcal{D}_{\mathcal{H}} = \mathbb{L}^{n-2} \cup \mathcal{T}(H, \mathcal{F}^d)$.

This is exemplified by $H_\infty \cap \mathcal{D}_{\{z_i=c\}} = \{z_i = 0\} \cup \{g(z_1, \ldots, z_n) = 0\}$.

4 Polar varieties and classes

Let us follow Fulton ([2]-example 14.4.15) and recall polar varieties. Suppose $\mathbf{V}^m \xrightarrow{\imath} \mathbb{P}^n_{\mathbb{C}}$ is a smooth irreducible algebraic subvariety of $\mathbb{P}^n_{\mathbb{C}}$ and let $\mathbb{L}^{n-m+j-2} \subset \mathbb{P}^n_{\mathbb{C}}$ be a linear subspace of dimension $n - m + j - 2$. Then, the j-th polar locus of $\mathbf{V}^m$ is defined by

$$\mathcal{P}_j(\mathbb{L}^{n-m+j-2}) = \left\{q \in \mathbf{V}^m \,|\, \dim\left(\mathrm{T}_q\mathbf{V}^m \cap \mathbb{L}^{n-m+j-2}\right) \geq j-1\right\}$$

for $0 \leq j \leq m$.

Let us digress briefly on the meaning of polar loci. Recall that a finite set $W_1, \ldots, W_k$, of pure dimensional subvarieties of $\mathbb{P}^n_{\mathbb{C}}$, is said to be in *general position* or to *intersect properly* provided

$$\sum_{i=1}^{k} \operatorname{codim} W_i = \operatorname{codim}\left(\bigcap_{i=1}^{k} W_i\right).$$

We have $\operatorname{codim} \mathrm{T}_q\mathbf{V}^m + \operatorname{codim} \mathbb{L}^{n-m+j-2} = n - j + 2$ and hence we would expect to have $\dim(\mathrm{T}_q\mathbf{V}^m \cap \mathbb{L}^{n-m+j-2}) = j-2$. Polar loci are then defined by demanding higher order contact between $\mathrm{T}_q\mathbf{V}^m$ and $\mathbb{L}^{n-m+j-2}$. A computation shows that, if $\mathbb{L}^{n-m+j-2}$ lies in a Zariski open subset of the Grassmanian $Gr(n-m+j-2; n)$, then the codimension of $\mathcal{P}_j(\mathbb{L}^{n-m+j-2})$ in $\mathbf{V}^m$ is precisely j. On the other hand, the subspaces $\mathbb{L}^{n-m+j-2}$ which intersect $\mathbf{V}^m$ properly also lie in a Zariski open set of this Grassmanian. We shall refer to the subspaces $\mathbb{L}^{n-m+j-2}$ which intersect $\mathbf{V}^m$ properly and for which $\operatorname{codim}(\mathcal{P}_j(\mathbb{L}^{n-m+j-2}), \mathbf{V}^m) = j$ as *generic with respect to* $\mathbf{V}^m$. The j-th class of $\mathbf{V}^m$, $\varrho_j(\mathbf{V}^m)$, is the degree of $\mathcal{P}_j(\mathbb{L}^{n-m+j-2})$ and, since the cycle associated to $\mathcal{P}_j(\mathbb{L}^{n-m+j-2})$ is

$$\left[\mathcal{P}_j(\mathbb{L}^{n-m+j-2})\right] = \sum_{i=0}^{j} (-1)^i \binom{m-i+1}{j-i} c_i(\mathbf{V}^m)\, c_1(\imath^*\mathcal{O}(1))^{j-i}$$

we have

$$\varrho_j(\mathbf{V}^m) = \int\limits_{\mathbf{V}^m} \sum_{i=0}^{j} (-1)^i \binom{m-i+1}{j-i} c_i(\mathbf{V}^m)\, c_1(\imath^*\mathcal{O}(1))^{m-i}, \quad 0 \leq j \leq m.$$

Lemma 3. *Suppose $\mathbf{V}^m$ is a smooth, irreducible algebraic variety, $\mathcal{F}^d$-invariant and not contained in $sing(\mathcal{F})$. Let $\mathcal{H}$ be a pencil with axis $\mathbb{L}^{n-2}$ generic with respect to $\mathbf{V}^m$ and such that $\mathcal{D}_{\mathcal{H}}$ is a hypersurface. Then*

$$\mathcal{P}_m(\mathbb{L}^{n-2}) \subset \mathcal{D}_{\mathcal{H}} \quad \textit{and} \quad \mathcal{P}_0(\mathbb{L}^{n-m-2}) = \mathbf{V}^m \not\subset \mathcal{D}_{\mathcal{H}}.$$

Proof. Let us first assume $\mathbf{V}^m$ is a linear subspace of $\mathbb{P}^n_{\mathbb{C}}$. In this case $\mathcal{P}_j = \emptyset$, for $j \geq 1$, so the first assertion of the lemma is meaningless. Assume then $\mathbf{V}^m$ is not a linear subspace and choose a pencil of hyperplanes $\mathcal{H} = \{H_t\}_{t\in\mathbb{P}^1_{\mathbb{C}}}$, with axis $\mathbb{L}^{n-2}$ generic with respect to $\mathbf{V}^m$, so that $\operatorname{codim}(\mathcal{P}_m(\mathbb{L}^{n-2}), \mathbf{V}^m) = m$. If $q \in \mathcal{P}_m(\mathbb{L}^{n-2})$, then $\mathrm{T}_q\mathbf{V}^m$ meets $\mathbb{L}^{n-2}$ in a subspace W of dimension at least $m-1$. If $\mathrm{T}_q\mathbf{V}^m \subset \mathbb{L}^{n-2}$ then any hyperplane $H_t \in \mathcal{H}$ contains $\mathrm{T}_q\mathbf{V}^m$, if not, a line $\ell \subset \mathrm{T}_q\mathbf{V}^m$, $\ell \not\subset \mathbb{L}^{n-2}$, $\ell \cap W$ consisting of a point determines, together with $\mathbb{L}^{n-2}$, a hyperplane $H_t \in \mathcal{H}$ such that $\mathrm{T}_q\mathbf{V}^m \subset H_t$. Since $\mathbf{V}^m$ is $\mathcal{F}^d$-invariant, we have $\mathrm{T}_q\mathcal{L} \subset \mathrm{T}_q\mathbf{V}^m \subset H_t$, in case q is not a singular point of $\mathcal{F}^d$, where $\mathcal{L}$ is the leaf of $\mathcal{F}^d$ through q. This implies $q \in \mathcal{T}(H_t, \mathcal{F}) \subset \mathcal{D}_{\mathcal{H}}$, so that $\mathcal{P}_m(\mathbf{V}^m) \subset \mathcal{D}_{\mathcal{H}}$. It remains to show that $\mathbf{V}^m \not\subset \mathcal{D}_{\mathcal{H}}$. Choose a hyperplane in the pencil which does not contain $\mathbf{V}^m$, call it the hyperplane at infinity and change coordinates in $\mathbb{P}^n_{\mathbb{C}}$ so that the pencil is given, in affine coordinates, by $z_n = c$, $c \in \mathbb{C}$. The foliation is represented, in these coordinates, by

$$\mathcal{X} = X_1\frac{\partial}{\partial z_1} + \cdots + X_n\frac{\partial}{\partial z_n}$$

and $\mathcal{D}_{\mathcal{H}}$ is given by $\{z \in \mathbb{C}^n : X_n = 0\}$, where, by hypothesis, $X_n \not\equiv 0$. If

$$\{z \in \mathbb{C}^n : X_n = 0\} \cap \mathbf{V}^m$$

contains an open subset of $\mathbf{V}^m$, then $\mathbf{V}^m \subset \mathcal{D}_{\mathcal{H}}$. But then, both $\mathbf{V}^m$ and $\mathbb{L}^{n-2}$ are contained in $\mathcal{D}_{\mathcal{H}}$, which is of dimension $n-1$. Hence, $\dim(\mathbf{V}^m \cap \mathbb{L}^{n-2}) \geq m-1$, which contradicts the fact that $\mathbb{L}^{n-2}$ is generic with respect to $\mathbf{V}^m$. It follows that $\{z \in \mathbb{C}^n : X_n = 0\} \cap \mathbf{V}^m$ has codimension 1 in $\mathbf{V}^m$, so $\mathbf{V}^m \not\subset \mathcal{D}_{\mathcal{H}}$. $\square$

5 Genericity assumptions

In the proof of the main result we shall make use of several genericity conditions as well as appropriate choices of linear subspaces of $\mathbb{P}^n_{\mathbb{C}}$. We comment on these now.

Consider the following arrangement of linear subspaces of $\mathbb{P}^n_{\mathbb{C}}$, all generic with respect to $\mathbf{V}^m$ (see section 4), where superscripts denote dimensions and arrows denote inclusions:

$$
\begin{array}{ccccccccc}
 & & & & \mathbb{L}^{n-m-1}_{1,\ldots,m} & & & & \\
 & & & \swarrow & \downarrow & \searrow & & & \\
 & & \mathbb{L}^{n-m}_{1,\ldots,m-1,\widehat{m}} & & & & \mathbb{L}^{n-m}_{\widehat{1},2,\ldots,m} & & \\
 & \swarrow & \downarrow & \searrow & & \swarrow & \downarrow & \searrow & \\
\mathbb{L}^{n-m+1}_{1,\ldots,\widehat{m-1},\widehat{m}} & & & & \mathbb{L}^{n-m+1}_{\widehat{1},2,\ldots,m-1,\widehat{m}} & & & & \mathbb{L}^{n-m+1}_{\widehat{1},\widehat{2},\ldots,m-1,m} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & & & & & & & \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
\mathbb{L}^{n-2}_{1} & \downarrow & \mathbb{L}^{n-2}_{2} & \cdots & \cdots & \cdots & \mathbb{L}^{n-2}_{m-1} & \downarrow & \mathbb{L}^{n-2}_{m} \\
 & \searrow \quad \swarrow & & & & & & \searrow \quad \swarrow & \\
 & \mathbb{L}^{n-1}_{1,2} & & \cdots & \cdots & \cdots & & \mathbb{L}^{n-1}_{m-1,m} &
\end{array}
$$

This diagram has the form of a Pascal's triangle and is obtained as follows: the $(n-2)$-spaces $\mathbb{L}^{n-2}_{i}$ and $\mathbb{L}^{n-2}_{i+1}$ generate the hyperplane $\mathbb{L}^{n-1}_{i,i+1}$ and cut each other along the $(n-3)$-space $\mathbb{L}^{n-3}_{i,i+1}$, $i = 1, \ldots, m-1$, so the three rows at the bottom of the diagram are:

$$
\begin{array}{ccccccc}
 & \mathbb{L}^{n-3}_{1,2} & & \cdots\cdots\cdots & & \mathbb{L}^{n-3}_{m-1,m} & \\
\swarrow & & \searrow & & \swarrow & & \searrow \\
\mathbb{L}^{n-2}_{1} & \downarrow & \mathbb{L}^{n-2}_{2} & \cdots\cdots\cdots & \mathbb{L}^{n-2}_{m-1} & \downarrow & \mathbb{L}^{n-2}_{m} \\
\searrow & & \swarrow & & \searrow & & \swarrow \\
 & \mathbb{L}^{n-1}_{1,2} & & \cdots\cdots\cdots & & \mathbb{L}^{n-1}_{m-1,m} &
\end{array}
$$

The fourth row, from bottom to top, is obtained as follows: the $(n-3)$-spaces $\mathbb{L}^{n-3}_{i,i+1}$ and $\mathbb{L}^{n-3}_{i+1,i+2}$ cut each other along the $(n-4)$-space $\mathbb{L}^{n-4}_{i,i+1,i+2}$, $i = 1, \ldots, m-2$, so we have

$$
\begin{array}{ccccccccc}
 & & \mathbb{L}^{n-4}_{1,2,3} & & \cdots & & \mathbb{L}^{n-4}_{m-2,m-1,m} & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow & \\
 & \mathbb{L}^{n-3}_{1,2} & & \downarrow & \cdots & \downarrow & & \mathbb{L}^{n-3}_{m-1,m} & \\
\swarrow & & \searrow & & & & \swarrow & & \searrow \\
\mathbb{L}^{n-2}_{1} & \downarrow & \mathbb{L}^{n-2}_{2} & & \cdots & & \mathbb{L}^{n-2}_{m-1} & \downarrow & \mathbb{L}^{n-2}_{m}.
\end{array}
$$

Continuing this way we obtain the diagram. The right edge of the diagram

$$
\mathbb{L}^{n-m-1}_{1,\ldots,m} \to \mathbb{L}^{n-m}_{\widehat{1},2,\ldots,m} \to \mathbb{L}^{n-m+1}_{\widehat{1},\widehat{2},\ldots,m-1,m} \to \cdots \to \mathbb{L}^{n-4}_{m-2,m-1,m} \to \mathbb{L}^{n-3}_{m-1,m} \to \mathbb{L}^{n-2}_{m}
$$

will be the reference subspaces for the following polar varieties of $\mathbf{V}^m$:

$$\mathcal{P}_1(\mathbb{L}^{n-m-1}_{1,\dots,m}) \supset \mathcal{P}_2(\mathbb{L}^{n-m}_{\widehat{1},2,\dots,m}) \supset \mathcal{P}_3(\mathbb{L}^{n-m+1}_{\widehat{1},\widehat{2},\dots,m-1,m}) \supset \\ \cdots \supset \mathcal{P}_{m-1}(\mathbb{L}^{n-3}_{m-1,m}) \supset \mathcal{P}_m(\mathbb{L}^{n-2}_m).$$

We assume $\mathbf{V}^m$ is $\mathcal{F}^d$-invariant, that $S = sing(\mathcal{F}^d) \cap \mathbf{V}^m$ is a finite set and let $\mathcal{D}_i$ be the polar divisor of $\mathcal{F}^d$ associated to the pencil $\mathcal{H}_i$, with axis $\mathbb{L}^{n-2}_i$ generic with respect to $\mathbf{V}^m$, $i = 1, 2, \dots, m$. We assume throughout that $\dim \mathcal{D}_i = n-1$ (recall lemma 2). The first genericity condition is:

(GC1). $\mathcal{D}_1, \dots, \mathcal{D}_m$ and $\mathbf{V}^m$ intersect properly, that is,

$$\sum_{i=1}^{m} \operatorname{codim} \mathcal{D}_i + \operatorname{codim} \mathbf{V}^m = \operatorname{codim}\left[\left(\bigcap_{i=1}^{m} \mathcal{D}_i\right) \cap \mathbf{V}^m\right]$$

and $\mathcal{D}_{j+1}, \dots, \mathcal{D}_m, \mathcal{P}_j(\mathbb{L}^{n-m+j-2}_{\widehat{1},\dots,\widehat{j-1},j,\dots,m})$, $j = 0, \dots, m-1$, also intersect properly.

Define the 0-cycles $A_0, A_1, \dots, A_{m-1}, A_m$ as follows: let

$$A^j = \mathcal{D}_{j+1} \cap \cdots \cap \mathcal{D}_m \cap \mathcal{P}_j(\mathbb{L}^{n-m+j-2}_{\widehat{1},\dots,\widehat{j-1},j,\dots,m}), \qquad j = 0, \dots, m-1$$

$$A^j_{i,i+1} = \mathcal{D}_{j+1} \cap \cdots \cap \widehat{\mathcal{D}_i} \cap \widehat{\mathcal{D}_{i+1}} \cap \cdots \mathcal{D}_m \cap \mathcal{P}_j(\mathbb{L}^{n-m+j-2}_{\widehat{1},\dots,\widehat{j-1},j,\dots,m}),\ j = 0, \dots, m-1$$

and set

$$A_j = A^j \setminus \bigcup_{i=j+1}^{m-1} A^j_{i,i+1}, \qquad j = 0, \dots, m-1, \qquad A_m = \mathcal{P}_m(\mathbb{L}^{n-2}_m).$$

The other genericity conditions are:

(GC2). $S \cap \mathbb{L}^{n-1}_{i,i+1} = \emptyset$, for $i = 1, \dots, m-1$.

(GC3$_j$). $\mathcal{D}_{j+1} \cap \cdots \cap \widehat{\mathcal{D}_i} \cap \widehat{\mathcal{D}_{i+1}} \cap \cdots \mathcal{D}_m \cap \mathcal{P}_j(\mathbb{L}^{n-m+j-2}_{\widehat{1},\dots,\widehat{j-1},j,\dots,m}) \cap \mathbb{L}^{n-3}_{i,i+1} = \emptyset$, for $j = 0, \dots, m-1$.

(GC4). $S \cap \mathcal{P}_1(\mathbb{L}^{n-m-1}_{1,\dots,m}) = \emptyset$.

(GC5$_j$). $A^j \cap (\cup_{i=1}^{j} \mathbb{L}^{n-1}_{i,i+1}) = \emptyset$, for $j = 1, \dots, m-1$.

(GC5$_m$). $A_m \cap \mathcal{D}_{m-1} = \emptyset$.

(GC6$_j$). $A^j \cap \mathbb{L}^{n-2}_{i,i+1} = \emptyset$ for $i \geq j+1$ and $j = 0, \dots, m-1$.

We now comment on these conditions. They can all be realized by transversality arguments. The following results can be found in [4]: let $\check{\mathbb{P}}^n$ be the dual of $\mathbb{P}^n_{\mathbb{C}}$, that is, the set of hyperplanes of $\mathbb{P}^n_{\mathbb{C}}$. All hyperplanes of $\mathbb{P}^n_{\mathbb{C}}$ which are tangent to $\mathbf{V}^m$ form a closed irreducible subvariety $\check{\mathbf{V}} \subset \check{\mathbb{P}}^n$ of dimension at most $n-1$. $\check{\mathbf{V}}$ is the dual variety of $\mathbf{V}^m$ and is, in general, a singular variety. The hyperplanes of $\mathbb{P}^n_{\mathbb{C}}$ which are in general position with $\mathbf{V}^m$ form the Zariski-open set $\check{\mathbb{P}}^n \setminus \check{\mathbf{V}}$. Let $\zeta \in \check{\mathbb{P}}^n \setminus \check{\mathbf{V}}$ (so the corresponding hyperplane of $\mathbb{P}^n_{\mathbb{C}}$ intersects $\mathbf{V}^m$ properly). All projective lines in $\check{\mathbb{P}}^n$ through ζ form an $n-1$ dimensional projective space E. If $\dim \check{\mathbf{V}} \leq n-2$, then the lines in E which do not meet $\check{\mathbf{V}}$ form a non-empty open subset of E, whereas in case $\dim \check{\mathbf{V}} = n-1$, the lines in E which avoid the singular set of $\check{\mathbf{V}}$ and intersect $\check{\mathbf{V}}$ transversally form a non-empty open subset of E. It follows from this that if $\mathbb{P}^1 \subset \check{\mathbb{P}}^n$ is a projective line which intersects $\check{\mathbf{V}}$ properly and avoids its singular set, then $\mathbb{P}^1$ defines a unique pencil $\mathcal{H}_{\mathbb{P}^1}$ in $\mathbb{P}^n_{\mathbb{C}}$, whose axis $\mathbb{L}^{n-2}$ is in general position with $\mathbf{V}^m$.

This is the procedure to obtain the axes $\mathbb{L}^{n-2}_i$: start by choosing a ${\mathbb{P}^1}_m$ that cuts $\check{\mathbf{V}}$ properly, avoids its singular set and then choose a point $\check{H}_m \in {\mathbb{P}^1}_m \setminus \check{\mathbf{V}}$. To $\check{H}_m$ corresponds the hyperplane $\mathbb{L}^{n-1}_{m-1,m}$ and $\mathbb{L}^{n-2}_m$ is then transverse to $\mathbf{V}^m$. Also, choose $\check{H}_m$ such that $\mathbb{L}^{n-1}_{m-1,m}$ does not intersect S. Now choose a ${\mathbb{P}^1}_{m-1}$ cutting ${\mathbb{P}^1}_m$ precisely at $\check{H}_m$, transverse to $\check{\mathbf{V}}$ and avoiding its singular set. To ${\mathbb{P}^1}_{m-1}$ corresponds a unique Lefschetz pencil whose axis $\mathbb{L}^{n-2}_{m-1}$ is transverse to $\mathbf{V}^m$ and is contained in $\mathbb{L}^{n-1}_{m-1,m}$. Clearly

$$\mathbb{L}^{n-2}_{m-1} \cap \mathbb{L}^{n-2}_m = \mathbb{L}^{n-3}_{m-1,m}$$

and by moving $\check{H}_m$, if necessary, along ${\mathbb{P}^1}_m$ we can assume that

$$\mathcal{P}_m(\mathbb{L}^{n-2}_m) \cap \mathcal{D}_{m-1} = \emptyset$$

and also, by invoking Bertini-Sard, that $\mathcal{D}_{m-1}$, $\mathcal{D}_m$ and $\mathbf{V}^m$ intersect properly. Continuing this way we obtain the assumed genericity conditions. Examples 2 and 3 in the next section illustrate these conditions.

6 Geometric calculation of the top Baum-Bott number

In this section we present the main result. We assume throughout that we are given a holomorphic foliation $\mathcal{F}^d$ on $\mathbb{P}^n_{\mathbb{C}}$, of degree $d \geq 2$, and a smooth irreducible $\mathcal{F}^d$-invariant variety $i : \mathbf{V}^m \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$. In order to clarify the arguments we will use, we give initially two examples, relative to dimensions one and two, respectively. The one-dimensional case is very simple and we present it for the sake of completness, but the main complications appearing in the case of arbitrary dimension are already present in the two dimensional case.

Example 2. Let $i : \mathbf{V}^1 \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$, $n \geq 2$, be a smooth irreducible $\mathcal{F}^d$-invariant curve, which is not contained in $sing(\mathcal{F}^d)$. Then

$$\int_{\mathbf{V}^1} c_1(\mathrm{T}\mathbf{V}^1 - i^*\mathrm{T}_{\mathcal{F}^d}) = \mathcal{D}_{\mathcal{H}} \cdot \mathbf{V}^1 - \varrho_1(\mathbf{V}^1) = \varrho_0(\mathbf{V}^1)(d+1) - \varrho_1(\mathbf{V}^1),$$

where $\mathcal{D}_{\mathcal{H}}$ is the polar divisor of $\mathcal{F}^d$ relative to a generic pencil $\mathcal{H}$ in $\mathbb{P}^n_{\mathbb{C}}$. To see this write $S = sing(\mathcal{F}^d) \cap \mathbf{V}^1$ and note that S is necessarily finite. Recall that, by section 2, that $\int_{\mathbf{V}^1} c_1(\mathrm{T}\mathbf{V}^1 - i^*\mathrm{T}_{\mathcal{F}^d}) = \mathcal{N}(i^*\mathcal{F}^d, \mathbf{V}^1)$. Consider the finite collection of lines $\{\mathrm{T}_x\mathbf{V}^1 \mid x \in S\}$ and choose a linear subspace $\mathbb{L}^{n-2}$ such that $\mathbb{L}^{n-2} \cap \mathrm{T}_x\mathbf{V}^1 = \emptyset$ for all $x \in S$ (this is always possible since we are just requiring $\mathbb{L}^{n-2}$ and $\mathrm{T}_x\mathbf{V}^1$ to be in general position). This tells us that $S \cap \mathcal{P}_1(\mathbb{L}^{n-2}) = \emptyset$. If $\mathcal{D}_{\mathcal{H}}$ is the polar divisor of $\mathcal{F}^d$ relative to the pencil $\mathcal{H}$ with axis $\mathbb{L}^{n-2}$ then, by lemma 3, we have $S \cup \mathcal{P}_1(\mathbb{L}^{n-2}) \subseteq \mathcal{D}_{\mathcal{H}} \cap \mathbf{V}^1$. On the other hand, let $x \in (\mathcal{D}_{\mathcal{H}} \cap \mathbf{V}^1) \setminus S$. Then $\mathrm{T}_x\mathbf{V}^1 = \mathrm{T}_x\mathcal{L}$, the tangent space to the leaf of $\mathcal{F}^d$ through x, is either contained in $\mathbb{L}^{n-2}$ or cuts $\mathbb{L}^{n-2}$ at a point, since both lie in a hyperplane of $\mathcal{H}$. Hence, $x \in \mathcal{P}_1(\mathbb{L}^{n-2})$ so that $\mathcal{D}_{\mathcal{H}} \cap \mathbf{V}^1 = S \cup \mathcal{P}_1(\mathbb{L}^{n-2})$. Taking degrees on both sides we get, by Bézout's theorem, $\mathcal{D}_{\mathcal{H}} \cdot \mathbf{V}^1 = \mathcal{N}(i^*\mathcal{F}^d, \mathbf{V}^1) + \deg \mathcal{P}_1(\mathbb{L}^{n-2})$.

Example 3. Let $i : \mathbf{V}^2 \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$, $n \geq 3$, be a smooth irreducible $\mathcal{F}^d$-invariant surface with $S = sing(\mathcal{F}^d) \cap \mathbf{V}^2$ a finite set. Then

$$\begin{aligned}\int_{\mathbf{V}^2} c_2(\mathrm{T}\mathbf{V}^2 - i^*\mathrm{T}_{\mathcal{F}^d}) &= \mathcal{D}_1 \cdot \mathcal{D}_2 \cdot \mathbf{V}^2 - \mathcal{T}(\mathbb{L}^{n-1}_{1,2}, \mathcal{F}^d) \cdot \mathbf{V}^2 - \mathcal{D}_2 \cdot \mathcal{P}_1(\mathbb{L}^{n-3}) + \varrho_2 \\ &= \varrho_0(d^2+d+1) - \varrho_1(d+1) + \varrho_2.\end{aligned}$$

where $\mathcal{D}_1$, $\mathcal{D}_2$ are polar divisors of $\mathcal{F}^d$ relative to pencils $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathbb{L}^{n-3} \subset \mathbb{L}^{n-1}_{1,2}$.

To show this let us consider the following arrangement of linear subspaces, where superscripts denote dimensions and arrows denote inclusions:

$$\begin{tikzcd} & \mathbb{L}^{n-3} \arrow[dl] \arrow[dr] \arrow[dd] & \\ \mathbb{L}^{n-2}_1 \arrow[dr] & & \mathbb{L}^{n-2}_2 \arrow[dl] \\ & \mathbb{L}^{n-1}_{1,2} & \end{tikzcd}$$

The $(n-2)$-planes $\mathbb{L}_1^{n-2}$, $\mathbb{L}_2^{n-2}$ are the axes of the pencils $\mathcal{H}_1$, $\mathcal{H}_2$, respectively, $\mathcal{D}_1$ and $\mathcal{D}_2$ are polar divisors of $\mathcal{F}^d$ associated to these pencils and $\mathbb{L}_1^{n-2}$, $\mathbb{L}_2^{n-2}$ are chosen in such a way that the following genericity conditions hold (we shall comment on these conditions in the proof of the main theorem bellow):

(GC1) $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathbf{V}^2$ intersect properly.

(GC2) $S \cap \mathbb{L}_{1,2}^{n-1} = \emptyset$.

(GC3) $\mathbb{L}^{n-3} = \mathbb{L}_1^{n-2} \cap \mathbb{L}_2^{n-2}$ is such that $\mathbf{V}^2 \cap \mathbb{L}^{n-3} = \emptyset$.

(GC4) $S \cap \mathcal{P}_1(\mathbb{L}^{n-3}) = \emptyset$.

(GC5) $\mathcal{T}(\mathbb{L}_{1,2}^{n-1}, \mathcal{F}^d) \cap \mathcal{P}_1(\mathbb{L}^{n-3}) = \emptyset$.

(GC6) $\mathcal{D}_1 \cap \mathcal{P}_2(\mathbb{L}_2^{n-2}) = \emptyset$.

Let $A^0 = \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathbf{V}^2$ and $A_{1,2}^0 = A^0 \cap \mathbb{L}_{1,2}^{n-1} = \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathbf{V}^2 \cap \mathbb{L}_{1,2}^{n-1}$. By (GC1), A_0 is a 0-cycle and by the remark in section 3 we have

$$A_{1,2}^0 = \mathbf{V}^2 \cap (\mathcal{T}(\mathbb{L}_{1,2}^{n-1}, \mathcal{F}^d) \cup \mathbb{L}^{n-3})$$

but, by (GC3), this reduces to $A_{1,2}^0 = \mathbf{V}^2 \cap \mathcal{T}(\mathbb{L}_{1,2}^{n-1}, \mathcal{F}^d)$. Now, set $A_0 = A^0 \setminus A_{1,2}^0$ and invoke (GC2) to conclude $S \subset A_0$. Let $A_1 = \mathcal{D}_2 \cap \mathcal{P}_1(\mathbb{L}^{n-3})$ and $A_2 = \mathcal{P}_2(\mathbb{L}_2^{n-2})$. By (GC4), $S \cap A_1 = \emptyset$ and, by (GC6), $A_2 \cap (A_0 \setminus S) = \emptyset$. We claim that

$$(A_0 \setminus S) = (A_1 \setminus A_2).$$

In fact, let $x \in (A_0 \setminus S)$. Then, the tangent space to the leaf of $\mathcal{F}^d$ passing through x satisfies: $\mathrm{T}_x\mathcal{L} \subset H_1 \cap H_2$, where H_i is a hyperplane belonging to the pencil $\mathcal{H}_i$, and $\mathrm{T}_x\mathcal{L} \subset \mathrm{T}_x\mathbf{V}^2$. Observe that $\mathbb{L}^{n-3} \subset H_1 \cap H_2$. If $\mathrm{T}_x\mathcal{L} \subset \mathbb{L}^{n-3}$ then $x \in \mathbb{L}^{n-3}$ and we obtain $x \in \mathbb{L}_i^{n-2}$, for $i = 1, 2$, which says that $x \in A_{1,2}^0$, a impossibility since $x \in A_0$. So we have $\mathrm{T}_x\mathcal{L} \not\subset \mathbb{L}^{n-3}$ and $\mathrm{T}_x\mathcal{L}$ cuts each axis $\mathbb{L}_i^{n-2}$ at exactly one point, $\mathrm{T}_x\mathcal{L} \cap \mathbb{L}_i^{n-2} = \{p_i\}$. Suppose $p_1 \neq p_2$. Then, because $\mathbb{L}_1^{n-2} \cap \mathbb{L}_2^{n-2} = \mathbb{L}^{n-3}$, $\mathrm{T}_x\mathcal{L}$ and $\mathbb{L}_1^{n-2}$ generate a hyperplane which coincides with the hyperplane generated by $\mathrm{T}_x\mathcal{L}$ and $\mathbb{L}_2^{n-2}$. But this hyperplane is then $\mathbb{L}_{1,2}^{n-1}$ and we get $\mathrm{T}_x\mathcal{L} \subset \mathbb{L}_{1,2}^{n-1}$, which means $x \in \mathbb{L}_{1,2}^{n-1}$, a contradiction. Hence $p_1 = p_2 = p$ say, and $p \in \mathbb{L}^{n-3} = \mathbb{L}_1^{n-2} \cap \mathbb{L}_2^{n-2}$, so that $p = \mathrm{T}_x\mathcal{L} \cap \mathbb{L}^{n-3}$ and $\dim(\mathrm{T}_x\mathbf{V}^2 \cap \mathbb{L}^{n-3}) \geq 0$. This gives $x \in \mathcal{P}_1(\mathbb{L}^{n-3})$ and therefore $x \in A_1$. This shows $(A_0 \setminus S) \subseteq (A_1 \setminus A_2)$. Suppose now we are given $x \in (A_1 \setminus A_2)$. By hypothesis, $\dim(\mathrm{T}_x\mathbf{V}^2 \cap \mathbb{L}^{n-3}) \geq 0$, $x \in \mathcal{D}_2$ and x is a regular point of $\mathcal{F}^d$. Since $\mathrm{T}_x\mathcal{L} \subset H_2 \in \mathcal{H}_2$ it cuts $\mathbb{L}_2^{n-2}$. If $\mathrm{T}_x\mathcal{L} \subset \mathbb{L}_2^{n-2}$, then $x \in \mathbb{L}_{1,2}^{n-1}$ which is absurd since $x \in A_1$. Hence $\mathrm{T}_x\mathcal{L} \cap \mathbb{L}_2^{n-2} = \{p\}$, a single point. Assume $p \notin \mathbb{L}^{n-3}$. Then, because $\dim(\mathrm{T}_x\mathbf{V}^2 \cap \mathbb{L}^{n-3}) \geq 0$ we can choose a point $q \in \mathrm{T}_x\mathbf{V}^2 \cap \mathbb{L}^{n-3}$. Consider the line $\overline{pq}$, passing through p and q. It is contained in both $\mathrm{T}_x\mathbf{V}^2$ and $\mathbb{L}_2^{n-2}$ so that $\dim(\mathrm{T}_x\mathbf{V}^2 \cap \mathbb{L}_2^{n-2}) \geq 1$ and we conclude $x \in A_2$, an absurd. Hence $p \in \mathbb{L}^{n-3}$ and therefore $\mathrm{T}_x\mathcal{L} \cap \mathbb{L}_1^{n-2} \neq \emptyset$. If $\mathrm{T}_x\mathcal{L} \subset \mathbb{L}_1^{n-2}$ then $x \in \mathbb{L}_{1,2}^{n-1}$ and this is forbidden by (GC5). The conclusion is that $\mathrm{T}_x\mathcal{L} \cap \mathbb{L}_1^{n-2}$ reduces to the point p, so that $\mathrm{T}_x\mathcal{L}$ and $\mathbb{L}_1^{n-2}$ generate a hyperplane $H_1 \in \mathcal{H}_1$ and hence $x \in \mathcal{D}_1$. This shows $x \in (A_0 \setminus S)$. From the equality $(A_0 \setminus S) = (A_1 \setminus A_2)$ we conclude, by counting degrees on both sides

$$\deg(A_0) - \mathcal{N}(\boldsymbol{i}^*\mathcal{F}^d, \mathbf{V}^2) = \deg(A_1) - \deg(A_2)$$

which is the same as

$$\mathcal{D}_1 \cdot \mathcal{D}_2 \cdot \mathbf{V}^2 - \mathcal{T}(\mathbb{L}_{1,2}^{n-1}, \mathcal{F}^d) \cdot \mathbf{V}^2 - \mathcal{N}(\boldsymbol{i}^*\mathcal{F}^d, \mathbf{V}^2) = \mathcal{D}_2 \cdot \mathcal{P}_1(\mathbb{L}^{n-3}) - \varrho_2.$$

We now state the main result. Recall the zero-cycles A_j defined in section 5.

Theorem. *Let* $\mathbf{V}^m \xrightarrow{\boldsymbol{i}} \mathbb{P}^n_{\mathbb{C}}$, $n \geq m+1$, *be a smooth irreducible algebraic variety which is invariant by a holomorphic one-dimensional foliation* $\mathcal{F}^d$, *of degree* $d \geq 2$. *Assume* $S = sing(\mathcal{F}^d) \cap \mathbf{V}^m$ *is zero dimensional and let*

$$\mathcal{N}(\boldsymbol{i}^*\mathcal{F}^d, \mathbf{V}^m) = \int_{\mathbf{V}^m} c_m(T\mathbf{V}^m - \boldsymbol{i}^*T_{\mathcal{F}^d}).$$

Then, for a proper choice of the subspaces in the diagram of section 5,

$$\mathcal{N}(\boldsymbol{i}^*\mathcal{F}^d, \mathbf{V}^m) = \sum_{j=0}^{m} (-1)^j \deg(A_j).$$

Moreover, $\deg(A_j) = \varrho_j(\mathbf{V}^m)(d^j + d^{j-1} + \cdots + d + 1)$ *so that*

$$\mathcal{N}(\boldsymbol{i}^*\mathcal{F}^d, \mathbf{V}^m) = \sum_{j=0}^{m} \left[\sum_{i=0}^{j} (-1)^i \varrho_i(\mathbf{V}^m) \right] d^{m-j}.$$

Proof. The proof of the theorem will consist of two parts: in the first part we characterize S and in the second one we use Bézout's theorem to count degrees.

Part I. *Characterization of the 0-cycle S.*

By (GC1) the varieties $\mathcal{D}_1, \dots, \mathcal{D}_m$ and $\mathbf{V}^m$ intersect properly. Let A^0 be the zero cycle $A^0 = \mathcal{D}_1 \cap \mathcal{D}_2 \cap \cdots \cap \mathcal{D}_m \cap \mathbf{V}^m$. We have $\deg(A^0) = \mathcal{D}_1 \cdot \mathcal{D}_2 \cdots \mathcal{D}_m \cdot \mathbf{V}^m = \varrho_0(\mathbf{V}^m)\,(d+1)^m$.

We start by considering the hyperplane sections of A^0 by $\mathbb{L}^{n-1}_{i,i+1}$. So set, for $i = 1, \dots, m$, $A^0_{i,i+1} = A^0 \cap \mathbb{L}^{n-1}_{i,i+1} = \cap_{i=1}^m \mathcal{D}_i \cap \mathbf{V}^m \cap \mathbb{L}^{n-1}_{i,i+1}$. Each $A^0_{i,i+1}$ is again a 0-cycle and, by the remark in section 3

$$A^0_{i,i+1} = \mathcal{D}_1 \cap \cdots \cap \widehat{\mathcal{D}_i} \cap \widehat{\mathcal{D}_{i+1}} \cap \cdots \mathcal{D}_m \cap \mathbf{V}^m \cap \left(\mathcal{T}(\mathbb{L}^{n-1}_{i,i+1}, \mathcal{F}^d) \cup \mathbb{L}^{n-3}_{i,i+1}\right).$$

Impose the genericity conditions

(GC2). $S \cap \mathbb{L}^{n-1}_{i,i+1} = \emptyset$, for $i = 1, \dots, m-1$.

(GC3$_0$). $\mathcal{D}_1 \cap \cdots \cap \widehat{\mathcal{D}_i} \cap \widehat{\mathcal{D}_{i+1}} \cap \cdots \mathcal{D}_m \cap \mathbf{V}^m \cap \mathbb{L}^{n-3}_{i,i+1} = \emptyset$, for $i = 1, \dots, m-1$.

(GC3$_0$) implies that

$$A^0_{i,i+1} = \mathcal{D}_1 \cap \cdots \cap \widehat{\mathcal{D}_i} \cap \widehat{\mathcal{D}_{i+1}} \cap \cdots \mathcal{D}_m \cap \mathbf{V}^m \cap \mathcal{T}(\mathbb{L}^{n-1}_{i,i+1}, \mathcal{F}^d)$$

We define A_0 by

$$A_0 = A^0 \setminus \bigcup_{i=1}^{m-1} A^0_{i,i+1}.$$

Observe that, by (GC2), $S \subset A_0$. Next we consider the 0-cycle $A^1 = \mathcal{D}_2 \cap \cdots \cap \mathcal{D}_m \cap \mathcal{P}_1(\mathbb{L}^{n-m-1}_{1,\dots,m})$. Impose the genericity condition

(GC4). $S \cap \mathcal{P}_1(\mathbb{L}^{n-m-1}_{1,\dots,m}) = \emptyset$.
This tells us that $A^1 \cap S = \emptyset$. The next genericity condition is

(GC5$_1$). $A^1 \cap \mathbb{L}^{n-1}_{1,2} = \emptyset$.
Cutting A^1 by $\mathbb{L}^{n-1}_{i,i+1}$, $i = 2, \dots, m-1$, give the 0-cycles

$$A^1_{i,i+1} = \mathcal{D}_2 \cap \cdots \cap \widehat{\mathcal{D}_i} \cap \widehat{\mathcal{D}_{i+1}} \cap \cdots \mathcal{D}_m \cap \mathcal{P}_1(\mathbb{L}^{n-m-1}_{1,\dots,m}) \cap (\mathcal{T}(\mathbb{L}^{n-1}_{i,i+1}, \mathcal{F}^d) \cup \mathbb{L}^{n-3}_{i,i+1}).$$

Impose the genericity condition

(GC3$_1$). $\mathcal{D}_2 \cap \cdots \cap \widehat{\mathcal{D}_i} \cap \widehat{\mathcal{D}_{i+1}} \cap \cdots \mathcal{D}_m \cap \mathcal{P}_1(\mathbb{L}^{n-m-1}_{1,\dots,m}) \cap \mathbb{L}^{n-3}_{i,i+1} = \emptyset$.

With this at hand we have

$$A^1_{i,i+1} = \mathcal{D}_2 \cap \cdots \cap \widehat{\mathcal{D}_i} \cap \widehat{\mathcal{D}_{i+1}} \cap \cdots \mathcal{D}_m \cap \mathcal{P}_1(\mathbb{L}^{n-m-1}_{1,\dots,m}) \cap \mathcal{T}(\mathbb{L}^{n-1}_{i,i+1}, \mathcal{F}^d).$$

Define

$$A_1 = A^1 \setminus \bigcup_{i=2}^{m-1} A^1_{i,i+1}.$$

The 0-cycles A_j, $j = 2, \dots, m-1$, are successively defined by:

Let $A^j = \mathcal{D}_{j+1} \cap \cdots \cap \mathcal{D}_m \cap \mathcal{P}_j(\mathbb{L}^{n-m+j-2}_{\widehat{1},\dots,\widehat{j-1},j,\dots,m})$ and

$$A^j_{i,i+1} = \mathcal{D}_{j+1} \cap \cdots \cap \widehat{\mathcal{D}_i} \cap \widehat{\mathcal{D}_{i+1}} \cap \cdots \mathcal{D}_m \cap \mathcal{P}_j(\mathbb{L}^{n-m+j-2}_{\widehat{1},\dots,\widehat{j-1},j,\dots,m}).$$

We impose throughout the genericity conditions

(GC3$_j$). $\mathcal{D}_{j+1} \cap \cdots \cap \widehat{\mathcal{D}_i} \cap \widehat{\mathcal{D}_{i+1}} \cap \cdots \mathcal{D}_m \cap \mathcal{P}_j(\mathbb{L}^{n-m+j-2}_{\widehat{1},\dots,\widehat{j-1},j,\dots,m}) \cap \mathbb{L}^{n-3}_{i,i+1} = \emptyset$ and

(GC5$_j$). $A^j \cap (\cup_{i=1}^{j} \mathbb{L}^{n-1}_{i,i+1}) = \emptyset$.

Observe that (GC3$_j$) tells us that

$$A^j_{i,i+1} = \mathcal{D}_{j+1} \cap \cdots \cap \widehat{\mathcal{D}_i} \cap \widehat{\mathcal{D}_{i+1}} \cap \cdots \mathcal{D}_m \cap \mathcal{P}_j(\mathbb{L}^{n-m+j-2}_{\widehat{1},\dots,\widehat{j-1},j,\dots,m}) \cap \mathcal{T}(\mathbb{L}^{n-1}_{i,i+1}, \mathcal{F}^d).$$

Set

$$A_j = A^j \setminus \bigcup_{i=j+1}^{m-1} A^j_{i,i+1}, \qquad j = 0, \dots, m-1.$$

Finally, define

$$A_m = \mathcal{P}_m(\mathbb{L}^{n-2}_m).$$

Since

$$\mathcal{P}_m(\mathbb{L}^{n-2}_m) \subset \cdots \subset \mathcal{P}_j(\mathbb{L}^{n-m+j-2}_{\widehat{1},\dots,\widehat{j-1},j,\dots,m}) \subset \cdots \subset \mathcal{P}_1(\mathbb{L}^{n-m-1}_{1,\dots,m}),$$

by (GC4) we have $S \cap A_j = \emptyset$ for $j = 1, \dots, m$. The genericity condition for A_m is

(GC5$_m$). $A_m \cap \mathcal{D}_{m-1} = \emptyset$.

To characterize S we need the following lemmas:

Lemma 4. $(A_0 \setminus S) \subset A_1$.

Proof. Let $x \in (A_0 \setminus S)$. Then the tangent space to the leaf of $\mathcal{F}^d$ passing through x satisfies $\mathrm{T}_x\mathcal{L} \subset H_1 \cap \cdots \cap H_m$, where H_i is a hyperplane belonging to the pencil $\mathcal{H}_i$, and $\mathrm{T}_x\mathcal{L} \subset \mathrm{T}_x\mathbf{V}^m$. Observe that $\mathbb{L}^{n-m-1}_{1,\ldots,m} \subset H_1 \cap \cdots \cap H_m$. If $\mathrm{T}_x\mathcal{L} \subset \mathbb{L}^{n-m-1}_{1,\ldots,m}$ then $x \in \mathbb{L}^{n-m-1}_{1,\ldots,m}$ and we obtain $x \in \mathbb{L}^{n-2}_i$, for $i = 1, \ldots m$, which says that $x \in A^0_{i,i+1}$, and this is not possible since $x \in A_0$. So we have $\mathrm{T}_x\mathcal{L} \not\subset \mathbb{L}^{n-m-1}_{1,\ldots,m}$ and $\mathrm{T}_x\mathcal{L}$ cuts each axis $\mathbb{L}^{n-2}_i$ at exactly one point, $\mathrm{T}_x\mathcal{L} \cap \mathbb{L}^{n-2}_i = \{p_i\}$. Suppose $\exists\, i, j$ with $p_i \neq p_j$. We may assume i and j are consecutive, say $j = i+1$. Then, because $\mathbb{L}^{n-2}_i \cap \mathbb{L}^{n-2}_{i+1} = \mathbb{L}^{n-3}_{i,i+1}$, $\mathrm{T}_x\mathcal{L}$ and $\mathbb{L}^{n-2}_i$ generate a hyperplane which coincides with the hyperplane generated by $\mathrm{T}_x\mathcal{L}$ and $\mathbb{L}^{n-2}_{i+1}$. But this hyperplane is then $\mathbb{L}^{n-1}_{i,i+1}$ and we get $\mathrm{T}_x\mathcal{L} \subset \mathbb{L}^{n-1}_{i,i+1}$, which means $x \in \mathbb{L}^{n-1}_{i,i+1}$, a contradiction. We conclude $p_1 = p_2 = \cdots = p_m = p$ say. But then $p \in \mathbb{L}^{n-m-1}_{1,\ldots,m} = \cap_{i=1}^m \mathbb{L}^{n-2}_i$, so that $\{p\} = \mathrm{T}_x\mathcal{L} \cap \mathbb{L}^{n-m-1}_{1,\ldots,m}$ and hence, $x \in \mathcal{P}_1(\mathbb{L}^{n-m-1}_{1,\ldots,m})$. This gives $x \in A_1$. □

Lemma 5. $(A_j \setminus A_{j-1}) \subset A_{j+1}$.

Proof. Let $x \in A_j$. By (GC4) x is a regular point of $\mathcal{F}^d$. If $\mathrm{T}_x\mathcal{L}$ is the tangent space to the leaf of $\mathcal{F}^d$ through x, then $\mathrm{T}_x\mathcal{L} \subset H_{j+1} \cap \cdots \cap H_m$, where H_i belongs to the pencil $\mathcal{H}_i$. $\mathrm{T}_x\mathcal{L}$ cuts each $\mathbb{L}^{n-2}_i$, for $i = j+1, \ldots, m$. In case $\mathrm{T}_x\mathcal{L} \subset \mathbb{L}^{n-2}_i$ for some such i, we conclude $x \in A^j_{i,i+1}$, an absurd since $x \in A_j$. Therefore $\mathrm{T}_x\mathcal{L} \cap \mathbb{L}^{n-2}_i = \{p_i\}$, $i = j+1, \ldots, m$. Assume $j < m-1$. Suppose $\exists\, i, k$ with $p_i \neq p_k$. We may assume i and k are consecutive, say $k = i+1$. Then, because $\mathbb{L}^{n-2}_i \cap \mathbb{L}^{n-2}_{i+1} = \mathbb{L}^{n-3}_{i,i+1}$, $\mathrm{T}_x\mathcal{L}$ and $\mathbb{L}^{n-2}_i$ generate a hyperplane which coincides with the hyperplane generated by $\mathrm{T}_x\mathcal{L}$ and $\mathbb{L}^{n-2}_{i+1}$. But this hyperplane is then $\mathbb{L}^{n-1}_{i,i+1}$ and we get $\mathrm{T}_x\mathcal{L} \subset \mathbb{L}^{n-1}_{i,i+1}$, which is a contradiction since $x \in A_j$. We conclude $p_{j+1} = \cdots = p_m = p$ say. But then

$$\mathrm{T}_x\mathcal{L} \cap (\cap_{i=j+1}^m \mathbb{L}^{n-2}_i) = \mathrm{T}_x\mathcal{L} \cap \mathbb{L}^{n-m+j-1}_{\widehat{1},\ldots,\widehat{j},j+1,\ldots,m} = \{p\}.$$

Now, $x \in \mathcal{P}_j(\mathbb{L}^{n-m+j-2}_{\widehat{1},\ldots,\widehat{j-1},j,\ldots,m})$ so that $\dim(\mathrm{T}_x\mathbf{V}^m \cap \mathbb{L}^{n-m+j-2}_{\widehat{1},\ldots,\widehat{j-1},j,\ldots,m}) \geq j-1$. If $p \in \mathbb{L}^{n-m+j-2}_{\widehat{1},\ldots,\widehat{j-1},j,\ldots,m}$ then $\mathrm{T}_x\mathcal{L}$ cuts $\mathbb{L}^{n-2}_j$ and thus we get: $x \in \mathcal{D}_j$ and

$$\dim(\mathrm{T}_x\mathbf{V}^m \cap \mathbb{L}^{n-m+j-3}_{\widehat{1},\ldots,\widehat{j-2},j-1,\ldots,m}) \geq j-2,$$

which means $x \in A_{j-1}$. By hypothesis this is not allowed. Hence,

$$p \in \mathbb{L}^{n-m+j-1}_{\widehat{1},\ldots,\widehat{j},j+1,\ldots,m} \setminus \mathbb{L}^{n-m+j-2}_{\widehat{1},\ldots,\widehat{j-1},j,\ldots,m}.$$

Choose a point $q \in \mathrm{T}_x\mathbf{V}^m \cap \mathbb{L}^{n-m+j-2}_{\widehat{1},\ldots,\widehat{j-1},j,\ldots,m}$. Then the line $\overline{pq}$, passing through p and q, is contained in both $\mathrm{T}_x\mathbf{V}^m$ and $\mathbb{L}^{n-m+j-1}_{\widehat{1},\ldots,\widehat{j},j+1,\ldots,m}$. Therefore $\dim(\mathrm{T}_x\mathbf{V}^m \cap \mathbb{L}^{n-m+j-1}_{\widehat{1},\ldots,\widehat{j},j+1,\ldots,m}) \geq j$, which furnishes $x \in \mathcal{P}_{j+1}(\mathbb{L}^{n-m+j-1}_{\widehat{1},\ldots,\widehat{j},j+1,\ldots,m})$, so that $x \in A_{j+1}$. It remains to consider the case $j = m-1$. Let $x \in A_{m-1}$. Then $x \in \mathcal{P}_{m-1}(\mathbb{L}^{n-3}_{m-1,m})$ so that $\dim(\mathrm{T}_x\mathbf{V}^m \cap \mathbb{L}^{n-3}_{m-1,m}) \geq m-2$. Since $\mathrm{T}_x\mathcal{L} \subset H_m \in \mathcal{H}_m$, it cuts $\mathbb{L}^{n-2}_m$. If $\mathrm{T}_x\mathcal{L} \subset \mathbb{L}^{n-2}_m$, then $x \in \mathbb{L}^{n-1}_{m-1,m}$ which is absurd since $x \in A_{m-1}$. Hence $\mathrm{T}_x\mathcal{L} \cap \mathbb{L}^{n-2}_m = \{p\}$, a single point. Assume $p \in \mathbb{L}^{n-3}_{m-1,m}$. Then $\mathrm{T}_x\mathcal{L} \cap \mathbb{L}^{n-2}_{m-1} \neq \emptyset$. If $\mathrm{T}_x\mathcal{L} \subset \mathbb{L}^{n-2}_{m-1}$ then $x \in \mathbb{L}^{n-1}_{m-1,m}$ and this is forbidden by (GC5_{m-1}). If $\mathrm{T}_x\mathcal{L} \cap \mathbb{L}^{n-2}_{m-1}$ reduces to the point p, then $x \in \mathcal{D}_{m-1}$ so that $x \in A_{m-2}$, which is not allowed by hypothesis. Therefore $p \in \mathbb{L}^{n-2}_m \setminus \mathbb{L}^{n-3}_{m-1,m}$. Choose a point $q \in \mathrm{T}_x\mathbf{V}^m \cap \mathbb{L}^{n-3}_{m-1,m}$. Then the line $\overline{pq}$, passing through p and q, is contained in both $\mathrm{T}_x\mathbf{V}^m$ and $\mathbb{L}^{n-2}_m$. But this tells us that

$$\dim(\mathrm{T}_x\mathbf{V}^m \cap \mathbb{L}^{n-2}_m) \geq m-1$$

and we conclude $x \in \mathcal{P}_m(\mathbb{L}^{n-2}_m) = A_m$. $\square$

Define $B_1 = A_1 \setminus (A_0 \setminus S)$, $B_j = A_j \setminus B_{j-1}$, for $2 \leq j \leq m-1$, and $B_m = A_m$.

Lemma 6. $B_{m-1} = B_m$.

Proof. Let $x \in B_m = A_m$. We have $x \in A_{m-1}$ and, by (GC5_m), $x \notin A_j$, for $j \leq m-2$. Hence $x \in B_{m-1}$. On the other hand, by lemmas 3 and 4, if $x \in B_{m-1}$, then $x \notin A_j$, for $j \leq m-2$. By lemma 5, $x \in A_m = B_m$. $\square$

This finishes the characterization of S.

Part II. *Counting degrees.*

Up to now we have obtained a set theoretical characterization of the singular set of $\mathcal{F}^d$ along the invariant variety $\mathbf{V}^m$, $S = \mathbf{V}^m \cap sing(\mathcal{F}^d)$, in terms of the sets A_j and B_j. By the properties of Grothendieck residues stated in section 2, this is in fact a characterization in terms of algebraic zero cycles. We now compute their degrees, starting with the following:

Combinatorial Lemma. *Let X be a variable. Then*

$$\frac{X^{m+1}-1}{X-1} = \sum_{k=0}^{\frac{m}{2}} (-1)^k \binom{m-k}{k} X^k (X+1)^{m-2k}$$

if m is even and

$$\frac{X^{m+1}-1}{X-1} = \sum_{k=0}^{\frac{m-1}{2}} (-1)^k \binom{m-k}{k} X^k (X+1)^{m-2k}$$

if m is odd.

Proof. Consider the sequence $a_m = \sum\limits_{k \le m/2} (-1)^k \binom{m-k}{k} y^{m-2k}$. The proof consists in finding a closed form for the generating function $F(x) = \sum\limits_{m \ge 0} a_m x^m$. We have:

$$F(x) = \sum_{m \ge 0} x^m \sum_{k \le m/2} (-1)^k \binom{m-k}{k} y^{m-2k}$$

which can be rewriten as

$$F(x) = \sum_k (-1)^k y^{-2k} \sum_{m \ge 2k} \binom{m-k}{k} x^m y^m.$$

But this is the same as

$$F(x) = \sum_k (-1)^k y^{-2k} x^k y^k \sum_{m \ge 2k} \binom{m-k}{k} x^{m-k} y^{m-k}$$

and so

$$F(x) = \sum_k (-1)^k x^k y^{-k} \sum_{r \ge k} \binom{r}{k} (xy)^r.$$

Invoking the elementary identity $\sum\limits_{r \ge 0} \binom{r}{k} s^r = \frac{s^k}{(1-s)^{k+1}}$ we are left with

$$F(x) = \sum_k (-1)^k x^k y^{-k} \frac{(xy)^k}{(1-xy)^{k+1}}$$

and then

$$F(x) = \frac{1}{1-xy} \sum_{k \ge 0} \left(\frac{-x^2}{1-xy} \right)^k = \frac{1}{1-xy} \frac{1}{1+\frac{x^2}{1-xy}} = \frac{1}{1-xy+x^2}.$$

Set $x_\pm = \frac{y \pm \sqrt{y^2-4}}{2}$. The last term in the equality above is precisely:

$$F(x) = \frac{1}{(1-xx_+)(1-xx_-)}.$$

Expand in partial fractions to get

$$F(x) = \frac{x_+}{(x_+ - x_-)(1 - xx_+)} - \frac{x_-}{(x_+ - x_-)(1 - xx_-)}.$$

Expanding in powers of x and equating with the initial expression of $F(x)$ we conclude that

$$a_m = \frac{1}{\sqrt{y^2-4}} \left\{ \left(\frac{y + \sqrt{y^2-4}}{2} \right)^{m+1} - \left(\frac{y - \sqrt{y^2-4}}{2} \right)^{m+1} \right\}.$$

Replace y by $x + (1/x)$. Then $\sqrt{y^2-4} = x - (1/x)$ and since $\sum_{m \geq 0} a_m \, x^m = \frac{1}{(1-xx_+)(1-xx_-)}$ we obtain

$$\sum_{k \leq m/2} (-1)^k \binom{m-k}{k} (x^2+1)^{m-2k} x^{2k} = \frac{1 - x^{2m+2}}{1 - x^2}, \qquad m \geq 0.$$

Finally, set $X = x^2$ to get

$$\sum_{k \leq m/2} (-1)^k \binom{m-k}{k} (X+1)^{m-2k} X^k = \frac{1 - X^{m+1}}{1 - X}, \qquad m \geq 0$$

which is the content of the lemma. $\square$

Let us calculate $\deg(A_0)$. Since

$$A^0_{i,i+1} = \mathcal{D}_1 \cap \cdots \cap \widehat{\mathcal{D}_i} \cap \widehat{\mathcal{D}_{i+1}} \cap \cdots \mathcal{D}_m \cap \mathbf{V}^m \cap \mathcal{T}(\mathbb{L}^{n-1}_{i,i+1}, \mathcal{F}^d)$$

we have $\deg(A^0_{i,i+1}) = \varrho_0(\mathbf{V}^m) d(d+1)^{m-2}$. We now look at the 0-cycle $\cup_{i=1}^{m-1} A^0_{i,i+1}$. Its degree is calculated through the combinatorial principle of exclusion:

$$\deg(\cup A^0_{i,i+1}) = \sum \deg(A^0_{i,i+1}) - \sum \deg(A^0_{i,i+1} \cap A^0_{j,j+1}) +$$

$$+ \sum \deg(A^0_{i,i+1} \cap A^0_{j,j+1} \cap A^0_{k,k+1}) + \cdots + (-1)^{m-1} \deg(\cap_{i=1}^{m-1} A^0_{i,i+1}).$$

Consider $A^0_{i,i+1} \cap A^0_{i+1,i+2} = A^0 \cap \mathbb{L}^{n-1}_{i,i+1} \cap \mathbb{L}^{n-1}_{i+1,i+2} = A^0 \cap \mathbb{L}^{n-2}_{i+1}$. Since $\operatorname{codim}(A^0) + \operatorname{codim}(\mathbb{L}^{n-2}_{i+1}) = n+2$ we impose the genericity condition

(GC6$_0$). $A^0_{i,i+1} \cap A^0_{i+1,i+2} = \emptyset$, for $i = 1, \ldots, m-1$.

Hence, ($GC6_0$) tells us that, if m is even and we intersect more than $\frac{m}{2}$ such cycles, then the intersection is void, whereas in case m is odd, the intersection of more than $\frac{m-1}{2}$ of these cycles will be empty. Therefore, for m even

$$\deg(\cup_{i=1}^{m-1} A^0_{i,i+1}) = \sum_{i=1}^{m-1} \deg(A^0_{i,i+1}) - \sum_{i+1<j} \deg(A^0_{i,i+1} \cap A^0_{j,j+1}) +$$

$$+ \sum_{\substack{i+1<j\\ j+1<k}} \deg(A^0_{i,i+1} \cap A^0_{j,j+1} \cap A^0_{k,k+1}) + \cdots$$

$$\cdots + (-1)^{\frac{m}{2}} \sum_{\substack{i_1+1<i_2\\ i_2+1<i_3\\ \vdots\\ i_{\frac{m}{2}-1}+1<i_{\frac{m}{2}}}} \deg(A^0_{i_1,i_1+1} \cap \cdots \cap A^0_{i_{\frac{m}{2}},i_{\frac{m}{2}}+1})$$

and, for m odd

$$\deg(\cup_{i=1}^{m-1} A^0_{i,i+1}) = \sum_{i=1}^{m-1} \deg(A^0_{i,i+1}) - \sum_{i+1<j} \deg(A^0_{i,i+1} \cap A^0_{j,j+1}) +$$

$$+ \sum_{\substack{i+1<j\\ j+1<k}} \deg(A^0_{i,i+1} \cap A^0_{j,j+1} \cap A^0_{k,k+1}) + \cdots$$

$$\cdots + (-1)^{\frac{m-1}{2}} \sum_{\substack{i_1+1<i_2\\ i_2+1<i_3\\ \vdots\\ i_{\frac{m-1}{2}-1}+1<i_{\frac{m-1}{2}}}} \deg(A^0_{i_1,i_1+1} \cap \cdots \cap A^0_{i_{\frac{m-1}{2}},i_{\frac{m-1}{2}}+1}).$$

To calculate these sums we consider intersections of two, three and successively up to $\frac{m}{2}$, for even m, (up to $\frac{m-1}{2}$, for odd m) such cycles. Intersections of two of them, $A^0_{i,i+1} \cap A^0_{j,j+1}$ with $i+1 < j$ give

$$\begin{aligned} A^0_{i,i+1} \cap A^0_{j,j+1} &= \mathcal{D}_1 \cap \mathcal{D}_2 \cap \cdots \cap \mathcal{D}_m \cap \mathbf{V}^m \cap \mathbb{L}^{n-1}_{i,i+1} \cap \mathbb{L}^{n-1}_{j,j+1} = \\ &= \mathcal{D}_1 \cap \cdots \cap \widehat{\mathcal{D}_i} \cap \widehat{\mathcal{D}_{i+1}} \cap \cdots \cap \widehat{\mathcal{D}_j} \cap \widehat{\mathcal{D}_{j+1}} \cap \cdots \cap \mathcal{D}_m \cap \mathbf{V}^m \cap \\ &\cap \big(\mathcal{T}(\mathbb{L}^{n-1}_{i,i+1}, \mathcal{F}^d) \cup \mathbb{L}^{n-3}_{i,i+1}\big) \cap \big(\mathcal{T}(\mathbb{L}^{n-1}_{j,j+1}, \mathcal{F}^d) \cup \mathbb{L}^{n-3}_{j,j+1}\big). \end{aligned}$$

Now, it follows from ($\mathrm{GC3}_0$) that

$$\begin{aligned} A^0_{i,i+1} \cap A^0_{j,j+1} = \mathcal{D}_1 \cap \cdots \cap \widehat{\mathcal{D}_i} \cap \widehat{\mathcal{D}_{i+1}} \cap \cdots \cap \widehat{\mathcal{D}_j} \cap \widehat{\mathcal{D}_{j+1}} \cap \cdots \cap \mathcal{D}_m \cap \\ \mathbf{V}^m \cap \cap \mathcal{T}(\mathbb{L}^{n-1}_{i,i+1}, \mathcal{F}^d) \cap \mathcal{T}(\mathbb{L}^{n-1}_{j,j+1}, \mathcal{F}^d) \end{aligned}$$

so its degree is $\deg(A^0_{i,i+1} \cap A^0_{j,j+1}) = \varrho_0(\mathbf{V}^m) d^2 (d+1)^{m-4}$. When we consider intersections of three, four, etc. such cycles, repeated use of ($\mathrm{GC3}_0$) leads to $\deg(A^0_{i_1,i_1+1} \cap \cdots \cap A^0_{i_\ell,i_\ell+1}) = \varrho_0(\mathbf{V}^m) d^\ell (d+1)^{m-2\ell}$. Now, there are $\binom{m-2}{2}$ intersections of two such cycles, with $i+1 < j$, and $\binom{m-\ell}{\ell}$ intersections of ℓ such cycles, with the constraint $i_1 + 1 < i_2, \ldots, i_{\ell-1} + 1 < i_\ell$. Therefore, for even m

$$\deg(\cup_{i=1}^{m-1} A^0_{i,i+1}) = \varrho_0(\mathbf{V}^m) \sum_{k=1}^{\frac{m}{2}} (-1)^{k+1} \binom{m-k}{k} d^k (d+1)^{m-2k}$$

and, for odd m

$$\deg(\cup_{i=1}^{m-1} A^0_{i,i+1}) = \varrho_0(\mathbf{V}^m) \sum_{k=1}^{\frac{m-1}{2}} (-1)^{k+1} \binom{m-k}{k} d^k (d+1)^{m-2k}.$$

The conclusion is that the degree of the cycle $A_0 = A^0 \setminus (\cup_{i=1}^{m-1} A^0_{i,i+1})$ is, by the Combinatorial Lemma,

$$\deg(A_0) = \deg(A^0) - \deg(\cup_{i=1}^{m-1} A^0_{i,i+1}) = \varrho_0(\mathbf{V}^m)(d^m + d^{m-1} + \cdots + d + 1).$$

To calculate the degrees of the cycles $A_1, \ldots, A_{m-1}$ we proceed verbatim as for A_0, imposing the following genericity condition for A_j, $j = 1, \ldots, m-1$:

($\mathbf{GC6}_j$). $A^j \cap \mathbb{L}^{n-2}_{i,i+1} = \emptyset$ for $i \geq j+1$.

We then obtain that the degree of A_j is

$$\begin{aligned} \deg(A_j) &= \deg(A^j) - \deg(\cup_{i=j+1}^{m-1} A^j_{i,i+1}) = \\ &= \varrho_j(\mathbf{V}^m)(d^{m-j} + d^{m-j-1} + \cdots + d + 1). \end{aligned}$$

Of course, the degree of A_m is $\deg(A_m) = \varrho_m(\mathbf{V}^m)$. Let $\mathcal{N}(\boldsymbol{i}^*\mathcal{F}^d, \mathbf{V}^m)$ denote the degree of S. Then the degrees of the cycles B_j are:

$$\begin{aligned}\deg(B_j) = \deg(A_j) - \deg(B_{j-1}) = \\ = (-1)^{j+1}\mathcal{N}(\boldsymbol{i}^*\mathcal{F}^d, \mathbf{V}^m) + \sum_{i=0}^{j}(-1)^i \deg(A_{j-i})\end{aligned}$$

for $j = 1, \ldots, m-1$ and since $B_m = B_{m-1}$, it follows that

$$\deg(B_m) = (-1)^m\mathcal{N}(\boldsymbol{i}^*\mathcal{F}^d, \mathbf{V}^m) + \sum_{i=0}^{m-1}(-1)^i \deg(A_{m-1-i})$$

which amounts to $\mathcal{N}(\boldsymbol{i}^*\mathcal{F}^d, \mathbf{V}^m) = \sum_{i=0}^{m}(-1)^i \deg(A_i)$.

This finishes the proof of the teorem. □

Acknowledgements. I'm grateful to Pronex-Dynamical Systems (Brazil) for financial support, to I. Vainsencher, P. Sad and M. Spira for valuable discussions. I would like also to thank the referee for many helpful suggestions.

References

[1] P. Baum & R. Bott, *Singularities of holomorphic foliations*, Journal of Differential Geometry **7** (1972), 279–342.

[2] W. Fulton, *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3.Folge - Band 2, Springer-Verlag (1984).

[3] P. Griffiths & J. Harris, *Principles of Algebraic Geometry*, John Wiley, New York (1978).

[4] K. Lamotke, *The Topology of Complex Projective Varieties after S. Lefschetz*, Topology **20** (1981), 15–51.

[5] P. Painlevé, *Leçons sur la Théorie Analytique des Équations Differentielles*, Hermann, Paris (1897).

[6] M.G. Soares, *Projective varieties invariant by one-dimensional foliations*, Annals of Mathematics **152** (2000), 369–382.

[7] M.G. Soares, *On Chern's proof of Baum-Bott's theorem*, C.R.Acad.Sci.Paris, **333**, Série I, (2001), 757–761.

[8] M.G. Soares, *Lectures on Point Residues*, Monografias del IMCA n. 28, IMCA, Peru (2002).

Marcio G. Soares
Departamento de Matemática - ICEx - UFMG
31270-901 - Belo Horizonte
BRASIL

E-mail: msoares@mat.ufmg.br