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PL Homeomorphisms of the circle which are piecewise C^1 conjugate to irrational rotations

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Abstract. For a PL homeomorphism f with irrational rotation number α, the following properties are equivalent

(i) f is conjugate to the rotation by α through a piecewise C^1 homeomorphism,

(ii) the number of break points of f^n is bounded by some constant that doesn't depend on n,

(iii) f is conjugate to an affine 2-intervals exchange transformation (with rotation number α) through a PL homeomorphism,

(iv) f is conjugate to the rotation by α through a piecewise analytic homeomorphism.

Keywords: PL Homeomorphisms of the circle, invariant measure, conjugacy.

Mathematical subject classification: 37E10, 37C15.

1 Introduction

We write $S^1 = \frac{\mathbb{R}}{\mathbb{Z}}$ for the circle. We have the natural projection $\Pi : \mathbb{R} \to S^1$. This provides a lift of a homeomorphism $f : S^1 \to S^1$ to a homeomorphism $\tilde{f} : \mathbb{R} \to \mathbb{R}$ with the property $f \circ \Pi = \Pi \circ \tilde{f}$.

An important characteristic of circle homeomorphism is "rotation number" defined by H. Poincaré ([12]) as

$$\rho(f) = \lim_{n \to +\infty} \frac{\tilde{f}^n(x) - x}{n} \pmod 1.$$

Assuming that f is a C^2-diffeomorphism and $\rho(f)$ is irrational, A. Denjoy ([3]) proved that f is topologically conjugate to the rotation by $\rho(f)$.

This classical result can be extended (with the same proof) to the case of class P homeomorphisms.

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Definitions. A *class P* homeomorphism is an orientation preserving homeomorphism of the circle with the following properties:
its lift $\tilde{f}$ is differentiable except in countably many points called *break points* admitting left and right derivatives; the derivative of $\tilde{f}$ is a 1-periodic function, its restriction to [0, 1] denoted by Df has the following properties:
— there exists some constants $0 < a < b$ such that:

$$a < Df(x) < b, \text{ for all } x \text{ where } Df \text{ exists,}$$
$$a < Df_+(c) < b \text{ and } a < Df_-(c) < b \text{ at the break points;}$$

— Df has bounded variations (on [0, 1]).

The ratio $\sigma_f(c) := \frac{Df_+(c)}{Df_-(c)}$ is called *the derivative jump* of f in c, or f-jump.

The maps we intent to investigate: the *piecewise linear (PL) homeomorphisms* (orientation preserving homeomorphisms of the circle with derivative piecewise constant) are the simplest examples of class P homeomorphisms that are not C^2-diffeomorphisms.

Notice that, if a homeomorphism is conjugate to an irrational rotation, the conjugating homeomorphism h is unique up to the normalization $h(0) = 0$, and the question of its regularity -originated by Arnol'd and Moser- naturally arises. A global result was proved by M. Herman ([7]): it states that if f is a C^2 diffeomorphism of the circle with irrational rotation number α of constant type and f is close to the rotation by α then f admits an invariant probability measure equivalent to the Haar measure. The global version of this result (the condition that f is close to the rotation by α is no longer required) is proved in [8].

Also, Khanin and Sinai ([9]) proved that if f is a $C^{2+\nu}$ diffeomorphism of the circle with irrational rotation number of constant type then f is $C^{1+\nu}$ conjugate to the rotation by $\rho(f)$.

In the case of homeomorphisms, the situation becomes opposite : PL homeomorphisms were considered by M. Herman as examples of homeomorphisms with irrational rotation number and without absolutely continuous invariant measure. The case of general class P homeomorphisms (non PL homeomorphisms) with one break point has been studied by A.A. Dzhalilov and K.M. Khanin [4]. Assuming the rotation number is irrational, they proved that the invariant probabilty measure is singular with respect to the Haar measure. In [11], the author proves the same conclusion for PL homeomorphisms that have the following properties:

- irrational rotation number of constant type,
- disjoint break points orbits,
- $\mathbb{Z}$-independant logarithms of sloops.

Yet, the case of PL homeomorphisms is double, there exist PL homeomorphisms with absolutely continuous invariant measure: consider the homeomorphisms that are obtained by conjugating a rotation through a PL homeomorphism. More over, M. Herman's result give rise to examples of PL homeomorphisms with absolutely continuous invariant measure but not PL conjugate to a rotation. In more detail, in section 7, chapter VI of [6], M. Herman studies the following family of PL homeomorphisms.

Definition of the Herman's examples. Let $\lambda > 1$ and $\beta > 0$ be two real numbers. We define, for $x \in [0, 1]$:

$$f_{\beta,\lambda}(x) = \begin{cases} \lambda x \text{ if } \; 0 \le x \le a \\ \lambda^{-\beta}(x-1)+1 \text{ if } \; a \le x \le 1, \end{cases}$$

with $\lambda a = \lambda^{-\beta}(a-1)+1$.

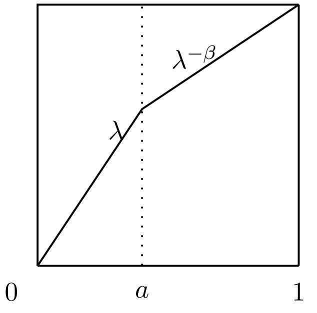

Figure 1: Herman examples $f_{\beta,\lambda}$.

Consider now the 1-parameter family of PL homeomorphisms of the circle $R_b \circ f_{\beta,\lambda}$ where $b \in [0, 1]$ and R_b denotes the rotation by b. By continuity of the rotation number, M. Herman proved that given an irrational number α, there exists a unique $b \in [0, 1]$ such that $R_b \circ f_{\beta,\lambda}$ has rotation number α, this homeomorphism is denoted by $f_{\alpha,\beta,\lambda}$.

Herman's result. *The following properties are equivalent:*

(i) $f_{\alpha,\beta,\lambda}$ *is conjugate to* R_α *through an absolutely continuous homeomorphism,*

(ii) $f_{\alpha,\beta,\lambda}$ *is conjugate to* R_α *through a lipschitz homeomorphism,*

(iii) $f_{\alpha,\beta,\lambda}$ *is conjugate to* R_α *through a piecewise* C^∞ *homeomorphism (but not PL),*

(iv) $\frac{\beta}{1+\beta} \in \mathbb{Z}\alpha \pmod 1$,

(v) *the break points* 0 *and* a *belong to the same* f*-orbit.*

Our purpose in this paper is to give characterizations of the PL homeomorphisms of the circle that are piecewise C^1 conjugate to irrational rotations. Notice that for a homeomorphism which is piecewise C^1 conjugate to rotation, the numbers of break points of the n-th iterates is bounded by some constant that does not depend on n. We'll see that this property is characteristic.

A very special family of Herman's examples has been studied by others authors ([1], [2], [10]) in the context of intervals exchange transformations. We call them *affine 2-intervals exchange transformations*, they are the Herman's examples with break points 0 and a satisfying $f(a) = 0$. Fixing the initial break point to be 0, these maps are uniquely determined by their loops (λ, λ') we denote them by $A_{\lambda,\lambda'}$ whithout making any distinction between the PL homeomorphism of the circle f and the associated interval exchange transformation (i.e. the bijection from [0, 1[given by $\tilde{f}(mod 1)$ that is drawn on picture 2).

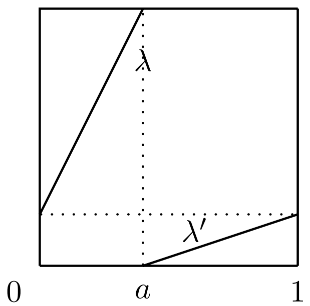

Figure 2: Affine 2-intervals exchange transformation $A_{\lambda,\lambda'}$.

We leave the reader verify the following properties (or consult [10]).

Properties of 2-intervals exchange transformations.

— The piecewise C^∞ (analytic) homeomorphism h with only one break point given by the restriction to [0, 1[of its lift $\tilde{h}$ (denoted also by h)

$$h(x) := \frac{(\frac{\lambda}{\lambda'})^x - 1}{\frac{\lambda}{\lambda'} - 1}$$

conjugate $A_{\lambda,\lambda'}$ to the rotation by $\frac{\log\lambda}{\log\lambda - \log\lambda'}$, more precisely we have $A_{\lambda,\lambda'} = h \circ R_\alpha \circ h^{-1}$.

— Conversely, the homeomorphism $h_w(x) := \frac{w^x-1}{w-1}$ conjugate any rotation to an affine 2-intervals exchange transformation.

— Two distincts (distincts pairs of loops) affine 2-intervals exchange transformations are not PL conjugate, we suppose that they have the same initial break point 0. Else, the only way for affine 2-intervals exchange transformations to be conjugate is to be conjugate through a rotation R_b, this operation translates the initial point 0 in b.

We'll see in the coming theorem that in some sense this family functions as normal forms for PL homeomorphisms that are piecewise C^1 conjugate to rotations.

Theorem. *Let f be a PL homeomorphsim with irrational rotation number α. The following properties are equivalent:*

(i) *f is conjugate to R_α through a piecewise C^1 homeomorphism,*

(ii) *the number of break points of f^n is bounded by some constant that doesn't depend on n,*

(iii) *f is conjugate to an affine 2-intervals exchange transformation $A_{\lambda,\lambda'}$ (with rotation number α) through a PL homeomorphism,*

(iv) *f is conjugate to R_α through a piecewise C^∞ (analytic) homeomorphism.*

Corollary. *Let f be a PL homeomorphsim with irrational rotation number α with only one orbit of break points then f is conjugate to a $A_{\lambda,\lambda'}$ through a PL homeomorphism and is conjugate to R_α through a piecewise C^∞ (analytic) homeomorphism.*

This corollary is related to the result of [5]: Dzhalilov considers piecewise $C^{2+\varepsilon}$ homeomorphisms f of the circle with break points $x_{p_0}, x_{p_1}, \dots, x_{p_m}$, $p_0 = 0 < p_1 < \cdots < p_m$, such that $x_{p_i} = f^{p_i}(x_0)$ and the product of the derivative jumps in this points is trivial. Assuming that the rotation number ρ of f is irrational and its chain fraction expansion is $\rho = [k_1, k_2, k_3, \dots]$ with $k_n \leq$ const, he proved that f is conjugate to a rotation through a piecewise $C^{1+\epsilon}$ homeomorphism.

Some implications of our theorem are obvious: $(i) \Longrightarrow (ii)$ and $(iv) \Longrightarrow (i)$. The implication $(iii) \Longrightarrow (iv)$ is a consequence of the recalled properties for affine 2-intervals exchange transformations. The only implication remaining is $(ii) \Longrightarrow (iii)$.

2 Proof of the theorem and its corollary

Lemma 1. *Let f and g be two class P homeomorphisms of the circle with the same irrational rotation number. If there exists an integer n such that $f^n = g^n$ then $f = g$.*

Lemma 2. *Assume A is an affine 2-intervals exchange transformation with rotation number α, N is a fixed integer and α_0 is such that $N\alpha_0 = \alpha (mod 1)$.*

There exists an affine 2-intervals exchange transformation with rotation number α_0 such that $(A_0)^N = A$.

Lemma 3. *Assume F is a PL homeomorphism with $2p$ break points $c_1, \dots, c_p$, $d_1, \dots, d_p$ satisfying the following properties:*

- $F(c_i) = d_i$,
- $\sigma_F(c_i).\sigma_F(d_i) = 1$,
- $F(d_i)$ *is not a break point of* F.

Then F is PL conjugate to an affine 2-intervals exchange transformation.

Lemma 4. *Let f be a PL homeomorphism of the circle with irrational rotation number. If the number of break points of f^n is bounded by some constant that does not depend on n then there exists an integer N such that f^N satisfies the hypothesis of lemma 3.*

Combining this 4 lemmas we obtain the implication $(ii) \Longrightarrow (iii)$. In more detail, assume f is a PL homeomorphism with irrational rotation number satisfying the condition that the number of break points of f^n is bounded by some constant that does not depend on n. By lemma 4 and lemma 3, it's possible to find an iterate $F = f^N$ of f that is PL conjugate to an affine 2-intervals exchange transformation A with rotation number $N\rho(f)$. By lemma 2, there exists A_0 an affine 2-intervals exchange transformation with rotation number $\rho(f)$ such that $A = (A_0)^N$.

Resuming this, there exists a PL homeomorphism H and an affine 2-intervals exchange transformation A_0 with rotation number $\rho(f)$ such that $f^N = H \circ (A_0)^N \circ H^{-1} = (H \circ A_0 \circ H^{-1})^N$. Finally, lemma 1 concludes the proof.

For the corollary, we have to prove that -under its hypothesis- the number of break points of f^n is bounded by some constant that doesn't depend on n, this fact has been already proved in [11], we recall it.

The set of the break points of f can be written as:

$$0 = c_1,\ c_2 = f^{-N_2}(0),\ \ldots,\ c_p = f^{-N_p}(0), \quad N_p \in \mathbb{N}, 0 < N_2 < \ldots < N_p,$$

with the following properties:

(*) the positive (resp. negative) orbit of 0 (resp. c_p) doesn't contain any break points of f,

(**) the product of the f-jumps in these points is trivial.

These properties will give rise to cancelations when we will compute the jumps of f^{n+1}.

In more detail, the "a priori" break points of the iterate f^{n+1} are the points $f^{-k}(c_i)$ with $0 \leq k \leq n$, that is, the points $f^{-k}(0)$ with $0 \leq k \leq n + N_p$. Computing the jump of f^{n+1} in the point $f^{-k}(0)$, we get:

$$\sigma_{f^{n+1}}(f^{-k}(0)) = \sigma_f(f^{n-k}(0)) \times \ldots \ldots \times \sigma_f(f^{-k}(0)).$$

Now, for n greater than N_p and $N_p \leq k \leq n$, the f-orbit segment

$$\{f^{n-k}(0), f^{n-k-1}(0) \ldots \ldots, f^{-k}(0)\}$$

contains 0 and $f^{-N_p}(0)$, it follows that

$$\sigma_{f^{n+1}}(f^{-k}(0)) = \sigma_f(0) \times \ldots \sigma_f(f^{-N_i}(0)) \ldots \times \ldots \sigma_f(f^{-N_p}(0)) = 1,$$

because of properties (*) and (**).

Conclusion. For n greater than N_p, the iterate f^{n+1} has at most $2N_p$ break points:

$$0,\ f^{-1}(0), \ldots,\ f^{-(N_p-1)}(0)$$
$$f^{-(n+1)}(c_p), \ldots,\ f^{-(n+N_p)}(0).$$

3 Proofs of the lemmas

Proof of Lemma 1. Since $\alpha = \rho(f) = \rho(g)$ is irrational, there exists h_1 and h_2 homeomorphisms such that $R_\alpha = h_1 \circ f \circ h_1^{-1} = h_2 \circ g \circ h_2^{-1}$ and $h_1(0) = h_2(0) = 0$.

By iterating n-times, we have:

$$R_{n\alpha} = h_1 \circ f^n \circ h_1^{-1} = h_2 \circ g^n \circ h_2^{-1} = h_2 \circ f^n \circ h_2^{-1}.$$

Since the normalized conjugacy between $R_{n\alpha}$ and f^n is unique, we have $h_1 = h_2$ and therefore $f = g$. □

Proof of Lemma 2. Let A be an affine 2-intervals exchange transformation. We've mentioned in the introduction that A is conjugate to R_α (with $\alpha = \frac{\log\lambda}{\log\lambda - \log\lambda'}$) through the homeomorphism $h(x) = \dfrac{(\frac{\lambda}{\lambda'})^x - 1}{\frac{\lambda}{\lambda'} - 1}$.

Now, consider $A_0 := h \circ R_{\alpha_0} \circ h^{-1}$. It is an affine 2-intervals exchange transformation and

$$(A_0)^N = (h \circ R_{\alpha_0} \circ h^{-1})^N = h \circ R_{N\alpha_0} \circ h^{-1} = h \circ R_\alpha \circ h^{-1} = A. \qquad \square$$

Proof of Lemma 3. Consider the PL homeomorphism H defined by:

- $d_1, d_2, \ldots, d_p$ are break points of H,
- the jumps of H in these points are $\sigma_H(d_i) := \sigma_F(d_i)$, for all $i \in \{1, \ldots, p\}$.

A necessary and sufficient condition for H to have exactly these break points is that

$$\prod_{i=1}^{p} \sigma_H(d_i) = 1.$$

When this identity holds, the right sloope λ_1 in d_1 and therefore all the sloops λ_i of H are uniquely determined by the identity

$$1 = \sum_{i=1}^{p} \lambda_i (d_{i+1} - d_i) = \lambda_1 \sum_{i=1}^{p} (\sigma_H(d_2) \ldots \sigma_H(d_i))(d_{i+1} - d_i),$$

with convention $d_{p+1} = d_1$.

When it's not the case, we have to add one break point c such that

$$\sigma_H(c) = \left(\prod_{i=1}^{p} \sigma_H(d_i) \right)^{-1},$$

we choose it with the additionnal properties:

- $c \in]d_p, d_1[$,
- $c \notin \{c_1, \ldots, c_p\}$,
- $c \notin F(\{d_1, \ldots, d_p\})$.

As previously, the right sloope λ_1 in d_1 and all the sloops λ_i of H are uniquely determined by the identity

$$\frac{1}{\lambda_1} = \sum_{i=1}^{p-1} \sigma_H(d_2)\ldots\sigma_H(d_i)|d_{i+1} - d_i| + \sigma_H(d_2)\ldots\sigma_H(d_p)|c - d_p| + \sigma_H(d_2)\ldots\sigma_H(d_p)\sigma_H(c)|d_1 - c|.$$

Now, we compute the break points of the conjugate $H \circ F \circ H^{-1}$ and the jumps in these points.

The "a priori" break points of $H \circ F \circ H^{-1}$ are:

- the break points of H^{-1} : $H(d_i)$, $H(c)$,
- the image by H of the break points of F : $H(d_i)$, $H(c_i)$,
- the image by $H \circ F^{-1}$ of the break points of H: $H \circ F^{-1}(d_i)$, $H \circ F^{-1}(c)$.

Since $F(c_i) = d_i$, the "a priori" break points of $H \circ F \circ H^{-1}$ are $H(d_i)$, $H(c_i)$, $H(c)$, $H \circ F^{-1}(c)$. The jumps in these points are:

$$\sigma_{H\circ F\circ H^{-1}}(H(d_i)) = \sigma_H(F(d_i))\sigma_F(d_i)\sigma_{H^{-1}}(H(d_i)) = \frac{\sigma_H(F(d_i))\sigma_F(d_i)}{\sigma_H(d_i)} = 1,$$

because of the choice $\sigma_H(d_i) := \sigma_F(d_i)$ and the fact that $F(d_i)$ is not a break point of H (this results froms the facts that $F(d_i) \neq c$ and $F(d_i)$ is not a break point of F, hence $F(d_i) \neq d_j$).

$$\sigma_{H\circ F\circ H^{-1}}(H(c_i)) = \frac{\sigma_H(F(c_i))\sigma_F(c_i)}{\sigma_H(c_i)} = \sigma_H(d_i)(\sigma_F(d_i))^{-1} = 1,$$

because of $\sigma_H(d_i) := \sigma_F(d_i) = (\sigma_F(c_i))^{-1}$ and the fact that c_i is not a break point of H.

Finally, the PL homeomorphism $H \circ F \circ H^{-1}$ has at most 2 (0 or 2) break points: $H \circ F^{-1}(c)$ and its image $H \circ F \circ H^{-1}(H \circ F^{-1}(c)) = H(c)$, it's a rotation or an affine 2-intervals exchange transformation (the initial point being $0 = H(c)$). □

Proof of Lemma 4. Assume f is a PL homeomorphism with irrational rotation number and suppose that the number of break points of the n-th iterate of f is bounded by some constant that doesn't depend on n. □

Definition. A *maximal f-connection* is a f-orbit segment:

$$a_1,\, f^{-1}(a_1), \dots, a_2 = f^{-N_2}(a_1), \dots, a_s = f^{-N_s}(a_1) \quad \text{where}$$

— a_i are pairwise distincts break points of f,
— N_i are integers $0 < N_2 < \dots < N_s$,
— The positive orbit of a_1 doesn't contain any break point of f,
— The negative orbit of a_s doesn't contain any break point of f.

Claim 1. Under the hypothesis of lemma 4, each break point d of f is contained in a maximal f-connection and the product of the f-jumps along this orbit segment is trivial.

Compute the jump of f^{n+1} in the point $f^{-k}(d)$, $k = 0, \dots, n$, we'll get:

$$\sigma_{f^{n+1}}(f^{-k}(d)) = \sigma_f(f^{n-k}(d)) \ \dots\ \sigma_f(d) \ \dots\ \sigma_f(f^{-k}(d)).$$

Since $\sigma_f(d) \neq 1$, if d is not contained in a f-orbit segment along which the product of f-jumps is trivial, all the points $f^{-k}(d)$, $k = 0, \dots, n$ are break points of f^{n+1} and the number of break points of f^{n+1} is greater than $n + 1$ so it can't be bounded by some constant that doesn't depend on n. Finally, since f doesn't admit any periodic orbit, we can extend the previous orbit segment to a maximal f-connection along which the product of f-jumps is trivial.

As result of claim 1, the set of break points of f can be written as an union of maximal f-connections along which the product of f-jumps is trivial:

$$\begin{array}{l} b_1, \dots,\ f^{-N_1(1)}(b_1), \dots \dots,\ f^{-N_{p_1}(1)}(b_1) = \beta_1 \\ b_2, \dots,\ f^{-N_1(2)}(b_2), \dots \dots,\ f^{-N_{p_2}(2)}(b_2) = \beta_2 \\ \quad . \\ \quad . \\ \quad . \\ b_m, \dots,\ f^{-N_1(m)}(b_m), \dots \dots,\ f^{-N_{pm}(m)}(b_m) = \beta_m. \end{array}$$

The positive (resp. negative) orbit of b_i (resp. β_i) doesn't contain any break point of f.

Consider N an integer greater than all $L_i := N_{p_i}(i)$, and compute the break points of $F = f^{N+1}$. As we've did in the proof of the corollary, we find the $2(L_1 + L_2 + ... + L_m)$ following points:

$$b_1, \dots\dots\dots\dots,\ f^{-L_1+1}(b_1) = f(\beta_1)$$

$$
\begin{aligned}
c_1 &= f^{-(N+1)+L_1}(\beta_1), \ldots\ldots\ldots\ldots\ldots\ldots, f^{-N}(\beta_1) \\
b_2, &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, f^{-L_2+1}(b_2) = f(\beta_2) \\
c_2 &= f^{-(N+1)+L_2}(\beta_2), \ldots\ldots\ldots\ldots\ldots\ldots, f^{-N}(\beta_2) \\
&\vdots \\
b_m, &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, f^{-L_m+1}(b_m) = f(\beta_m) \\
c_m &= f^{-(N+1)+L_m}(\beta_m), \ldots\ldots\ldots\ldots\ldots\ldots, f^{-N}(\beta_m)
\end{aligned}
$$

with the following suitable properties for lemma 3.

The couple (b_1, c_1) satisfies:

$$
\begin{aligned}
&F(c_1) = f^{N+1}(f^{-(N+1)+L_1}(\beta_1)) = f^{L_1}(\beta_1) = b_1 \\
&\sigma_F(b_1) = \sigma_{f^{n+1}}(b_1) = \sigma_f(b_1)\ldots\sigma_f(f^n(b_1) = \sigma_f(b_1) \\
&\sigma_F(c_1) = \sigma_{f^{n+1}}(f^{-(N+1)+L_1}(\beta_1)) = \sigma_f(f^{-(N+1)+L_1}(\beta_1))\ldots\sigma_f(f^{L_1-1}(\beta_1)) \\
&\qquad = \sigma_f(f^{-(N+1)}(b_1))\ldots\sigma_f(f^{-1}(b_1)) = \frac{1}{\sigma_f(b_1)} = \frac{1}{\sigma_F(b_1)}.
\end{aligned}
$$

The same holds for the couples

$$
(f^{-1}(b_1), f^{-1}(c_1)), \ldots, (f^{-L_1+1}(b_1), f^{-L_1+1}(c_1)).
$$

$$
F(b_1) = f^{N+1}(b_1), F(f^{-1}(b_1) = f^N(b_1), \ldots, F(f(\beta_1)) = f^{(N+1)-L_1}(b_1)
$$

are not break points of F: the positve f-orbit of b_1 doesn't contain any break point of f because of the maximality of the connection.

Since the same holds for all others value of $i = 2, \ldots, m$, we can conclude that the hypothesis of lemma 3 are realized.

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