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Markov subshifts and partial representation of $\mathbb{F}_n$

Danilo Royer*

Abstract. In this paper we fix a set Λ^* of positive elements of the free group $\mathbb{F}_n$ (e.g. the set of finite words occurring in a Markov subshift) as well as n partial isometries on a Hilbert space H. Based on these we define a map $S : \mathbb{F}_n \to \mathcal{L}(H)$ which we prove to be a partial representation of $\mathbb{F}_n$ on H under certain conditions studied by Matsumoto.

Keywords: Markov subshift, partial representation.

Mathematical subject classification: 47D99, 37B10.

1 Introduction

Considering a Markov subshift on an alphabet $\{g_1, \dots, g_n\}$, R. Exel proved in [3] that n partial isometries on a Hilbert space H, satisfying the corresponding Cuntz–Krieger relations, give rise to a partial representation of the free group $\mathbb{F}_n$ on H, that is, a map $S : \mathbb{F}_n \longrightarrow \mathcal{L}(H)$, satisfing $S(t^{-1}) = S(t)^*$ and $S(tr)S(r^{-1}) = S(t)S(r)S(r^{-1})$ for all r, t in $\mathbb{F}_n$.

In this work we fix a set Λ^* of positive elements of $\mathbb{F}_n$ which, among other requirements is assumed to be closed under sub-words, and we take a set $\{S_1, \dots, S_n\}$ of partial isometries on H. We define a map $S : \mathbb{F}_n \longrightarrow \mathcal{L}(H)$ by $S(r_1 \dots r_k) = S(r_1) \dots S(r_k)$, where $S(r_i) = S_j$ if $r_i = g_j$, $S(r_i) = S_j^*$ if $r_i = g_j^{-1}$ and $r = r_1 \dots r_k$ is in reduced form.

Under certain conditions studied by Matsumoto in [1], we prove that the map S is a partial representation of $\mathbb{F}_n$ on H. Since Matsumoto's conditions generalize the Cuntz-Krieger relations our result is a generalization of Exel's result mentioned above.

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2 Partial Representations of $\mathbb{F}_n$

Let us consider the Free Group $\mathbb{F}_n$ generated by a set of n elements, $G = \{g_1, \dots, g_n\}$. The elements of $\mathbb{F}_n$ can be written in the form $r = r_1 \dots r_k$ where each $r_i \in G \cup G^{-1}$. We say that r is in reduced form if $r_i \neq r_{i+1}^{-1}$, for each i. Two elements $r = r_1 \dots r_k$ and $s = s_1 \dots s_l$ of $\mathbb{F}_n$, in reduced form, are equal if and only if $l = k$ and $r_i = s_i$, for all i. In this way, each element, in reduced form, have unique representation and we define its lenght by the number of components, that is, if $r = r_1 \dots r_k$ is in reduced form then r have lenght k, wich will be denoted by $|r| = k$. A element $r = r_1 \dots r_k$ of $\mathbb{F}_n$, in reduced form, is called a positive element if $r_i \in G$, for all i, and the set of all positive elements will be called P. We consider e a element of P.

Let us fix a set $\Lambda^* \subseteq P$ with the following properties:

- $e \in \Lambda^*$,
- $G = \{g_1, \dots, g_n\} \subseteq \Lambda^*$,
- Λ^* is closed under sub-words, that is, if $\nu = \nu_1 \dots \nu_k \in \Lambda^*$ then each element of the form $\nu_i \dots \nu_{i+j}$ with $i = 1 \dots k$, $j \in \mathbb{N}$ is a element of Λ^*.

For all $\mu \in \Lambda^*$ we define the following sets:

$$L_\mu^1 = \{g_j \in G | j = 1, \dots, n,\ \ \mu g_j \notin \Lambda^*\},$$

$$L_\mu^k = \{\nu = \nu_1 \dots \nu_k \in \Lambda^* | \mu\nu_1 \dots \nu_{k-1} \in \Lambda^*, \mu\nu \notin \Lambda^*\}, \quad \forall k \in \mathbb{N}.$$

Lemma 1. *Let $\mu \in \Lambda^*$ and $r, s \in P$. If $vr = v's$, where $v \in L_\mu^k$ and $v' \in L_\mu^l$, then $v = v'$.*

Proof. Suppose by contradiction that $v \neq v'$. Then $|v| \neq |v'|$, because otherwise, $v_1 \dots v_k r = v_1' \dots v_k' s$, from where it follows that $v = v'$. Without loss of generality suppose $|v| > l$, write $v = v_1 \dots v_l \dots v_k$ and $v' = v_1' \dots v_l'$. Since $v_1 \dots v_l \dots v_k r = vr = v's = v_1' \dots v_l' s$, then $v_1 \dots v_l = v_1' \dots v_l'$, and therefore $v = v' v_{l+1} \dots v_k$. Since $v' \in L_\mu^l$, by definition of L_μ^l, $\mu v' \notin \Lambda^*$, hence $\mu v_1 \dots v_{k-1} = \mu v' v_{l+1} \dots v_{k-1} \notin \Lambda^*$. That is a contradiction, because $v \in L_\mu^k$ and so $v = v'$. $\square$

Let us consider a Hilbert space H and a set of partial isometries $\{S_1, \dots, S_n\} \subseteq \mathcal{L}(H)$. Recall that S_i is a partial isometry if $S_i S_i^* S_i = S_i$.

Define a map

$$S : \mathbb{F}_n \longrightarrow \mathcal{L}(H)$$
$$r = r_1 \dots r_k \mapsto S(r_1)\dots S(r_k)$$

where r is in reduced form, $S(r_i) = S_j$ if $r_i = g_j$ and $S(r_i) = S_j^*$ if $r_i = g_j^{-1}$. By convention, $S(e) = I$, where I is the identity operator on H. In this way, for all $r \in \mathbb{F}_n$ we have an operator $S(r) \in \mathcal{L}(H)$.This operator will also be called S_r. We will suppose that our set of partial isometries $\{S_1, \dots, S_n\} \subseteq \mathcal{L}(H)$ generated a map S which satisfies:

(M_1) $\sum_{i=1}^{n} S_i S_i^* = I$;

(M_2) For all μ and ν in Λ^* the operators $S_\mu S_\mu^*$ and $S_\nu^* S_\nu$ commute;

(M_3) $I - S_i^* S_i = \sum_{k=1}^{\infty} \sum_{\nu \in L_i^k} S_\nu S_\nu^*, i = 1, \dots, n.$

Note that for all i, $S_i S_i^*$ is idempotent and self-adjoint, and so a projection. By (M_1), $\sum_{i=1}^{n} S_i S_i^*$ is a projection and therefore $S_i S_i^*$ and $S_j S_j^*$ are orthogonal, for all $i \neq j$. So

$$S_i^* S_j = (S_i^* S_i S_i^*)(S_j S_j^* S_j) = S_i^*(S_i S_i^* S_j S_j^*) S_j = 0$$

whenever $i \neq j$.

Lemma 2. *For all $\mu \in \Lambda^*$, $S_\mu = S_\mu S_\mu^* S_\mu$.*

Proof. The proof will be by induction on $|\mu|$. For $|\mu| = 1$, $S_\mu = S_\mu S_\mu^* S_\mu$ by hypothesis. Suppose $S_\mu = S_\mu S_\mu^* S_\mu$ for all $\mu \in \Lambda^*$ with $|\mu| = k$, and consider $\nu \in \Lambda^*$, with $|\nu| = k + 1$. Then $\nu = \alpha g_j$, with $|\alpha| = k$, and

$$\begin{aligned} S_\nu S_\nu^* S_\nu = &\ S_{\alpha g_j} S_{\alpha g_j}^* S_{\alpha g_j} = S_\alpha S_{g_j} S_{g_j}^* S_\alpha^* S_\alpha S_{g_j} = \\ = &\ S_\alpha S_\alpha^* S_\alpha S_{g_j} S_{g_j}^* S_{g_j} = S_\alpha S_{g_j} = S_\nu. \end{aligned}$$ □

Lemma 3. *Let $\alpha \in P$ and $\nu \in \Lambda^*$.*

a) If $|\alpha| \geq |\nu|$ then $S_\nu S_\nu^* S_\alpha = \begin{cases} S_\alpha & \text{if } \alpha = \nu r \text{ for some } r \in P \\ 0 & \text{otherwise} \end{cases}$

b) If $|\alpha| < |\nu|$ then $S_\nu S_\nu^* S_\alpha = \begin{cases} S_\nu S_r^* & \text{if } \nu = \alpha r \text{ for some } r \in P \\ 0 & \text{otherwise} \end{cases}$

Proof.

a) Supposing that there exists r in P such that $\alpha = \nu r$, we have

$$S_\nu S_\nu^* S_\alpha = S_\nu S_\nu^* S_{\nu r} = S_\nu S_\nu^* S_\nu S_r = S_\nu S_r = S_\alpha.$$

On the other hand, if $\alpha \neq \nu r$ for all $r \in P$, write $\alpha = \alpha_1 \dots \alpha_l \dots \alpha_k$, $\nu = \nu_1 \dots \nu_l$ and take the smallest index i such that $\alpha_i \neq \nu_i$. Then we have $\alpha_1 \dots \alpha_{i-1} = \nu_1 \dots \nu_{i-1}$, and so

$$\begin{aligned} S_\nu S_\nu^* S_\alpha = {} & S_{\nu_1 \dots \nu_{i-1} \nu_i \dots \nu_l} S^*_{\nu_1 \dots \nu_{i-1} \nu_i \dots \nu_l} S_{\alpha_1 \dots \alpha_{i-1} \alpha_i \dots \alpha_k} = \\ = {} & S_{\nu_1 \dots \nu_{i-1}} S_{\nu_i \dots \nu_l} S^*_{\nu_i \dots \nu_l} S^*_{\nu_1 \dots \nu_{i-1}} S_{\nu_1 \dots \nu_{i-1}} S_{\alpha_i \dots \alpha_k} = \\ = {} & S_{\nu_1 \dots \nu_{i-1}} S^*_{\nu_1 \dots \nu_{i-1}} S_{\nu_1 \dots \nu_{i-1}} S_{\nu_i \dots \nu_l} S^*_{\nu_i \dots \nu_l} S_{\alpha_i \dots \alpha_k} = 0 \end{aligned}$$

because $S^*_{\nu_i} S_{\alpha_i} = 0$.

b) Suppose $\nu = \alpha r$ for some $r \in P$. Then

$$\begin{aligned} S_\nu S_\nu^* S_\alpha = {} & S_{\alpha r} S^*_{\alpha r} S_\alpha = S_\alpha S_r S_r^* S_\alpha^* S_\alpha = \\ = {} & S_\alpha S_\alpha^* S_\alpha S_r S_r^* = S_\alpha S_r S_r^* = S_{\alpha r} S_r^* = S_\nu S_r^*. \end{aligned}$$

If $\nu \neq \alpha r$, for all $r \in P$ as in (a), take the smallest index i such that $\nu_i \neq \alpha_i$. Then $\nu_1 \dots \nu_{i-1} = \alpha_1 \dots \alpha_{i-1}$ and

$$\begin{aligned} S_\nu S_\nu^* S_\alpha = {} & S_{\nu_1 \dots \nu_{i-1} \nu_i \dots \nu_k} S^*_{\nu_1 \dots \nu_{i-1} \nu_i \dots \nu_k} S_{\alpha_1 \dots \alpha_{i-1} \alpha_i \dots \alpha_l} = \\ = {} & S_{\nu_1 \dots \nu_{i-1}} S_{\nu_i \dots \nu_k} S^*_{\nu_i \dots \nu_k} S^*_{\nu_1 \dots \nu_{i-1}} S_{\nu_1 \dots \nu_{i-1}} S_{\alpha_i \dots \alpha_l} = \\ = {} & S_{\nu_1 \dots \nu_{i-1}} S^*_{\nu_1 \dots \nu_{i-1}} S_{\nu_1 \dots \nu_{i-1}} S_{\nu_i \dots \nu_k} S^*_{\nu_i \dots \nu_k} S_{\alpha_i \dots \alpha_l} = 0 \end{aligned}$$

because $S^*_{\nu_i} S_{\alpha_i} = 0$. □

Theorem 1. *If $\nu \in P \setminus \Lambda^*$ then $S_\nu = 0$.*

Proof. Write $\nu = g_j \alpha$, and in this way,

$$S_\nu^* S_\nu = S_\alpha^* S_{g_j}^* S_{g_j} S_\alpha = S_\alpha^* S_\alpha - \sum_{k=1}^{\infty} \sum_{\mu \in L^k_{g_j}} S_\alpha^* S_\mu S_\mu^* S_\alpha.$$

We will analyse the summands of $\sum\limits_{k=1}^{\infty} \sum\limits_{\mu \in L^k_{g_j}} S_\alpha^* S_\mu S_\mu^* S_\alpha$ in the following way:

Case 1: $|\mu| > |\alpha|$
By Lemma 3, $S_\mu S_\mu^* S_\alpha \neq 0$ only if $\mu = \alpha r$, for some $r \in P$. We will show that there exists no such r. Suppose $\mu \in L_{g_j}^k$ is such that $\mu = \alpha r$, with $|r| = l$. By definition of $L_{g_j}^k$, $g_j \mu_1 \dots \mu_{k-1} \in \Lambda^*$, but $g_j \mu_1 \dots \mu_{k-1} = g_j \alpha r_1 \dots r_{l-1}$, and so $\nu = g_j \alpha \in \Lambda^*$. This is a contradiction, because we are supposing $\nu \notin \Lambda^*$. Therefore $\mu \neq \alpha r$, for all $r \in P$, and so, by Lemma 3, $S_\alpha^* S_\mu S_\mu^* S_\alpha = S_\alpha^*(S_\mu S_\mu^* S_\alpha) = 0$ for all μ with $|\mu| > |\alpha|$.

Case 2: $|\mu| \leq |\alpha|$
By Lemma 3, $S_\mu S_\mu^* S_\alpha \neq 0$, only if $\alpha = \mu r$, for some r em P, and by Lemma 1 if there exists such $\mu \in \cup L_{g_j}^k$, it is unique. In this case we have by Lemma 3 that $S_\alpha^* S_\mu S_\mu^* S_\alpha = S_\alpha^*(S_\mu S_\mu^* S_\alpha) = S_\alpha^* S_\alpha$.

In this way, $S_\nu^* S_\nu = z S_\alpha^* S_\alpha$, where $z = 0$ if there exists $\mu \in \bigcup_{k \in \mathbb{N}} L_{g_j}^k$ such that $\alpha = \mu r$ for some $r \in P$, and $z = 1$ otherwise.
Write $\nu = \nu_1 \dots \nu_k$ and take the smallest index i such that $\nu_{i+1} \dots \nu_k \in \Lambda^*$. So,

$$S_\nu^* S_\nu = z_1 S_{\nu_2 \dots \nu_k}^* S_{\nu_2 \dots \nu_k} = \dots = z_1 \dots z_{i-1} S_{\nu_i \dots \nu_k}^* S_{\nu_i \dots \nu_k},$$

where z_i are 0 or 1. We will show that $S_{\nu_i \dots \nu_k}^* S_{\nu_i \dots \nu_k} = 0$. Since $\nu_i \dots \nu_k \notin \Lambda^*$, by case 1 and case 2 above, we need to show that there exist some $\mu \in \bigcup_{k \in \mathbb{N}} L_{\nu_i}^k$ such that $\nu_{i+1} \dots \nu_k = \mu r$ for some $r \in P$.
Take the index j such that $\nu_i \dots \nu_j \in \Lambda^*$ but $\nu_i \dots \nu_j \nu_{j+1} \notin \Lambda^*$. Such index exists because $\nu_i \in \Lambda^*$ and $\nu_i \dots \nu_k \notin \Lambda^*$. Moreover, $\nu_{i+1} \dots \nu_{j+1} \in \Lambda^*$ because $\nu_{i+1} \dots \nu_k \in \Lambda^*$, and so, $\nu_{i+1} \dots \nu_{j+1} \in L_{\nu_i}^{j+1-i}$. Thereby $S_{\nu_i \dots \nu_k}^* S_{\nu_i \dots \nu_k} = 0$, and so $S_\nu^* S_\nu = 0$, in other words, $S_\nu = 0$. □

Observe that if $r = r_1 \dots r_k$ is in reduced form, with $r_i \in G^{-1}$ and $r_{i+1} \in G$, then $S(r_i r_{i+1}) = S(r_i) S(r_{i+1}) = 0$, from where $S(r) = 0$. Also, if $r = r_1 \dots r_k$ and $s = s_1 \dots s_l$ are elements of $\mathbb{F}_n$ in reduced form and $r_k \neq s_1^{-1}$, then the reduced form of rs is $r_1 \dots r_k s_1 \dots s_l$, and so $S(rs) = S(r)S(s)$ by definition of S.

Definition 1. *Given a group $\mathbb{G}$ and a Hilbert space H, a map $S : \mathbb{G} \to \mathcal{L}(H)$ is a partial representation of the group $\mathbb{G}$ on H if:*

- P_1) *$S(e) = I$, where e is the neutral element of $\mathbb{G}$ and I is the identity operator on H,*
- P_2) *$S(t^{-1}) = S(t)^*$, $\forall t \in \mathbb{G}$,*

P_3) $S(t)S(r)S(r^{-1}) = S(tr)S(r^{-1})$, $\forall t, r \in \mathbb{G}$,

Theorem 2. *If the map $S : \mathbb{F}_n \to \mathcal{L}(H)$ defined before satisfies M_1,M_2 and M_3, then S is a partial representation of the group $\mathbb{F}_n$ on H.*

Proof. Property P_1 is trivial. The proof of P_2 will be by induction on $|t|$. If $|t| = 1$, the equality between $S(t^{-1})$ and $S(t^*)$ is obviously true. Suppose $S(t^{-1}) = S(t^*)$ for all $t \in \mathbb{F}_n$ with $|t| = k$. Take $t \in \mathbb{F}_n$ with $|t| = k + 1$ and write $t = \tilde{t}x$, where $|\tilde{t}| = k$. Using the induction hypothesis and the fact that the equality is true for $|x| = 1$,

$$\begin{aligned} S(t^{-1}) = {} & S((\tilde{t}x)^{-1}) = S(x^{-1}\tilde{t}^{-1}) = S(x^{-1})S(\tilde{t}^{-1}) \\ = {} & S(x)^*S(\tilde{t})^* = (S(\tilde{t})S(x))^* = S(\tilde{t}x)^* = S(t)^*. \end{aligned}$$

To verify property P_3 we will prove the following:

Claim. *For all r in $\mathbb{F}_n$ and t in $G \cup G^{-1}$, $E(r) = S(r)S(r)^*$ and $E(t) = S(t)S(t)^*$ commute.*

If $r = r_1 \dots r_k$ where r is in its reduced form, with $r_i \in G^{-1}$ and $r_{i+1} \in G$ for some i, then $S(r) = 0$ and so the claim is trivial. Therefore let $r = \alpha\beta^{-1}$, where r is in reduced form and $\alpha, \beta \in P$. If $\beta \notin \Lambda^*$, by Theorem 1, $S_\beta = 0$ from where we again see that the claim follows. Thus let us consider $\beta \in \Lambda^*$.

Case 1: *If $t \in G$, that is, $t = g_j$, for some j.*

a) $|\alpha| \neq 0$.
Write $\alpha = \alpha_1 \dots \alpha_l$. If $\alpha_1 \neq g_j$, then $S(g_j)^*S(\alpha) = 0$ and so $E(t)E(r) = 0 = E(r)E(t)$. If $\alpha_1 = g_j$ we have

$$\begin{aligned} S(\alpha)^*S(g_j)S(g_j)^* = {} & S(\alpha_2 \dots \alpha_l)^*S(\alpha_1)^*S(g_j)S(g_j)^* \\ = {} & S(\alpha_2 \dots \alpha_l)^*S(\alpha_1)^*S(\alpha_1)S(\alpha_1)^* = S(\alpha_2 \dots \alpha_l)^*S(\alpha_1)^* \\ = {} & (S(\alpha_1)S(\alpha_2 \dots \alpha_k))^* = S(\alpha)^* \end{aligned}$$

and similarly $S(g_j)S(g_j)^*S(\alpha) = S(\alpha)$. It follows that $E(t)$ and $E(r)$ commute.

b) $|\alpha| = 0$.
We have $r = \beta^{-1}$. Since $\beta \in \Lambda^*$, using M_2,

$$\begin{aligned} E(r)E(t) = {} & S(r)S(r)^*S(t)S(t)^* = S(\beta)^*S(\beta)S(g_j)S(g_j)^* \\ = {} & S(g_j)S(g_j)^*S(\beta)^*S(\beta) = S(t)S(t)^*S(r)S(r)^* = E(t)E(r). \end{aligned}$$

Case 2: *If $t \in G^{-1}$, namely, $t = g_j^{-1}$, with $g_j \in G$.*

Note that

$$E(r)E(t) = E(r)S_t S_t^* = E(r)S_{g_j}^* S_{g_j} = E(r)\left(I - \sum_{k=1}^{\infty} \sum_{\mu \in L_{g_j}^k} S_\mu S_\mu^*\right) =$$

$$= E(r) - E(r)\left(\sum_{k=1}^{\infty} \sum_{\mu \in L_{g_j}^k} S_\mu S_\mu^*\right)$$

and similarly,

$$E(t)E(r) = S_{g_j}^* S_{g_j} E(r) = E(r) - \left(\sum_{k=1}^{\infty} \sum_{\mu \in L_{g_j}^k} S_\mu S_\mu^*\right) E(r).$$

To prove that $E(t)$ and $E(r)$ commute, it is enough to show that

$$E(r)S_\mu S_\mu^* = S_\mu S_\mu^* E(r) \quad \forall \mu \in L_{g_j}^k, \ \forall k \in \mathbb{N}.$$

a) $|\alpha| \neq 0$.

i) $|\alpha| \geq |\mu|$.
By Lemma 3, if $\alpha = \mu s$ for some s in P then $S_\alpha^* S_\mu S_\mu^* = S_\alpha^*$. Therefore,

$$E(r)S_\mu S_\mu^* = S_\alpha S_\beta^* S_\beta S_\alpha^* S_\mu S_\mu^* = S_\alpha S_\beta^* S_\beta S_\alpha^* = E(r),$$

and similarly $S_\mu S_\mu^* E(r) = E(r)$, and this proves that $E(r)S_\mu S_\mu^* = S_\mu S_\mu^* E(r)$. Also by Lemma 3, if $\alpha \neq \mu s$ for all $s \in P$, then $S_\alpha^* S_\mu S_\mu^* = 0 = S_\mu S_\mu^* S_\alpha$ and also in this case $E(r)$ and $S_\mu S_\mu^*$ commute.

ii) $|\alpha| < |\mu|$.
By Lemma 3, if $\mu \neq \alpha s$ $\forall s \in P$, then $S_\alpha^* S_\mu S_\mu^* = 0 = S_\mu S_\mu^* S_\alpha$, from where the equallity follows. If $\mu = \alpha s$ for some $s \in P$, also by Lemma 3, $S_\alpha^* S_\mu S_\mu^* = S_s S_\mu^*$ and $S_\mu S_\mu^* S_\alpha = S_\mu S_s^*$, from where

$$E(r)S_\mu S_\mu^* = S_\alpha S_\beta^* S_\beta S_\alpha^* S_\mu S_\mu^* = S_\alpha S_\beta^* S_\beta S_s S_\mu^* = S_\alpha S_\beta^* S_\beta S_s S_s^* S_\alpha^*,$$

and

$$S_\mu S_\mu^* E(r) = S_\mu S_\mu^* S_\alpha S_\beta^* S_\beta S_\alpha^* = S_\mu S_s^* S_\beta^* S_\beta S_\alpha^* = S_\alpha S_s S_s^* S_\beta^* S_\beta S_\alpha^*.$$

Since $\beta \in \Lambda^*$, by M_2,

$$S_s S_s^* S_\beta^* S_\beta = S_\beta^* S_\beta S_s S_s^*,$$

and this shows that $E(r) S_\mu S_\mu^* = S_\mu S_\mu^* E(r)$.

b) $|\alpha| = 0$
Since $\beta \in \Lambda^*$, the equality between $E(r) S_\mu S_\mu^*$ and $S_\mu S_\mu^* E(r)$ follows from M_2.

This proves our claim. Let us now return to the proof of P_3, that is,

$$S(t)S(r)S(r^{-1}) = S(tr)S(r^{-1}), \forall t, r \in \mathbb{F}_n.$$

To do this we use induction on $|t| + |r|$. The equality is obvious if $|t| + |r| = 1$. Suppose the equality true for all $t, r \in \mathbb{F}_n$ such that $|t| + |r| < k$. Take $t, r \in \mathbb{F}_n$, with $|t| + |r| = k$, write $t = \tilde{t}x$, $r = y\tilde{r}$, with $x, y \in G \cup G^{-1}$. If $y \neq x^{-1}$, we have $S(tr) = S(t)S(r)$, from where $S(tr)S(r^{-1}) = S(t)S(r)S(r^{-1})$. Let us consider the case $x = y^{-1}$.

$$\begin{aligned} S(t)S(r)S(r^{-1}) &= S(\tilde{t}x)S(y\tilde{r})S((y\tilde{r})^{-1}) = \\ &= S(\tilde{t})S(x)S(y)S(\tilde{r})S(\tilde{r}^{-1})S(y^{-1}) = \\ &= S(\tilde{t})S(x)S(x^{-1})S(\tilde{r})S(\tilde{r}^{-1})S(x). \end{aligned}$$

Using the claim and the fact that $S(x)$ is a partial isometry,

$$\begin{aligned} S(\tilde{t})S(x)S(x^{-1})S(\tilde{r})S(\tilde{r}^{-1})S(x) &= S(\tilde{t})S(\tilde{r})S(\tilde{r}^{-1})S(x)S(x^{-1})S(x) = \\ &= S(\tilde{t})S(\tilde{r})S(\tilde{r}^{-1})S(x) \end{aligned}$$

and by the induction hypothesis,

$$S(\tilde{t})S(\tilde{r})S(\tilde{r}^{-1})S(x) = S(\tilde{t}\tilde{r})S(\tilde{r}^{-1})S(x).$$

On the other hand,

$$\begin{aligned} S(tr)S(r^{-1}) &= S(\tilde{t}xy\tilde{r})S((y\tilde{r})^{-1}) = \\ &= S(\tilde{t}\tilde{r})S(\tilde{r}^{-1}y^{-1}) = S(\tilde{t}\tilde{r})S(\tilde{r}^{-1})S(x). \end{aligned}$$

This concludes the proof of P_3, and also of the theorem. □

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Danilo Royer
Departamento de Matemática
Universidade Federal de Santa Catarina
88040-900 Florianópolis SC
BRASIL
E-mail: royer@mtm.ufsc.br