BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 35(3), 377-385

The set of smooth metrics in the torus without continuous invariant graphs is open and dense in the C^1 topology

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Abstract. We show that the set of C^∞ metrics in the two dimensional torus with no continuous invariant graphs of the geodesic flow is open and dense in the C^1 topology. The generic nonexistence of invariant graphs with rational rotation numbers was known in the C^∞ topology for metrics, and in general the generic nonexistence in the C^∞ topology of invariant graphs with Liouville rotation numbers is known for twist maps and Hamiltonian flows in the torus. The main idea of the proof is that small C^1 bumps are enough to prevent the existence of invariant graphs.

Keywords: Invariant graphs of the geodesic flow, C^1 bump, converse KAM theory.

Mathematical subject classification: 37J50, 37J30, 70H07.

Introduction

Let g be a C^∞ Riemannian metric in the torus T^2, let (T^2, g) be the torus endowed with the Riemannian metric g, and let T_1T^2 be the unit tangent bundle of the torus. We shall denote by (T_1T^2, g) the unit tangent bundle endowed by the Sasaki metric induced by g. A subset $S \subset T_1T^2$ is called an invariant graph if the set S is invariant by the action of the geodesic flow of g and the canonical projection $\pi : T_1T^2 \longrightarrow T^2$ restricted to S is a homeomorphism. The graph S is of class C^k, $k \geq 0$, if S is a submanifold of T_1T^2 of class C^k. Invariant graphs are examples of invariant tori of Euler-Lagrange flows defined in the tangent space of the torus, whose study is one of the central subjects of classical mechanics and mathematical physics. One of the most appealing aspects of the theory of invariant graphs is the interplay between dynamics and calculus of variations,

Received 1 December 2003.
Partially supported by CNPq, FAPERJ, TWAS

which turns invariant graphs into natural counterparts of homotopically nontrivial, closed, invariant curves of measure preserving twist maps of the annulus. In the present note we are interested in the generic nonexistence of invariant graphs. Some of the main results about the nonexistence of invariant graphs are due to Mather for twist maps and billiards [15], [16], MacKay [10], MacKay-Percival [14] for Hamiltonians and twist maps, and Bangert [2] who gave examples of metrics in the torus with the so-called big bumps whose geodesic flows cannot have any invariant graph. There are also many results concerning the destruction of especific families of invariant graphs by perturbations of the system. Here we should mention the work of Mather [17] proving the C^∞-genericity of the nonexistence of invariant graphs of measure preserving twist maps of the annulus with Liouville rotation numbers, the works of MacKay and many authors about the destruction of certain invariant graphs using renormalization ideas (for instance see [11], [12] also with many references on the subject, [13]). A very interesting example due to Bangert [2] of a flat metric in T^2 that is approached in the C^1 topology by a sequence of metrics with no invariant graphs at all is perhaps the best known answer to the following question: given a metric in the torus, what is the highest $k \in \mathbb{N}$ such that one can prevent the existence of invariant graphs in the geodesic flow by perturbing the metric in the C^k topology? The work of Kolmogorov, Arnold and Moser implies that $k \leq 4$, and suggests that the destruction of all invariant graphs by perturbations might be very difficult. None of the results existing in the literature implies the genericity in some topology of the nonexistence of invariant graphs. Notice that the deformation of a metric in T^2 by a big bump cannot be attained by C^0 perturbations of the given metric. Our main result is the following.

Theorem 1. *The set of C^∞ metrics in T^2 with no continuous invariant graphs of the geodesic flow is open and dense in the C^1 topology.*

In fact, what we show is that the set of C^∞ metrics in T^2 for which there exists a point in the torus that is not contained in any globally minimizing geodesic is open and dense in the C^1 topology. Recall that a geodesic $\gamma \in T^2$ of the metric g is called globally minimizing for g if any lift $\bar{\gamma}$ of γ by the covering map is a global minimizer of the pullback g^* of the metric g in the universal covering: the g^*-length of $\bar{\gamma}[t, s]$ equals the distance (in the metric g^*) $d_{g^*}(\bar{\gamma}(t), \bar{\gamma}(s))$ for every $s, t \in \mathbb{R}$. The proof of Theorem 1 is based in two main ideas. The first one comes from calculus of variations and the work of Weierstrass about fields of minimizers. The canonical projection of a continuous invariant graph of the geodesic flow of (T^2, g) is a continuous flow in T^2 whose orbits are globally g-minimizing geodesics (see for instance [23]), a well known fact if the invariant graph is a C^1 submanifold (see [4], [19] for instance), or if the invariant graph is continuous and contains no periodic orbits [3]. The second idea is to show that we can make small C^1 bumps in the metric g supported in arbitrarily small neighborhoods of T^2 which create conjugate points in these neighborhoods. The idea of creating conjugate points in small neighborhoods by C^1 perturbations of the metric does not work with C^2 perturbations. In fact, if new conjugate points appear after C^2 perturbations of the metric they are typically far from each other (see for instance [10], [21] or [7]). We would like to thank the IMCA in Lima, Perú, the Catholic University of Lima, and Professors C. Camacho and A. Poirier for their kind hospitality while part of this work was in progress.

1 C^1 perturbations of the Euclidean metric in a disk which create conjugate points

Let us first introduce some notations. An open disk in $\mathbb{R}^2$ with radius $r > 0$ centered at $(0, 0)$ will be denoted by D_r, its closure will be $\bar{D}_r$, the circle of radius r centered at $(0, 0)$ will be S_r. A conic sector $C_{\alpha,r}$ of the disk D_r is the set of points $(x, y) \in D_r$ such that the angle between (x, y) and $(1, 0)$ is less than α. The Euclidean metric in $\mathbb{R}^2$ will be called g_0. The main result of the section is the following:

Lemma 1.1. *Given $r > 0$, $0 < s < r$, there exists a C^∞ metric $g_{s,r}$ in the disk D_r such that*

1. *There exists an open neighborhood $D_{\epsilon(s,r)}$ of $(0, 0)$ such that every $g_{s,r}$-geodesic $\gamma : [a, b] \longrightarrow D_r$ satisfying*
   - *(a) The endpoints $\gamma(a) \neq \gamma(b)$ belong to $D_{\frac{r}{2}}$,*
   - *(b) $\gamma[a, b] \cap D_{\epsilon(s,r)} \neq \emptyset$,*

   is not $g_{s,r}$-minimizing.
2. *The metric $g_{s,r}$ coincides with the Euclidean metric g_0 outside the disk $D_{s'}$, where $s' = s + \frac{r-s}{2} < r$,*
3. $\lim_{s\to r} \| g_{s,r} - g_0 \|_{C^1} = 0.$

Proof. The idea is to endow D_r with a metric induced by a small cone where a small neighborhood of its vertex has been removed and replaced by a smooth cap. To construct the cone, let $r > 0$, $0 < s < r$, and consider the function $f_{s,r} : \bar{D}_s \longrightarrow \mathbb{R}$ given by

$$f_{s,r}(x, y) = \sqrt{r^2 - s^2}\left(1 - \frac{1}{s}\sqrt{x^2 + y^2}\right).$$

The graph of $f_{s,r}$ is the cone $A_{s,r}$ generated by the rotation around the z-axis of the segment $L_{s,r}$ defined by

$$L_{s,r} = \{(x, 0, z), x \in [0, s], z = \sqrt{r^2 - s^2}\big(1 - \frac{x}{s}\big)\}.$$

The segment $L_{s,r}$ has lenght r, its slope is

$$-\frac{\sqrt{r^2 - s^2}}{s} = -\sqrt{\frac{r^2}{s^2} - 1},$$

and it is clear that as $s \to r$ the segment $L_{s,r}$ tends to the horizontal segment $\{(t, 0, 0), t \in [0, r]\}$. The function $f_{s,r}$ is continuous, and differentiable at every point of D_s but $(0, 0)$. The variation of the first derivatives of $f_{s,r}$ is bounded above by $2\sqrt{\frac{r^2}{s^2} - 1}$. Let us endow the cone $A_{s,r}$ with the restriction of the Euclidean metric of $\mathbb{R}^3$. The cone $A_{s,r}$ is of course a singular surface, but since $A_{s,r}$ is compact it is a complete metric, geodesic space. The curvature of $A_{s,r}$ can be calculated at every point but the vertex, and it is equal to zero; the geodesics in $A_{s,r}$ are the straight lines through the vertex and the curves satisfying the corresponding Clairaut equation of surfaces of revolution. The following claim is inspired in [2]:

Claim 1: The union of two straight lines in $A_{s,r}$ containing the vertex is not a minimizing geodesic.

A short proof of this fact is the following. Observe first of all that by removing the straight line $L_{s,r}$ from the cone $A_{s,r}$ we obtain a subset $B_{s,r} = A_{s,r} - L_{s,r}$ that is isometric to an open conic sector $C_{\alpha(s,r),r}$ in $\mathbb{R}^2$, where $\alpha(s, r) < 2\pi$, endowed with the Euclidean metric. Let $\Phi : B_{s,r} \longrightarrow C_{\alpha(s,r),r}$ be an isometry between these metric spaces. Consider two straight segments R_1, R_2 in $A_{s,r}$ joining the vertex of $A_{s,r}$ with its boundary, which make an angle $\theta(R_1, R_2)$ in $A_{s,r}$ that has to be strictly less than $\alpha(s, r) < \pi$. The union of the segments R_1, R_2 separates $A_{s,r}$ in two subcones, let us call by $A(R_1, R_2)$ the smallest one (if the union of R_1 and R_2 divides $A_{s,r}$ in two subcones of the same size, we choose any of them as $A(R_1, R_2)$). By the rotational symmetry of $A_{s,r}$ we can assume without loss of generality that the segment $L_{s,r}$ is not contained in $A(R_1, R_2)$. Under this assumption, the isometry Φ sends the lines $R_1 \cap B_{s,r}$, $R_2 \cap B_{s,r}$ to two lines L_1, L_2 in $C_{\alpha(s,r),r}$ whose closures contain $(0, 0)$ and make an angle that is strictly less than π. Moreover, the set $\Phi(A(R_1, R_2))$ is a cone bounded by L_1 and L_2 that is isometric to $A(R_1, R_2)$. Clearly, the union of the closures of L_1 and L_2 is not a minimizing geodesic in the closure of $C_{\alpha(s,r),r}$. Because if $p \in L_1$, $q \in L_2$, the segment $[p, q]$ contained in $\Phi(A(R_1, R_2)) \subset C_{\alpha(s,r),r}$ minimizes the distance between p and q. And since Φ is an isometry, the curve $\Phi^{-1}([p, q])$ minimizes the distance between $\Phi^{-1}(p) \in R_1$ and $\Phi^{-1}(q) \in R_2$, thus proving that the union of R_1 and R_2 cannot be minimizing in $A_{s,r}$.

By the Claim we have that there exists an open ball $U_{s,r}$ with center at the vertex $(0, 0, f_{s,r}(0, 0))$ in the cone $A_{s,r}$ such that no minimizing geodesic segment in $A_{s,r}$ with endpoints in $D_{\frac{r}{2}}$ meets $U_{s,r}$. Indeed, since the set of minimizing geodesics in $A_{s,r}$ is closed in the C^0 topology, a convergent sequence of such minimizing geodesics approaching the vertex would converge to a minimizing curve formed by the union of a pair of lines in $A_{s,r}$ containing the vertex. Let $D_{\epsilon(s,r)}$ be the disk around $(0, 0)$ such that the graph of $f_{s,r}$ restricted to $D_{\epsilon(s,r)}$ is $U_{s,r}$. Next, let us extend the function $f_{s,r}$ to a continuous function $\bar{f}_{s,r} : f_{s,r} : D_r \longrightarrow \mathbb{R}$ which assumes the value 0 at the points of $D_r - D_s$, and coincides with $f_{s,r}$ in D_s.

Claim 2: We can approach the function $\bar{f}_{s,r}$ by a C^∞ function $F_{s,r} : D_r \longrightarrow \mathbb{R}$ with the following properties:

(1) If $\delta = \frac{r-s}{2}$, the function $F_{s,r}$ coincides with $\bar{f}_{s,r}$ outside the union of $D_{\epsilon(s,r)}$ and a δ-tubular neighborhood of $\{x^2 + y^2 = s^2\}$,

(2) There exists an open neighborhood $\bar{U}_{s,r}$ of $(0, 0, F_{s,r}(0, 0))$ in the graph of $F_{s,r}$ that is avoided by minimizing geodesics in the graph of $F_{s,r}$ with endpoints outside the set $D(s, \frac{r}{2}) = \{(x, y, F_{s,r}(x, y)), (x, y) \in D_{\frac{r}{2}}\}$.

(3) $\| \bar{f}_{s,r} - F_{s,r} \| \leq 2\sqrt{\frac{r^2}{s^2} - 1}$.

The crucial points regarding the proof of the main theorem are assertions (2) and (3) in Claim 2, which say essentially that $F_{s,r}$ is C^1-close to the zero function in D_r and that there exists a small neighborhood $\bar{U}_{s,r}$ of $(0, 0, F_{s,r}(0, 0))$ in the graph of $F_{s,r}$ such that every minimizing geodesic segment whose endpoints are suitably far away from $(0, 0, F_{s,r}(0, 0))$ does not meet $\bar{U}_{s,r}$. The proof of Claim 2 is straightforward from the above construction and elementary analysis. Item

(3) is due to the fact that the variation of the first derivatives of $\bar{f}_{s,r}$ is bounded above by $2\sqrt{\frac{r^2}{s^2} - 1}$.

Now, let $\Psi : D_r \longrightarrow graph(F_{s,r})$ be the map $\Psi(x, y) = (x, y, F_{s,r}(x, y))$. It is clear that Ψ is a diffeomorphism that is C^1-close to the identity, and let $g_{s,r}$ be the metric defined in D_r by the pullback by Ψ of the restriction of the Euclidean metric to the graph of $F_{s,r}$. The metric $g_{s,r}$, $s \in (0, r)$ is the metric in the statement of Lemma 1.1. □

2 The nonexistence of invariant graphs is open and dense in the C^1 topology

Lemma 2.1. *The set of C^∞ Riemannian metrics in T^2 without continuous invariant graphs is dense in the C^1 topology.*

Proof. We shall show that given a C^∞ Riemannian metric g in T^2, and $\epsilon > 0$, there exists a C^∞ Riemannian metric g_ϵ in T^2 that is ϵ-close to g in the C^1 topology whose geodesic flow has no invariant graphs. Let $p \in (T^2, g)$ be a point where the Gaussian g-curvature is zero. The point p always exists due to the Gauss-Bonet Theorem. Given $\epsilon > 0$ small, there exists $\delta > 0$ and a Riemannian structure $(T^2, \bar{g})$ that is $\frac{1}{2}\epsilon$-close to (T^2, g) in the C^2 topology, with the property that the Gaussian curvature in the ball $B_\delta(p)$ of g-radius δ centered at p is zero. By Cartan's Theorem [6], there exists an isometry $T : B_\delta(p) \longrightarrow D_\delta$, where D_δ is the disk in $\mathbb{R}^2$ of radius δ centered at $(0, 0)$. Choose $0 < s < \delta$, and consider the metric $g_{s,\delta}$ constructed in Lemma 1.1 in the disk D_δ. Define a new metric g^s in T^2 by

$$g^s_q = \bar{g}_q$$

if $q \notin B_\delta(p)$, and

$$g^s_q = T^* g_{s,\delta}|_{T(q)}$$

if $q \in B_\delta(p)$, where $T^* g_{s,\delta}$ is the pullback of the metric $g_{s,\delta}$ by the map T. The metric g_s is clearly C^∞ in $B_\delta(p)$ and in the interior of the complement of $B_\delta(p)$. Item (2) in Lemma 1.1 implies that g^s is C^∞ in T^2: in fact, the metric $g_{s,\delta}$ coincides with the Euclidean metric in D_δ when restricted to the annulus $\{s' <\| (x, y) \|< \delta\}$, where $s' = s + \frac{\delta - s}{2}$, and hence $T^* g_{s,\delta}$ is just the metric $\bar{g}$ outside a small ball $B_{r(s)}(p) \subset B_\delta(p)$. By Lemma 1.1, we can choose $s < \delta$ such that $\| \bar{g} - g^s \|_{C^1} < \frac{1}{2}\epsilon$, and therefore $\| g - g^s \|_{C^1} < \epsilon$.

Claim: The geodesic flow of (T^2, g^s) has no continuous invariant graphs.

Indeed, a continuous invariant graph $S \subset (T_1T^2, g^s)$ would define a continuous flow $\psi_t : T^2 \longrightarrow T^2$ without singularities by globally minimizing g^s-geodesics. In particular, there would exist a globally minimizing geodesic γ of (T^2, g_s) such that $\gamma(0) = p$ that is an orbit of the flow ψ_t. We can assume without loss of generality that the parameter t of the flow ψ_t is the g^s arc length. It is clear that the connected component of the intersection of the orbit $O(p) = \{\psi_t(p), t \in \mathbb{R}\}$ through p with the closure of $B_\delta(p)$ is of the form $\psi_{[a,b]}(p)$, with $a < 0 < b$, and $\psi_a(p)$, $\psi_b(p)$ in the boundary of $B_\delta(p)$. To show this assertion, observe first that this connected component is diffeomorphic to an open segment of the real line. And this segment has to be bounded; otherwise we would get that either $\psi_{[0,\infty)}(p)$ or $\psi_{(-\infty,0]}(p)$ is contained in $B_\delta(p)$ which would imply, by Poincaré-Bendixson theorem, that ψ_t has singularities. But this impossible by the assumptions on the flow ψ_t.

Thus, we can apply Lemma 1.1 to the geodesic $\psi_{[a,b]}(p)$ to get a contradiction: by Lemma 1.1 there exists an open small neighborhood of p that is avoided by every minimizing geodesic segment with endpoints in the boundary of $B_\delta(p)$. This finishes the proof of the Claim.

The Claim and the estimate $\| g - g^s \|_{C^1} < \epsilon$ finish the proof of the lemma. □

Lemma 2.2. *The set of C^∞ metrics in T^2 without continuous invariant graphs in open in the C^1 topology.*

Proof. The proof of this lemma follows from standard arguments of the theory of globally minimizing objects which are invariant by Lagrangian flows (see for instance [2]). However, we give a sketch of proof for the sake of completeness. The point is that the set of metrics with continuous invariant graphs is closed in the C^1 topology. In fact, let (T^2, g_n) be a sequence of C^∞ metrics having continuous invariant graphs $S_n \subset (T_1T^2, g_n)$ such that the metrics g_n converge to a C^∞ metric g_∞ in the C^1 topology. The geodesic flows of the metrics g_n converge uniformly on compact subsets of the arc length parameter to the geodesic flow of g_∞. Each graph S_n defines a g_n-unit, continuous vector field $X_n : T^2 \longrightarrow TT^2$ whose integral orbits are globally g_n-minimizing geodesics. Moreover, each vector field X_n has an associated homological direction $h_n \in \mathbb{R}^2$, according to the work of Hedlund [8]. Then, if $h_{n_k} \to h_\infty$ is a convergent subsequence of homological directions, it is not difficult to show that there exists a limit vector field X_∞ in T^2 by globally g_∞-minimizing geodesics approached by a subsequence of the vector fields X_n whose homological direction is h_∞. The torus $\{(p, X_\infty(p)), p \in T^2\}$ is a continuous invariant graph of the geodesic flow of (T^2, g_∞). $\square$

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