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Dense solutions to the Cauchy problem for minimal surfaces

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Abstract. We show a general way to produce in explicit coordinates complete minimal surfaces in $\mathbb{R}^3$ that lie densely in the whole space. This construction relies on solving the Björling problem for adequate initial data.

Keywords: minimal surfaces, Björling problem, dense surfaces.

Mathematical subject classification: 53C42.

1 Introduction

The existence of complete minimal surfaces in $\mathbb{R}^3$ that are dense in the whole space has motivated in the last few years some work, and opened new problems in the theory [BJO, BeJo, And2]. First, Rosenberg provided an example of a complete minimal surface with bounded curvature lying densely in $\mathbb{R}^3$, constructed by Schwarzian reflection on a fundamental domain.

Inspired by this example and a question by L.P. Jorge, P. Andrade described in [And2] a complete minimal surface that is dense in a large open subset of $\mathbb{R}^3$, but not in the whole space. To do so, he used a parametrization of the Weierstrass formulae derived in [And1]. The main features of Andrade's example are, first, that it has bounded curvature and, second, that it is given in explicit coordinates. Unfortunately, this is just an isolated example.

Finally, in [BeJo] it was proposed a line of research that can be summarized as follows: to what extent does a complete non-proper minimal surface with bounded curvature need to be dense? Of course, in this question one has to

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leave apart some trivial cases, like the universal coverings of complete minimal surfaces with finite total curvature.

Motivated by these facts, the aim of the present work is to show a general procedure for constructing complete minimal surfaces in $\mathbb{R}^3$ that lie densely in the whole space. All these minimal surfaces will be given in explicit coordinates, in terms of suitable elliptic functions.

The construction that we develop here is completely different from Rosenberg's approach, and relies on solving the Cauchy problem for minimal surfaces with certain adequate initial data.

Just as the Dirichlet problem for minimal surfaces is usually called the *Plateau problem*, this Cauchy problem is classically known as *Björling problem* [DHKW]. It asks for the construction of a minimal surface passing through a given curve, and with prescribed tangent plane at each point of the curve, and was solved by H.A. Schwarz in the 19th century (see also [ACM] for the situation in the Minkowski 3-space setting). The present work seems to be the first time that Björling problem is applied to study the global behaviour of complete minimal surfaces in $\mathbb{R}^3$.

We have organized this paper as follows. In Section 2 we will construct a general family of connected regular curves in the x_1, x_2-plane, with the property that the only minimal surface that has any of these curves as a planar geodesic is complete, and its projection over the x_1, x_2-plane is dense in it. Moreover, the general solution to Björling problem will provide explicit coordinates for these minimal surfaces.

In Section 3 we shall prove that among these examples there exist some of them which are dense in $\mathbb{R}^3$.

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2 Complete solutions to Björling problem

Let Λ be the rectangular lattice $\Lambda = \{m + in : m, n \in \mathbb{Z}\}$, denote by $\mathbb{C}_\infty$ the Riemann sphere, and consider an elliptic function $h : \mathbb{C}/\Lambda \to \mathbb{C}_\infty$ on the torus $\mathbb{C}/\Lambda$ satisfying

C.1 $h(z) \in \mathbb{R} \cup \{\infty\}$ for all $z \in \mathbb{R}$,

C.2 there is some $b \in \mathbb{R}$ such that $f = \sqrt{b + h'}$ is a well defined elliptic function on $\mathbb{C}/\Lambda$,

C.3 all zeroes and poles of f lie in $\mathbb{Q} + i\mathbb{Q}$.

Later on we will produce elliptic functions satisfying these conditions.

Choose $q \in \mathbb{Q}$ such that $f(q) \neq 0, \infty$, and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. If we let $g(z) = bz + h(z)$, the curve $\beta : \mathbb{R} \to \mathbb{R}^2 \equiv \mathbb{C}$ defined as $\beta(s) = g(q + (1 + i\alpha)s)$ is regular by **C.3**, and real analytic. By identifying $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$, we shall regard β as a plane curve lying in the x_1, x_2-plane of $\mathbb{R}^3$. Let us consider the meromorphic functions g_1, g_2 given by

$$g_1(z) = g(q + (1 + i\alpha)z), \qquad g_2(z) = g(q + (1 - i\alpha)z).$$

In the same way we can define $h_1(z), h_2(z)$ and $f_1(z), f_2(z)$ in terms of h and f, respectively. With this, one can easily check that f_1, h_1 are actually elliptic functions on the torus $\mathbb{C}/\Gamma$, where Γ is the lattice $\Gamma = \Lambda/(1 + i\alpha)$, and that f_2, h_2 are elliptic functions on $\mathbb{C}/\Upsilon$, where $\Upsilon = \Lambda/(1 - i\alpha)$.

Then, we have

Theorem 1. *The only minimal surface that contains $\beta(s)$ as a planar geodesic is complete, and can be explicitly parametrized as $\psi : \widetilde{\mathbb{C} \setminus S} \to \mathbb{R}^3$,*

$$\begin{aligned}\psi(z) \;=\; \frac{1}{2}\Bigg(&\operatorname{Re}(g_1(z) + g_2(z)),\ \operatorname{Im}(g_1(z) - g_2(z)), \\ &2\sqrt{1+\alpha^2}\,\operatorname{Im}\int^z f_1(w) f_2(w)\,dw\Bigg),\end{aligned} \tag{1}$$

where S denotes the set of poles of $h_1 h_2$ in $\mathbb{C}$ and $\widetilde{\mathbb{C} \setminus S}$ is the universal covering of $\mathbb{C} \setminus S$.

This minimal surface is symmetric with respect to the x_1, x_2-plane, and its projection over that plane is dense.

Proof. For any regular, real analytic curve $\beta(s)$ in the x_1, x_2-plane, the classical solution to Björling problem shows that the only minimal surface in $\mathbb{R}^3$ containing $\beta(s)$ as a planar geodesic is given near $\beta(s)$ by (see [DHKW])

$$\psi(z) = \left(\operatorname{Re}\beta_1(z), \operatorname{Re}\beta_2(z), \operatorname{Im}\int^z \sqrt{\beta_1'(w)^2 + \beta_2'(w)^2}\,dw\right). \tag{2}$$

Here $\beta_i(z)$ is a holomorphic extension of $\beta_i(s)$ to a simply connected open subset of $\mathbb{C}$, and the integral is taken along an arbitrary path joining z and a fixed base point $s_0 \in \mathbb{R}$.

In our case, it follows from **C.1** and the identification $\mathbb{C} \equiv \mathbb{R}^2$ that

$$2\,(\beta_1(z), \beta_2(z)) = \big(g_1(z) + g_2(z), -i(g_1(z) - g_2(z))\big).$$

From this expression we obtain that (2) turns into (1). In addition it is clear that, if $\Sigma = \widetilde{\mathbb{C} \setminus S}$, then $\psi : \Sigma \to \mathbb{R}^3$ is well defined. Moreover, $\beta(s)$ is a planar geodesic of this minimal surface, and so $\psi(\Sigma)$ is symmetric with respect to the x_1, x_2-plane.

Let ds^2 denote the metric of the minimal surface, i.e. $ds^2 = \langle d\psi, d\psi \rangle$. Since

$$4\frac{\partial \psi}{\partial z} = \Bigg((1+i\alpha) f_1^2 + (1-i\alpha) f_2^2, -i\Big((1+i\alpha) f_1^2 - (1-i\alpha) f_2^2 \Big), \\ - 2i\sqrt{1+\alpha^2} f_1 f_2 \Bigg)$$

we obtain that $8\langle \psi_z, \psi_{\bar{z}} \rangle = (1+\alpha^2)\left(|f_1(z)|^2 + |f_2(z)|^2 \right)^2$, and therefore the metric is written as

$$ds^2 = \frac{1+\alpha^2}{4} \left(|f_1(z)|^2 + |f_2(z)|^2 \right)^2 |dz|^2. \tag{3}$$

Note that ds^2 is well defined on $\mathbb{C} \setminus S$. Since α is irrational, the condition **C.3** ensures that f_1, f_2 cannot vanish simultaneously. Thus the metric (3) is regular on $\mathbb{C} \setminus S$.

It is obvious that the metric ds^2 is complete about any point in S. In addition, let $\sigma(u)$ be a divergent curve in $\mathbb{C}$ not meeting S. Since f_1 is elliptic, we can choose small disks about its zeroes so that

1. the Euclidean length of $\sigma(u)$ in the exterior of these disks is infinite, and

2. $|f_1| \geq c$ for some $c > 0$ in the exterior of these disks.

Thus the length of $\sigma(u)$ with respect to ds^2 is infinite. This ensures that ds^2 is complete.

We have only left to check the assertion about the projection of $\psi(\Sigma)$. For this, we begin by noting that, under the identification $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$, the projection of ψ over the x_1, x_2-plane is

$$(\psi_1(z), \psi_2(z)) = \psi_1(z) + i\psi_2(z) = \frac{1}{2}\left(g_1(z) + \overline{g_2(z)} \right).$$

If we denote $z = s + it$, this equation is written as

$$2\left(\psi_1(s+it) + i\psi_2(s+it)\right) = 2b(q + (1+i\alpha)s) + h_1(z) + \overline{h_2(z)}. \tag{4}$$

Now, since due to **C.1** we have $h(\bar{z}) = \overline{h(z)}$, we find that $\overline{h_2(z)} = h_1(\bar{z})$, and (4) turns into

$$2\left(\psi_1 + i\psi_2\right)(s+it) = 2b(q + (1+i\alpha)s) + h_1(s+it) + h_1(s-it). \quad (5)$$

Let $s_0 \in \mathbb{R}$ be fixed and arbitrary, and let us define the meromorphic map

$$G(w) = h_1(s_0 + iw) + h_1(s_0 - iw).$$

It is clear that $G(w)$ is elliptic on $\mathbb{C}/\Gamma$. On the other hand, if $w = u + iv$, the curve $[(u,0)] : \mathbb{R} \to \mathbb{C}/\Gamma$ is dense over the torus $\mathbb{C}/\Gamma$, due to the fact that α is irrational. Since $G(w)$ is elliptic and non-constant, the curve

$$G(u,0) = h_1(s_0 + iu) + h_1(s_0 - iu)$$

is dense on the Riemann sphere $\mathbb{C}_\infty$. Now, by (5), the map $(\psi_1 + i\psi_2)(s_0 + it)$ is a (possibly non-connected) dense curve in $\mathbb{C}$. This finishes the proof. $\square$

Remark 2. It is not difficult to obtain elliptic functions $h : \mathbb{C}/\Lambda \to \mathbb{C}_\infty$ satisfying conditions **C.1**, **C.2** and **C.3**. First, note that from **C.2** h must have odd degree. Let $\wp$ be the Weierstrass function of the torus $\mathbb{C}/\Lambda$, and $a > 0$ the real number such that $\wp(1/2) = a = -\wp(i/2)$. If we search among elliptic functions in $\mathbb{C}/\Lambda$ of degree three, the choices

$$\begin{cases} h = \wp' & b = 2a^2 \quad f = \sqrt{6}\wp \\ h = \wp'/\wp^2 & b = 2 \quad\;\;\, f = \sqrt{6}a/\wp \\ h = (1/(\wp - a))' & b = 1 \quad\;\;\, f = \sqrt{3}(\wp + a)/(\wp - a) \end{cases}$$

satisfy **C.1**, **C.2** and **C.3**. This follows from the identities

$$\wp'^2 = 4\wp\left(\wp^2 - a^2\right), \qquad \wp'' = 6\wp^2 - 2a^2. \quad (6)$$

In general, any elliptic function h on $\mathbb{C}/\Lambda$ can be expressed as

$$h = R_1(\wp) + \wp' R_2(\wp)$$

for rational functions R_1, R_2. This fact together with (6) make it possible to obtain many more examples with the three desired conditions.

3 Dense examples

In this Section we shall show that some of the complete minimal surfaces constructed in Theorem 1 are dense in $\mathbb{R}^3$.

First of all, assume that h has a pole of order l at $d \in \mathbb{C}$, what means that f has a pole of order $k = (l+1)/2$ at d. In particular, all poles of h must be of odd order. Then f_1 has a pole of order k at $z_0 = (d-q)/(1+i\alpha)$, and f_2 is holomorphic at z_0. Let us compute the residue of $f_1 f_2$ at z_0.

If $\sum_{n=1}^{k} a_{-n}(z-d)^{-n}$ is the principal part of f at the pole $d \in \mathbb{C}$, then a direct computation shows that

$$\text{Res}\,(f_1 f_2, z_0) = \sum_{n=1}^{k} \frac{a_{-n}(1-i\alpha)^{n-1} f^{(n-1)}\left(q + \frac{1-i\alpha}{1+i\alpha}(d-q)\right)}{\left((n-1)!\,(1+i\alpha)^n\right)}. \tag{7}$$

In the same way, f_2 has a pole of order k at $\widetilde{z}_0 = (d-q)/(1-i\alpha)$, f_1 is holomorphic at $\widetilde{z}_0$ and

$$\text{Res}\,(f_1 f_2, \widetilde{z}_0) = \sum_{n=1}^{k} \frac{a_{-n}(1+i\alpha)^{n-1} f^{(n-1)}\left(q + \frac{1+i\alpha}{1-i\alpha}(d-q)\right)}{\left((n-1)!\,(1-i\alpha)^n\right)}. \tag{8}$$

Let us consider the real function $\mathcal{A}$ that assigns to every $\alpha \in \mathbb{R}$ the value

$$\mathcal{A}(\alpha) = \frac{\text{Im}\,(\text{Res}\,(f_1 f_2, z_0))}{\text{Im}\,(\text{Res}\,(f_1 f_2, \widetilde{z}_0))} \in \mathbb{R}, \tag{9}$$

defined whenever the lower part of the quotient is non-zero.

It follows from (7) and (8) that the function $\mathcal{A}$ is smooth. Note that $\mathcal{A}$ can be constant, as the choice $h = \wp'$, $d = 0$ shows. Indeed, whenever $d \in \mathbb{R}$, we find from **C.1**, (7) and (8) that $\mathcal{A}(\alpha) \equiv -1$ if the quotient is well defined.

However, $\mathcal{A}$ is not constant in general. For instance, if we make the choices $h = \wp'$ and $d = i$, the graphic of $\mathcal{A}(\alpha)$ is shown in Figure 1.

So, let us choose h satisfying **C.1**, **C.2** and **C.3**, and assume that h has a pole at $d \in \mathbb{C}$ such that $\mathcal{A} = \mathcal{A}(\alpha, d)$ is not constant with respect to α. Since $\mathcal{A}$ is continuous, there is some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ so that $\mathcal{A}(\alpha) \in \mathbb{R} \setminus \mathbb{Q}$.

For this α we consider $\beta(s) = g(q + (1+i\alpha)s)$, and thus we obtain via Theorem 1 a complete minimal surface given by (1).

If we regard $\psi(z)$ as parametrized in $\mathbb{C} \setminus S$, then its first two coordinates are well defined, but the integral of the holomorphic 1-form $f_1(z) f_2(z) dz$ that gives

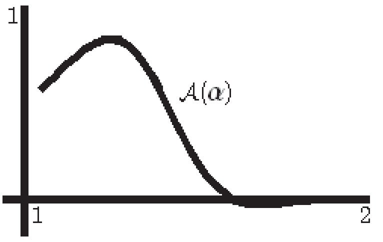

Figure 1: The graphic of $\mathcal{A}(\alpha)$ for $h = \wp'$, $d = i$ and $q = 1/2$.

the third coordinate has non-zero residue at $z_0 \in S$ and $\widetilde{z}_0 \in S$. Moreover, if we denote

$$A = 2\sqrt{1+\alpha^2}\ \mathrm{Im}\,(\mathrm{Res}\,(f_1 f_2, z_0)), \quad B = 2\sqrt{1+\alpha^2}\ \mathrm{Im}\,(\mathrm{Res}\,(f_1 f_2, \widetilde{z}_0)),$$

then A, B are non-zero, since $\mathcal{A}(\alpha)$ is irrational. Thus, the integration over a homotopically non-trivial loop about z_0 produces a translational symmetry of the minimal surface with vector $(0, 0, A)$. In the same way, by integrating about $\widetilde{z}_0$ one obtains a translational symmetry of vector $(0, 0, B)$.

This ensures that the minimal surface is invariant under all translations of $\mathbb{R}^3$ in the direction of the x_3-axis with vector $(0, 0, \lambda A + \mu B)$, $\lambda, \mu \in \mathbb{Z}$. Besides, all these planes are symmetry planes of the surface, and at each height $x_3 = \lambda A + \mu B$ the minimal surface is an exact replic of its intersection with the x_1, x_2-plane.

Finally, since $\mathcal{A}(\alpha) \in \mathbb{R} \setminus \mathbb{Q}$, it holds $A/B \in \mathbb{R} \setminus \mathbb{Q}$. This ensures that $\{\lambda A + \mu B : \lambda, \mu \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. But in addition, the projection of ψ over the x_1, x_2-plane is dense in that plane. All of this implies that the minimal surface is dense in the whole $\mathbb{R}^3$. Summarizing, we have proved the following.

Theorem 3. *Let $h : \mathbb{C}/\Lambda \to \mathbb{C}_\infty$ satisfy* **C.1**, **C.2**, **C.3**, *and assume that the function $\mathcal{A}$ in* (9) *is not constant for some pole d of h. Then there exist infinitely many $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $\mathcal{A}(\alpha) \in \mathbb{R} \setminus \mathbb{Q}$. For any such pair (h, α), the complete minimal surface constructed in Theorem 1 is dense in $\mathbb{R}^3$, and symmetric with respect to a dense family of parallel planes in $\mathbb{R}^3$.*

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