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Generic diffeomorphisms away from homoclinic tangencies and heterodimensional cycles

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Abstract. The C^1 density conjecture of Palis asserts that diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle are C^1 dense in the complement of the C^1 closure of hyperbolic systems. In this paper we prove some results towards the conjecture.

Keywords: hyperbolic diffeomorphism, homoclinic tangency, heterodimensional cycle, generic property, dominated splitting.

Mathematical subject classification: 37D30.

§1. Introduction

Let M be a compact manifold without boundary, and $\mathrm{Diff}(M)$ be the set of diffeomorphisms of M, endowed with the C^1 topology.

A diffeomorphism $f : M \to M$ is called *hyperbolic* if the limit set $L(f)$ of f is a hyperbolic set, where $L(f)$ is by definition the closure of the union of the ω-limit set $\omega(x)$ and α-limit set $\alpha(x)$, for all $x \in M$. Hyperbolic systems include many nice systems such as structurally stable systems, Axiom A systems, etc. However, contrary to a common expectation, hyperbolic systems are found not dense in $\mathrm{Diff}(M)$. That is, there are diffeomorphisms that can not be C^1 approximated by hyperbolic systems (see [AS] [N1] [Si] for early examples of this nature). Which bifurcation phenomena could be typical in this robustly non-hyperbolic world? Palis [P,PT] has the following famous conjecture:

The C^1 Density Conjecture. Diffeomorphisms of M exhibiting either a homoclinic tangency or a heterodimensional cycle are C^1 dense in the complement of the C^1 closure of hyperbolic systems.

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Here a point $x \in M$ is called a *homoclinic tangency* of f, if there is a hyperbolic periodic orbit P of f such that $x \in W^u(P) \cap W^s(P) - P$, and such that the intersection of $W^u(P)$ and $W^s(P)$ at x is not transverse. A *heterodimensional cycle* of f consists of two hyperbolic periodic orbits P and Q of f of different indices such that $W^u(P) \cap W^s(Q) \neq \emptyset$, and $W^u(Q) \cap W^s(P) \neq \emptyset$. Here by the *index* of a hyperbolic periodic orbit P we mean the integer $\dim(W^s(P))$. Note that at least one of the two intersections, $W^u(P) \cap W^s(Q)$ or $W^u(Q) \cap W^s(P)$, is not transverse, due to inadequate dimensions. Homoclinic tangency and heterodimensional cycle are two bifurcation phenomena that go beyond the Kupka-Smale systems. This makes the conjecture of Palis even more striking.

In dimension 2 the conjecture is proved recently by Pujals and Sambarino in a remarkable paper [PS]. Note that in dimension 2, a priori, there can be no heterodimensional cycles (a heterodimensional cycle involves periodic saddles of different indices, while in dimension 2 all periodic saddles have the same index 1). Hence in dimension 2, what was conjectured by Palis, and now proved by Pujals and Sambarino, is that any diffeomorphism can be C^1 approximated either by a hyperbolic diffeomorphism, or by one with a homoclinic tangency.

In dimension higher than 2, there are systems that can be C^1 approximated neither by hyperbolic systems, nor by systems with a homoclinic tangency [Sh1] [M1] [BD] [BV], and the conjecture says such a system must be C^1 approximated by one with a heterodimensional cycle. In this paper we prove some results towards the conjecture for higher dimensions. Note that the conjecture can be alternately stated as that *Hyperbolic diffeomorphisms are C^1 dense in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle*. We will prove in this paper that there is a C^1 residual subset $\mathcal{R}$ in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle such that every $f \in \mathcal{R}$ is "nearly" hyperbolic, in the sense as described in Theorem A and B below. The statements use the notion of the so called minimally non-hyperbolic set we now introduce.

A compact invariant set Λ of f is called *minimally non-hyperbolic* of f if Λ is not a hyperbolic set of f but any nonempty compact invariant proper subset of Λ is a hyperbolic set of f. This notion plays an important role in Pujals and Sambarino [PS]. It resembles the notion of minimally rambling sets, studied intensively by Liao [L2].

Proposition 1.1 ([L2, P.4], [PS, P.983]). *Any non-empty non-hyperbolic compact invariant set of f contains at least one minimally non-hyperbolic set of f.*

Proof. Let A be a non-empty non-hyperbolic compact invariant set of f. Let C be the set of non-empty non-hyperbolic compact invariant subset of A, respecting f. C is partially ordered by the inclusion. It is easy to see any linearly ordered subset of C has a lower bound in C. By Zorn's lemma, C has a minimal element Λ. It is easy to see Λ is a minimally non-hyperbolic set of f, contained in A. This proves Proposition 1.1.

We follow Liao [L2] to divide minimally non-hyperbolic sets into two types, simple and non-simple, in a slightly different way by using the following characterization of hyperbolic sets that can be found in Selgrade [S], Sacker and Sell [SS], Mañé [M2] and Liao [L1]. Denote

$$\begin{aligned} D^s(x) &= \big\{v \in T_x(M) \mid \ \|Df^n(v)\| \to 0,\ n \to +\infty\big\}, \\ D^u(x) &= \big\{v \in T_x(M) \mid \ \|Df^{-n}(v)\| \to 0,\ n \to +\infty\big\}. \end{aligned}$$

These are Df-invariant (as family) linear subspaces of T_xM. By definition, vectors of D^s and D^u are asymptotic to zero under forward or backward iterates, respectively, but not necessarily exponentially fast. However, if the two subspaces form a direct sum at every point of a compact invariant set, exponential rates will follow:

Proposition 1.2 ([S, SS, M2, L1]). *A compact invariant set A of f is hyperbolic if and only if $D^s(x) \oplus D^u(x) = T_xM$, for all $x \in A$.*

Let us call a point $x \in M$ *resisting* of f if the equality $D^s(x) \oplus D^u(x) = T_xM$ does not hold. This means either $D^s(x) + D^u(x) \neq T_xM$, or $D^s(x) \cap D^u(x) \neq \{0\}$. The set of resisting points of f is f-invariant, but generally not closed. A minimally non-hyperbolic set Λ will be called of *simple type* if there is a resisting point $a \in \Lambda$ such that both $\omega(a)$ and $\alpha(a)$ are proper subsets of Λ. Otherwise the minimally non-hyperbolic set will be called of *non-simple type*. The following proposition describes the structure of a simple type minimally non-hyperbolic set.

Proposition 1.3. *A simple type minimally non-hyperbolic set Λ of f can be written as $\Lambda = \omega(a) \cup \mathrm{Orb}(a) \cup \alpha(a)$, where $a \in \Lambda$ is a resisting point of f, such that $\omega(a)$ and $\alpha(a)$ are both hyperbolic, and $a \notin \omega(a) \cup \alpha(a)$.*

Proof. Since $\omega(a)$ and $\alpha(a)$ are both proper subsets of Λ, they are hyperbolic. Hence $a \notin \omega(a) \cup \alpha(a)$. Also, being a non-hyperbolic (with a resisting) compact invariant subset, $\omega(a) \cup \mathrm{Orb}(a) \cup \alpha(a)$ must be the whole Λ. This proves Proposition 1.3.

Thus a simple type minimally non-hyperbolic set Λ has a clear structure. It is a non-transverse (more correctly, non-direct-sum) heteroclinic (if $\omega(a) \cap \alpha(a) = \emptyset$) or homoclinic (if $\omega(a) \cap \alpha(a) \neq \emptyset$) connection of two hyperbolic sets. In particular, in case $\omega(a) = \alpha(a) = \{p\}$ is a hyperbolic fixed point, this will be the familiar picture of homoclinic tangency.

On the other hand, there has been no general structure theorem available for a non-simple type minimally non-hyperbolic set. By definition a non-simple type minimally non-hyperbolic set Λ must be topologically transitive. Indeed, by definition, every resisting point $a \in \Lambda$ satisfies either $\omega(a) = \Lambda$, or $\alpha(a) = \Lambda$. A trivial example for a non-simple type minimally non-hyperbolic set would be a non-hyperbolic fixed point, or a non-hyperbolic periodic orbit, or any non-hyperbolic minimal set. Liao proves under the conditions of [L2] that a non-simple type minimally non-hyperbolic set must not be a minimal set. Pujals-Sambarino prove under the conditions of [PS] that the only non-simple type minimally non-hyperbolic set is an invariant circle with an irrational rotation. These highly non-trivial results form a key step in their work, and also justify the use of a general principle of Liao:

Principle. *To prove that a compact invariant set A is hyperbolic, it suffices to rule out the possibility of the existence of simple type and non-simple type minimally non-hyperbolic sets contained in A.*

This principle has an advantage that, to prove that a compact invariant set A (for instance the nonwandering set) is hyperbolic, we do not have to handle the whole set A globally, but only have to rule out the possibility of the existence of the two types of minimally non-hyperbolic sets in A, which may be of relatively less global nature.

Now we state the main results of this paper. Recall that Palis conjecture can be alternately stated as that *Hyperbolic diffeomorphisms are C^1 dense in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle*.

Theorem A. *There is a C^1 residual subset $\mathcal{R}$ in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle, such that every $f \in \mathcal{R}$ has no simple type minimally non-hyperbolic sets contained in $L(f)$.*

To prove Palis conjecture it remains to rule out the possibility of the existence of non-simple type minimally non-hyperbolic sets Λ (contained automatically in $L(f)$ since Λ is topologically transitive), for a dense (or residual, if possible) subset of diffeomorphisms in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle.

The following Theorem B has not achieved this, but asserts that such a non-simple type minimally non-hyperbolic set Λ, if it exists, must look somewhat special. We hope this would help to rule out its existence eventually. Since the intersection of (countably many) residual subsets is residual, we will use the same notation for several different residual subsets of the same space ($\mathcal{R}$ for a local one, and $\mathcal{A}$ for a global one).

Theorem B. *There is a C^1 residual subset $\mathcal{R}$ in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle, such that any non-simple type minimally non-hyperbolic set Λ of any $f \in \mathcal{R}$ has the following feature:*

(1) *Λ is the common Hausdorff limit of two sequences of hyperbolic periodic orbits of different indices. More precisely, there are two sequences of hyperbolic periodic orbits $\{P_k\}$ and $\{Q_k\}$ of f that both converge to Λ in the Hausdorff metric, such that $\mathrm{ind}(Q_k) = \mathrm{ind}(P_k) + 1$ and $W^u(P)$ intersects $W^s(Q)$ transversely. In particular, Λ can not be contained in any normally hyperbolic arc or circle of f.*

(2) *Λ is partially hyperbolic with central bundle at most 2-dimensional. More precisely, either there is a three-ways Df-invariant splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$, where E^s is dominated by E^c, and E^c is dominated by E^u, such that E^s is contracting, E^u is expanding, and E^c is 1-dimensional and is neither contracting nor expanding, or, there is a four-ways Df-invariant splitting $T_\Lambda M = E^s \oplus E^{cs} \oplus E^{cu} \oplus E^u$, where E^s is dominated by E^{cs}, E^{cs} is dominated by E^{cu}, and E^{cu} is dominated by E^u, such that E^s is contracting, E^u is expanding, and E^{cs} and E^{cu} are each 1-dimensional and neither contracting nor expanding.*

Theorem A and B will be proved in §2 and §3, respectively. Theorem A can be obtained quickly from the results of [W2] and [GW] (which are actually preparations for the present paper). The proof of Theorem B will depend in addition the elegant selecting lemma of Liao [L2], reviewed in §3.

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§2. The proof for Theorem A

We start with recalling a characterization for diffeomorphisms that are C^1 away from homoclinic tangencies, by dominated splittings on the so called preperiodic sets. A point $x \in M$ is C^1 *preperiodic*, if for any C^1 neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}(M)$ and any neighborhood U of x in M, there is $g \in \mathcal{U}$ and $y \in U$ such that y is periodic of g (see [W1]). We denote the set of C^1 preperiodic points by $P_*(f)$. It is easy to see that $P_*(f)$ is closed and f-invariant. Also,

$$\Omega(f) \subset P_*(f) \subset R(f),$$

where $\Omega(f)$ and $R(f)$ denote the nonwandering and chain recurrent sets of f, respectively. The first inclusion is by the C^1 closing lemma of Pugh [Pu, PR], and the second inclusion is just by definitions.

Note that in the definition of preperiodic points it is equivalent to replace the term "periodic" by "hyperbolic periodic", because any periodic point can be made hyperbolic by an arbitrarily small C^r perturbation. We call a point $x \in M$ C^1 *i-preperiodic* of f, $0 \le i \le d$, if for any C^1 neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}(M)$ and any neighborhood U of x in M, there is $g \in \mathcal{U}$ and $y \in U$ such that y is a hyperbolic periodic point of g of index i. Denote by $P_*^i(f)$ the set of C^1 i-preperiodic points of f. Then

$$P_*(f) = \bigcup_{i=0}^{d} P_*^i(f).$$

$P_*^i(f)$ is closed and f-invariant, for each $0 \le i \le d$. Generally $P_*^i(f)$ and $P_*^j(f)$ are not disjoint for $i \ne j$. In fact, according to Liao [L2] and Mañé [M3], $P_*^i(f)$ are mutually disjoint for $0 \le i \le d$ if and only if f is Axiom A and no-cycle.

Let Λ be a compact invariant set of f. A continuous invariant splitting $T_\Lambda M = \Delta^s \oplus \Delta^u$ on Λ is called *dominated of index* i, $1 \le i \le d-1$, if $\dim \Delta^s(x) = i$ for all $x \in \Lambda$, and if there are two constants $0 < \lambda < 1$ and $C > 0$ such that

$$\|Df^n|_{\Delta^s(x)}\| \cdot \|Df^{-n}|_{\Delta^u(f^n(x))}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. We may call $\Delta^s \oplus \Delta^u$ specifically a (C, λ)-dominated splitting. This is equivalent to that for some constants $\iota \in I\!N$ and $0 < \mu < 1$,

$$\|Df^\iota|_{\Delta^s(x)}\| \cdot \|Df^{-\iota}|_{\Delta^u(f^\iota(x))}\| \le \mu$$

for all $x \in \Lambda$. We may also call $\Delta^s \oplus \Delta^u$ specifically an (ι, μ)-dominated splitting. A compact invariant set may have more than one dominated splitings. Nevertheless, for fixed i, dominated splitting of index i is unique.

Proposition 2.1 ([W2]). *Let $f : M \to M$ be a diffeomorphism. The following three conditions are equivalent:*

(1) *f has dominated splitting of index i on its C^1 i-preperiodic set $P_*^i(f)$, for all $1 \leq i \leq d-1$.*

(2) *f can not be C^1 approximated by a system that exhibits a homoclinic tangency associated with some hyperbolic periodic point of some index $1 \leq i \leq d-1$.*

(3) *There is a C^1 neighborhood $\mathcal{U}$ of f and a number $\gamma > 0$ such that for any hyperbolic periodic point p of any $g \in \mathcal{U}$ of any index $1 \leq i \leq d-1$, $\angle(E^s(p,g), E^u(p,g)) \geq \gamma$.*

Another major issue to us will be C^1 generic properties about orbits-connecting. There are a number of recent work along this direction, see [Ab1], [Ab2], [Ar], [CMP] and [GW]. We state three propositions of this type. The first one concerns a weak form of transitivity. It is to generalize the notion of topological transitive sets as much as possible, but still to keep certain recurrence so that the C^1 connecting lemma applies on such a set to yield various connections. We say $y \in M$ is *attainable from $x \in M$ respecting f*, if for any neighborhood U of x in M and any neighborhood V of y in M, there is $z \in U$ such that $f^n(z) \in V$ for some integer $n \geq 1$. Thus a nonwandering point of f is attainable from itself. We say x and y are *bi-attainable* to each other if y is attainable from x and x is attainable from y. A compact invariant set Λ is called *weakly transitive* if every pair of points of Λ are bi-attainable to each other. The main examples in our mind for weakly transitive sets are a single ω-limit set $\Lambda = \omega(x)$ or α-limit set $\Lambda = \alpha(x)$ (this is more general than a transitive set because x may not be in Λ), or the Hausdorff limit Λ of a sequence of periodic orbits P_k of f.

Proposition 2.2 ([Ar], [GW]). *There is a C^1 residual subset $\mathcal{A} \subset \mathrm{Diff}(M)$ such that for every $f \in \mathcal{A}$, bi-attainability is a closed equivalence relation on the nonwandering set $\Omega(f)$.*

Thus for every $f \in \mathcal{A}$, $\Omega(f)$ decomposes into closed, f-invariant equivalence classes (generally infinite in number). We call each equivalence class a *weakly transitive component* of f. Note that bi-attainability is not an equivalence relation in general.

The second generic property concerns the possibility of creation of a heterodimensional cycle by C^1 perturbations. Recall for a hyperbolic set H with constant dimension i of stable subspaces, we define its *index* to be i. Any hyperbolic set decomposes into at most $\dim(M) + 1$ pieces of hyperbiolic subsets, of which the index is well defined.

Proposition 2.3 ([GW]). *There is a C^1 residual subset $\mathcal{A} \subset \mathrm{Diff}(M)$ such that for every $f \in \mathcal{A}$, if a weakly transitive set Λ of f contains two hyperbolic sets H_1 and H_2 of different indices, then f can be C^1 approximated by g that has a heterodimensional cycle.*

Note that H_1 and H_2 in the assumption must be both of saddle type. That is, $1 \leq \mathrm{ind}(H_1) \leq d-1$, $1 \leq \mathrm{ind}(H_2) \leq d-1$. This is because a hyperbolic set of index 0 (or d) consists of finitely many periodic sources (or sinks), and because if a weakly transitive set Λ contains a periodic source (or sink) P, Λ must reduce to P.

The problem stated in Proposition 2.3 is very natural. Being contained in the same weakly transitive set Λ, the two hyperbolic sets of different indices are loosely connected each other. The problem is then to create a true connection between two hyperbolic periodic orbits of different indices. Interference of orbits appears in the perturbations however, which makes it unclear if the problem could be solved by the C^1 connecting lemma alone, even if the set Λ in question is not only weakly transitive, but transitive. See [GW] for a detailed illustration about this subtle point. Some C^1 generic assumptions then are added to avoid the interference of orbits.

The third generic property concerns how much a homoclinic class spreads. It involves a similar subtle point and needs some generic assumptions, as stated in Proposition 2.4 next. Recall that the *homoclinic class* $H(P)$ of a hyperbolic periodic orbit P of f is defined to be the closure of the union of hyperbolic periodic orbits of f that are H-related to P. Here two hyperbolic periodic orbits P and Q of f are called *H-related* if $W^u(P) \cap W^s(Q) \neq \emptyset$ with a transverse intersection, and $W^u(Q) \cap W^s(P) \neq \emptyset$ with a transverse intersection. A homoclinic class $H(P)$ is called *trivial* if it consists of the orbit P only. If $H(P)$ is non-trivial, it coincides with the closure of transverse homoclinic points of P. If $\mathrm{ind}(P) = i$, we will simply call $H(P)$ an *i-homoclinic class*. Note that i may not be uniquely assigned to $H(P)$, because generally it is possible that $H(P) = H(Q)$ for a hyperbolic periodic orbit Q with $\mathrm{ind}(P) \neq \mathrm{ind}(Q)$.

Proposition 2.4 ([Ar], [GW]). *There is a C^1 residual subset $\mathcal{A} \subset \mathrm{Diff}(M)$ such that for every $f \in \mathcal{A}$, if a weakly transitive set Λ of f contains a hyperbolic set K of index i, then $\Lambda \subset H(P)$ for a hyperbolic periodic orbit P of index i. In particular, if this Λ is a weakly transitive component, then $\Lambda = H(P)$.*

Note that K must be of saddle type. That is, $1 \leq i \leq d-1$. This is because otherwise Λ would reduces to a periodic source or sink, contradicting that Λ is non-trivial.

In particular, Proposition 2.4 says that, for generic f, any homoclinic class $H(P)$ is itself a weakly transitive component, $-$ the component that contains

P. Of course, being transitive, a homoclinic class can not go beyond the weakly transitive component it lies in. Thus, generically, every homoclinic class spreads as much as it can.

We also need the following two simple facts. Recall $D^s(x)$ and $D^u(x)$ are the two linear subspaces of vectors in T_xM that tend to zero asymptotically under positive and negative Df-iterates, respectively, defined in §1. Also, note that if $\omega(x)$ happens to be hyperbolic for some $x \in M$, then $\mathrm{ind}(\omega(x))$ is well defined.

Lemma 2.5. *Let a be any point of M. If $\omega(a)$ is hyperbolic, then* $\dim D^s(a) = \mathrm{ind}(\omega(a))$. *Likewise, if $\alpha(a)$ is hyperbolic, then* $\dim D^u(a) = d - \mathrm{ind}(\alpha(a))$.

Proof. The proof is easy by the shadowing lemma. We take the case of $\omega(a)$. Write $\mathrm{ind}\,\omega(a) = i$. Take a large integer m such that the positive f-orbit of $b = f^m(a)$ remains close to $\omega(a)$. Together with the negative f-orbit of a point $y \in \omega(a)$ that is close to b, it gives a pseudo orbit, which is hence shadowed by the f-orbit of some point z, which remains entirely in a small neighborhood of $\omega(a)$ hence is hyperbolic of index i. Moreover, by shadowing, $b \in W^s(z)$. Hence $a \in W^s(z)$. Then it is easy to see $\dim D^s(a) = i$. This proves Lemma 2.5.

The following elementary lemma is due to Liao. We omit the proof, which is cited in [W2, Lemma 2.2].

Lemma 2.6 ([L2]). *Assume Λ is a compact invariant set of f with a dominated splitting $\Delta^s \oplus \Delta^u$ on Λ, and $x \in \Lambda$. Then either $\Delta^s(x) \subset D^s(x)$, or $D^s(x) \subset \Delta^s(x)$. Likewise, either $\Delta^u(x) \subset D^u(x)$, or $D^u(x) \subset \Delta^u(x)$.*

Now we prove Theorem A.

Theorem A. *There is a C^1 residual subset $\mathcal{R}$ in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle, such that every $f \in \mathcal{R}$ has no simple type minimally non-hyperbolic sets contained in $L(f)$.*

Proof. Let $\mathcal{R}$ be the set of diffeomorphisms in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle that satisfy the C^1 generic conditions stated in Propositions 2.2 through 2.4, as well as the well known C^1 generic condition $L(f) = \overline{P(f)}$ (a consequence of the C^1 closing lemma of Pugh). Let $f \in \mathcal{R}$. We prove f has no simple type minimally non-hyperbolic set contained in $L(f) = \overline{P(f)}$. Suppose f has a simple type minimally non-hyperbolic set Λ in $\overline{P(f)}$. By Proposition 1.3,

$$\Lambda = \omega(a) \cup \mathrm{Orb}(a) \cup \alpha(a),$$

where $a \in \Lambda$ is a resisting point of f such that $\omega(a)$ and $\alpha(a)$ are both hyperbolic, and $a \notin \omega(a) \cup \alpha(a)$. Since $a \in \overline{P(f)}$, there is a sequence of periodic orbits of f that converge in the Hausdorff metric to a compact f-invariant set Γ with $a \in \Gamma$. Thus $\Gamma \supset \Lambda$. Being the Hausdorff limit of a sequence of periodic orbits, Γ is weakly transitive. If $\omega(a)$ and $\alpha(a)$ have different indices, by Proposition 2.3, a heterodimensional cycle can be created by C^1 perturbation, giving a contradiction. Thus $\omega(a)$ and $\alpha(a)$ have the same index, say i. Note that Γ is non-trivial, because even the subset Λ of it does not reduce to a periodic orbit. In particular, $1 \leq i \leq d-1$. By Proposition 2.4, Γ is contained in an i-homoclinic class of f, hence contained in $\overline{P^i(f)} \subset P^i_*(f)$ and hence, by Proposition 2.1, has a dominated splitting

$$T_\Gamma(M) = \Delta^s \oplus \Delta^u$$

of index i. By Lemma 2.5,

$$\dim D^s(a) = i, \quad \text{and} \quad \dim D^u(a) = d - i.$$

Hence by Lemma 2.6, $D^s(a) = \Delta^s(a)$, $D^u(a) = \Delta^u(a)$. Thus

$$T_a(M) = D^s(a) \oplus D^u(a),$$

contradicting that a is a resisting point. This proves Theorem A.

§3. The proof for Theorem B

First we quote two results from [W2] about periodic orbits for systems C^1 away from homoclinic tangencies. The first one says that, for such a system, non-hyperbolic periodic orbits appear in a very restricted way. Let $\pi(p)$ denote the period of a periodic point p.

Lemma 3.1 ([W2]). *Assume f can not be C^1 approximated by systems with homoclinic tangencies. Then there is a C^1 neighborhood $\mathcal{U}$ of f such that for any periodic point p of any $g \in \mathcal{U}$, $Dg^{\pi(p)}|_{T_pM}$ can have at most one eigenvalue of modulus 1 which, if exists, is real and has multiplicity 1.*

Alternately, let us call a non-hyperbolic periodic orbit P of f *sole-neutral partially hyperbolic* if T_PM splits into a three ways Df-invariant splitting $T_PM = G^s \oplus G^c \oplus G^u$ with G^c 1-dimensional, such that $Df|_{G^s}$ is contracting, $Df|_{G^u}$ is expanding, and $Df^{\pi(P)}|_{G^c}$ is either id, or $-id$. Then Lemma 3.1 just says that any non-hyperbolic periodic orbit of any $g \in \mathcal{U}$ is sole-neutral partially hyperbolic. If $\dim G^s = i$, we will call P *i-sole-neutral partially hyperbolic*. Thus i runs from 0 to $d-1$.

Lemma 3.2 ([W2]). *Assume f can not be C^1 approximated by systems with homoclinic tangencies. Then there is a C^1 neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ (where $\mathcal{U}$ is given as in Lemma 3.1) of f and a number $K \geq 2$, such that for any hyperbolic periodic point p of any $g \in \mathcal{U}_0$,*

$$\|Dg^{\pi(p)}|_{E^s(p)}\| \leq K,$$
$$\|Dg^{-\pi(p)}|_{E^u(p)}\| \leq K,$$

where $E^s(p)$ and $E^u(p)$ are the stable and unstable subspaces of p, respectively.

Here we adopt the usual convention that E^s or E^u could be trivial. That is, we regard a trivial subbundle as to automatically satisfy the inequality. Thus whenever we say such an inequality is not satisfied below, the subbundle in question has to be non-trivial.

Let us call an eigenvalue λ of some linear automorphism *δ-neutral* if $1 - \delta \leq |\lambda| \leq 1 + \delta$.

Lemma 3.3. *Assume f can not be C^1 approximated by systems with homoclinic tangencies. Then there is a C^1 neighborhood $\mathcal{V}$ of f, together with three numbers $K \geq 2$, $0 < \lambda < 1$ and $\delta > 0$ such that the following conditions hold.*

(1) *Any periodic point p of any $g \in \mathcal{V}$ can have for $Dg^{\pi(p)}$ at most one δ-neutral eigenvalue λ_p which, if exists, must be real and of multiplicity 1.*

(2) *If a periodic point p of $g \in \mathcal{V}$ has no δ-neutral eigenvalue for $Dg^{\pi(p)}$, then (it is hyperbolic and)*

$$\|Dg^{\pi(p)}|_{E^s(p)}\| \leq K\lambda^{\pi(p)}, \quad \text{and}$$
$$\|Dg^{-\pi(p)}|_{E^u(p)}\| \leq K\lambda^{\pi(p)}.$$

(3) *If a periodic point p of $g \in \mathcal{V}$ has a δ-neutral eigenvalue λ_p for $Dg^{\pi(p)}$, then T_pM splits into a $Dg^{\pi(p)}$-invariant partially hyperbolic splitting $G^s(p) \oplus G^c(p) \oplus G^u(p)$, where $G^c(p)$ is the 1-dimensional eigenspace associated with λ_p, such that*

$$\|Dg^{\pi(p)}|_{G^s(p)}\| \leq K\lambda^{\pi(p)}, \quad \text{and}$$
$$\|Dg^{-\pi(p)}|_{G^u(p)}\| \leq K\lambda^{\pi(p)}.$$

In particular, this is true for any non-hyperbolic periodic orbit of $g \in \mathcal{V}$.

Proof. Since the idea of the proof is simple but the details are tedious, we only illustrate the idea of the proof. See [W2] for more details.

Let the C^1 neighborhood $\mathcal{U}_1$ and the three numbers $K \geq 2$, $\delta_1 > 0$ and $\lambda = 1/(1 + \delta_1)$ be determined as in [W2, Lemma 3.6]. Rewrite $\mathcal{U}_1$ as $\mathcal{V}$, and δ_1 as δ, for short. This gives the required $\mathcal{V}$, K, λ and δ. We remark that the procedure to determine these items in [W2, Lemma 3.6] is quite long, hence is omitted. The idea is that, roughly, $\delta > 0$ and $\mathcal{V} \subset \mathcal{U}_0$ (where $\mathcal{U}_0$ is given as in Lemma 3.2) are chosen small enough so that, given any hyperbolic periodic point p of any $g \in \mathcal{V}$, with perturbations such as $(1 + \delta)$-stretching or $(1 - \delta)$-depressing on the tangent spaces $T_{g^i p}M$ along the orbit of p (By Franks lemma [F], these can indeed be realized as C^1 perturbations of g), one will never get out of $\mathcal{U}_0$, the neighborhood where Lemma 3.1 and 3.2 with (K, λ)-estimates hold. Now we verify $\mathcal{V}$, K, λ and δ satisfy Lemma 3.3.

First note that any δ-neutral eigenvalue λ_p of any periodic point p of any $g \in \mathcal{V}$ is real of multiplicity 1, because otherwise small stretching (or compressing) on the tangent spaces along the g-orbit of p will be allowed to reach a contradiction to Lemma 3.1. More precisely, this means if we define

$$T_j : Dg^j(T_pM) \to Dg^j(T_pM)$$

to be

$$T_j = (1 + t\delta)id$$

(or $(1 - t\delta)id$), where $t \in [0, 1]$, then there is $g_t \in \mathcal{U}_0$ that keeps the orbit of p unchanged such that Dg_t at the point $g_t^j(p)$ is just $T_j \circ Dg_{g^j(p)}$. With a suitable choice of t, g_t would have a non-real eigenvalue of absolute value 1, or a multiple root of real eigenvalue of absolute value 1, contradicting (3) or (2) of Lemma 3.1. This proves that any δ-neutral eigenvalue λ_p of any periodic point p of any $g \in \mathcal{V}$ is real of multiplicity 1. Second, we prove any periodic point p of any $g \in \mathcal{V}$ can have for $Dg^{\pi(p)}$ at most one δ-neutral eigenvalue λ_p. Suppose a periodic point p of a $g \in \mathcal{V}$ has two different real δ-neutral eigenvalues λ_1 and λ_2 with $|\lambda_1| < |\lambda_2|$. Let V_1 and V_2 be the two 1-dimensional eigenspaces associated with λ_1 and λ_2. We claim the angle between V_1 and V_2 (as well as their iterates under Dg) is bounded below by the constant $\gamma > 0$ given in Proposition 2.1. In fact, if $|\lambda_1| < 1 < |\lambda_2|$, then V_1 belongs to the stable subspace and V_2 belongs to the unstable subspace, and Proposition 2.1 applies. On the other hand, if $|\lambda_2| \leq 1$ (or if $|\lambda_1| \geq 1$), a $(1+\delta)$-stretching (or a $(1-\delta)$-compressing) on the whole tangent space will keep the angle unchanged, and make V_1 stable but V_2 unstable, for a perturbation in $\mathcal{U}_0$. Thus Proposition 2.1 applies anyway. This proves that the angle between V_1 and V_2 is bounded below by $\gamma > 0$. Then a simultaneous $(1+\delta)$-stretching on V_1 and $(1-\delta)$-compressing on V_2 (this is the real perturbation, while the stretching or compressing just mentioned is merely a testing one to get the conclusion about the angle) will be allowed. This would give rise to a $g' \in \mathcal{U}_0$ with two independent eigenvectors of real eigenvalue of absolute value 1, contradicting (1) of Lemma 3.1. This proves (1).

Next let p be a periodic point of $g \in \mathcal{V}$ that has no δ-neutral eigenvalue for Dg^{τ}. It is then hyperbolic. With $(1+\delta)$-stretching on $T_{g^i p}M$ along the orbit we get a $g_1 \in \mathcal{U}_0$ that keeps the orbit of p unchanged such that $E^s(g^i p)$ is still contracting with respect to g_1. By Lemma 3.2,

$$(1+\delta)^n \|Dg^{\pi(p)}|_{E^s_p}\| = \|Dg_1^{\pi(p)}|_{E^s_p}\| \le K.$$

Hence

$$\|Dg^{\pi(p)}|_{E^s_p}\| \le 1/(1+\delta)^{\pi(p)} K = \lambda^{\pi(p)} K.$$

The other inequality is proved similarly. This proves (2).

Now we prove (3). Let p be a periodic point of $g \in \mathcal{V}$ that has a δ-neutral eigenvalue λ_p for $Dg^{\pi(p)}$. By (1), T_pM splits into a $Dg^{\pi(p)}$-invariant splitting $G^s(p) \oplus G^c(p) \oplus G^u(p)$, where $G^c(p)$ is the 1-dimensional eigenspace $Dg^{\pi(p)}$ associated with λ_p, and $G^s(p)$ and $G^u(p)$ are generated by the rest of the stable and unstable eigenvalues of $Dg^{\pi(p)}$, respectively. Thus the orbit P of p is partially hyperbolic with 1-dimensional central direction. Since $G^s(p)$ contains no δ-neutral eigenvalue, the same proof by $(1+\delta)$-stretching on $T_{g^i p}M$ shows that

$$\|Dg^{\pi(p)}|_{G^s_p}\| \le 1/(1+\delta)^{\pi(p)} K = \lambda^{\pi(p)} K.$$

The other inequality for $G^u(p)$ is proved the same way. This proves (3), hence Lemma 3.3.

Note that, in Lemma 3.3, while the two constants K and λ are independent of the periodic point p, the inequalities in item (2) are only known to hold at the $\pi(p)$-th iterate, where $\pi(p)$ depends on the periodic point p. Thus this does not necessarily imply (uniform) hyperbolicity on the set of all hyperbolic periodic points. A similar remark holds for item (3).

Lemma 3.3 allows the following improvement.

Lemma 3.4. *Assume f can not be C^1 approximated by systems with homoclinic tangencies. Then there is a C^1 neighborhood $\mathcal{V}$ of f, together with three numbers $K \ge 2$, $0 < \lambda < 1$ and $\delta > 0$ and a positive integer ι such that the following conditions hold.*

(1) *Any periodic point p of any $g \in \mathcal{V}$ can have for $Dg^{\pi(p)}$ at most one δ-neutral eigenvalue λ_p which, if exists, must be real and of multiplicity 1.*

(2) *If a periodic point p of $g \in \mathcal{V}$ has no δ-neutral eigenvalue for $Dg^{\pi(p)}$, then (it is hyperbolic and)*

$$\prod_{j=0}^{[\pi(p)/\iota]-1} \|Dg^{\iota}|_{E^s(g^{j\iota}p)}\| \le K\lambda^{[\pi(p)/\iota]},$$

and

$$\prod_{j=0}^{[\pi(p)/\iota]-1} \|Dg^{-\iota}|_{E^u(g^{-j\iota}p)}\| \le K\lambda^{[\pi(p)/\iota]}.$$

(3) *If a periodic point p of $g \in \mathcal{V}$ has a δ-neutral eigenvalue λ_p for $Dg^{\pi(p)}$, then T_pM splits into a $Dg^{\pi(p)}$-invariant partially hyperbolic splitting $G^s(p) \oplus G^c(p) \oplus G^u(p)$, where $G^c(p)$ is the 1-dimensional eigenspace associated with λ_p, such that*

$$\prod_{j=0}^{[\pi(p)/\iota]-1} \|Dg^{\iota}|_{G^s(g^{j\iota}p)}\| \le K\lambda^{[\pi(p)/\iota]},$$

and

$$\prod_{j=0}^{[\pi(p)/\iota]-1} \|Dg^{-\iota}|_{G^u(g^{-j\iota}p)}\| \le K\lambda^{[\pi(p)/\iota]}.$$

In particular, this is true for any non-hyperbolic periodic orbit of $g \in \mathcal{V}$.

(4) *For any hyperbolic periodic point p of any $g \in \mathcal{V}$ of index i, $1 \le i \le d-1$ (saddle type),*

$$\|Dg^{\iota}|_{E^s(p)}\| \cdot \|Dg^{-\iota}|_{E^u(g^{\iota}(p))}\| \le \lambda.$$

Proof. Item (1) repeats that of Lemma 3.3 just for completeness. Items (2) and (3) are improvement of those of Lemma 3.3. Such an improvement (from the norm of product to the product of norms) has been standard in the work of Liao and Mañé, for instance see [M3, Page 528]. We omit the proof. Item (4) is just Proposition 2.1, with the domination constants (ι, λ) specified. Note that an (ι_1, λ_1)-dominated splitting is automatically an (ι_2, λ_2)-dominated splitting if

$\iota_1 \leq \iota_2$ and $\lambda_1 \leq \lambda_2$. Thus (ι, λ) can be chosen to work for items (1) through (4) simultaneously. This proves Lemma 3.4.

We add two more C^1 generic properties, Lemma 3.5 and 3.6. According to Liao [L2], a compact set Λ is called a *fundamental i-limit* of f, if there is a sequence of diffeomorphisms g_k that converges to f in the C^1 topology, together with a sequence P_k, where P_k is a periodic orbit of g_k of index i, such that $P_k \to \Lambda$ in the Hausdorff metric. Note that a fundamental i-limit Λ of f is f-invariant. When the integer i is not specified in question, we will simply speak of a *fundamental limit* of f.

Lemma 3.5. *There is a C^1 residual subset $\mathcal{A} \subset \mathrm{Diff}(M)$ such that for every $f \in \mathcal{A}$, any fundamental i-limit Λ of f is the Hausdorff limit of a sequence of hyperbolic periodic orbits of index i of f itself.*

Proof. Let C be the set of non-empty compact subsets of M, endowed with the Hausdorff metric. It is well known that C is a compact metric space. Take a countable basis $V_1, V_2, \ldots, V_n, \ldots$ of C. For each n, let $\mathcal{N}_n$ be the set of C^1 diffeomorphisms g such that g has a C^1 neighborhood $\mathcal{U}$ in $\mathrm{Diff}(M)$ such that every $g_1 \in \mathcal{U}$ does not have any hyperbolic periodic orbit of index i that is an element of V_n, and let $\mathcal{H}_n$ be the set of C^1 diffeomorphisms g such that g has a C^1 neighborhood $\mathcal{U}$ in $\mathrm{Diff}(M)$ such that every $g_1 \in \mathcal{U}$ has a hyperbolic periodic orbit of index i that is an element of V_n. By definition $\mathcal{N}_n \cup \mathcal{H}_n$ is C^1 open in $\mathrm{Diff}(M)$. It is also clear that $\mathcal{N}_n \cup \mathcal{H}_n$ is C^1 dense in $\mathrm{Diff}(M)$. In fact, if $g \notin \mathcal{N}_n$, then g can be C^1 approximated by g_1 that has a hyperbolic periodic orbit of index i that is an element of V_n. Since a hyperbolic periodic orbit of index i survives under C^1 perturbations, $g_1 \in \mathcal{H}_n$. This verifies that $\mathcal{N}_n \cup \mathcal{H}_n$ is C^1 dense in $\mathrm{Diff}(M)$. Let

$$\mathcal{A} = \bigcap_{n=1}^{\infty} (\mathcal{N}_n \cup \mathcal{H}_n).$$

Then $\mathcal{A}$ is C^1 residual in $\mathrm{Diff}(M)$. We prove that any $f \in \mathcal{A}$ satisfies Lemma 3.5.

Let f be any diffeomorphism in $\mathcal{A}$, and Λ be the Hausdorff limit of a sequence of hyperbolic periodic orbits (P_n, g_n) of index i, where $g_n \to f$ in the C^1 topology. Take any neighborhood W of Λ in C (here Λ is treated as an element of C). There is an integer k such that $\Lambda \in V_k \subset W$. Then $f \notin \mathcal{N}_k$. But $f \in \mathcal{A}$. Hence $f \in \mathcal{H}_k$, which means, in particular, that f itself has a hyperbolic periodic orbit of index i that is an element of V_k. This proves Lemma 3.5.

Lemma 3.6. *There is a C^1 residual subset $\mathcal{A} \subset \mathrm{Diff}(M)$ such that for every $f \in \mathcal{A}$, every transitive set Λ of f is the Hausdorff limit of a sequence of hyperbolic periodic orbits of f.*

Proof. Let Λ be a transitive set of any diffeomorphism f. By Lemma 3.5, it suffices to prove that Λ is a fundamental limit. We may assume there is $a \in \Lambda$ such that $\omega(a) = \Lambda$. By the C^1 closing lemma, for any C^1 neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}(M)$, any neighborhood U of Λ in M, and any neighborhood V of a in M, there is $g \in \mathcal{U}$ that has a periodic orbit $Q \subset U$ such that Q intersects V. Up to an arbitrarily small perturbation we may assume that Q is hyperbolic. Thus there is a sequence of diffeomorphisms g_n together with a sequence Q_n, where Q_n is a hyperbolic periodic orbit of g_n, such that g_n converges to f in the C^1 topology and Q_n converges to a nonempty compact subset $\Delta \subset \Lambda$ in the Hausdorff metric with $a \in \Delta$. By construction Δ is f-invariant, hence $\Delta = \Lambda$. This proves Lemma 3.6.

The heart of the proof for Theorem B is the following elegant selecting lemma of Liao [L2]. Recall from §2 that an invariant splitting $\Delta^s \oplus \Delta^u$ on a compact invariant set Λ is called (ι, λ)-*dominated of index* i, where $\iota \in IN$ and $0 < \lambda < 1$, if $\dim \Delta^s = i$, and if

$$\|Df^{\iota}|_{\Delta^s(x)}\| \cdot \|Df^{-\iota}|_{\Delta^u(f^{\iota}(x))}\| \leq \lambda$$

for all $x \in \Lambda$.

Proposition 3.7 (Liao). *Let Λ be a compact invariant set of f with (ι, λ)-dominated splitting $\Delta^s \oplus \Delta^u$ of index i, $1 \leq i \leq d-1$. Assume*

(1) *There is a point $b \in \Lambda$ satisfying*

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta^s(f^{j\iota}b)}\| \geq 1$$

for all $n \geq 1$.

(2) (The tilda condition) *There are λ_1 and λ_2 with $\lambda < \lambda_1 < \lambda_2 < 1$ such that for any $x \in \Lambda$ satisfying*

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta^s(f^{j\iota}x)}\| \geq \lambda_2^n$$

for all $n \geq 1$, $\omega(x)$ *contains a point* $c \in \Lambda$ *satisfying*

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta^s(f^{j\iota}c)}\| \leq \lambda_1^n$$

for all $n \geq 1$.

Then for any λ_3 *and* λ_4 *with* $\lambda_2 < \lambda_3 < \lambda_4 < 1$, *and any neighborhood* U *of* Λ, *there exists a hyperbolic periodic orbit* Q *of* f *of index* i *contained entirely in* U *with a point* $q \in Q$ *such that*

$$\prod_{j=0}^{m-1} \|Df^{\iota}|_{E^s(f^{j\iota}q)}\| \leq \lambda_4^m,$$

$$\prod_{j=m-1}^{[\pi(q)/\iota]-1} \|Df^{\iota}|_{E^s(f^{j\iota}q)}\| \geq \lambda_3^{[\pi(q)/\iota]-m+1},$$

for $m = 1, \cdots, [\pi(q)/\iota]$. *Here, as usual,* $\pi(q)$ *denotes the period of* q, *and* E^s *denotes the stable subbundle over* Q.

Similar assertions for Δ^u *hold respecting* f^{-1}.

Let us make some informal illustrations. We take the case of Δ^s. The selecting lemma concerns the (product of) norms of iterates of Df^{ι} restricted to Δ^s, a quantity that measures the contraction on Δ^s. We may briefly refer to it below as Δ^s-rate. Of course smaller rates give better contractions. Roughly, the selecting lemma says if there is a point $b \in \Lambda$ of very bad Δ^s-rate (≥ 1) and if things are turning back in the sense that every $x \in \Lambda$ of nearly bad Δ^s-rate ($\geq \lambda_2$) is approaching to a point $c \in \Lambda$ of a good Δ^s-rate ($\leq \lambda_1$), then there must be hyperbolic periodic orbits Q of f of index $\dim \Delta^s$ arbitrarily near Λ with E^s-rates between any λ_3 and λ_4 in $(\lambda_2, 1)$ (ignoring the tip). We will use the selecting lemma in two ways. One is to fix some λ_3 and λ_4 to get periodic orbits Q with E^s-rates of contraction not too weak (for instance in the proof of Lemma 3.8 below). The other way is to choose λ_3 and λ_4 arbitrarily close to 1, that is, to get periodic orbits Q with arbitrarily weak E^s-rates of contraction (for instance in the proof of Proposition 3.11 below).

The selecting lemma is the combination of the sifting lemma and a generalized shadowing lemma of Liao. The sifting lemma first appears in [L3] (in English) for the flow version, and then in [L2] for the discrete version. The generalized shadowing lemma can be found in [L4], also see [G]. The reference [L5] is the English translation of a Chinese book that collects some papers of Liao (including among others the papers [L1], [L2] and [L3]). The term “tilda” for condition (2) comes from the proof of the selecting lemma where condition (2) gives rise to a portion of a graph of shape tilda.

A direct consequence of the selecting lemma is about homoclinic classes.

Lemma 3.8. *Let Λ be a compact invariant set of f with (ι, λ)-dominated splitting $\Delta^s \oplus \Delta^u$ of index i, $1 \le i \le d-1$. If the two conditions in the selecting lemma are satisfied, then Λ intersects an i-homoclinic class $H(P)$ (for fome P) of f. Likewise for Δ^u respecting f^{-1}.*

Proof. We only prove for Δ^s. Assume for some $\lambda < \lambda_1 < \lambda_2 < 1$, the two conditions in the selecting lemma are satisfied. Fix any $\lambda_3 < \lambda_4$ in $(\lambda_2, 1)$. By the selecting lemma, for any $k \ge 1$, there is a hyperbolic periodic orbit Q_k of f of index i contained in the $1/k$-neighborhood of Λ with a point $q_k \in Q_k$ such that

$$\prod_{j=0}^{m-1} \|Df^{\iota}|_{E^s(f^{j\iota}(q_k))}\| \le \lambda_4^m,$$

$$\prod_{j=m-1}^{[\pi(q_k)/\iota]-1} \|Df^{\iota}|_{E^s(f^{j\iota}(q_k))}\| \ge \lambda_3^{[\pi(q_k)/\iota]-m+1},$$

for $m = 1, \cdots, [\pi(q_k)/\iota]$. Since ι and λ_4 are fixed, and since the first inequality holds for consecutive iterates from $m = 1$ to $m = [\pi(q_k)/\iota]$, it is easy to see the size of the local stable manifolds of q_k are uniformly (respecting k) bounded from below. On the other hand, since Δ^s is (ι, λ)-dominated by Δ^u, when k is large, $E^s(Q_k)$ will be (ι, λ_1)-dominated by $E^u(Q_k)$, where E^u is the expanding subbundle over Q_k. Hence the second inequality gives

$$\prod_{j=m-1}^{[\pi(q_k)/\iota]-1} \|Df^{-\iota}|_{E^u(f^{(j+1)\iota}(q_k))}\| \le (\lambda_1/\lambda_3)^{[\pi(q_k)/\iota]-m+1},$$

for $m = 1, \cdots, [\pi(q_k)/\iota]$. Then the size of the local unstable manifolds of q_k are also bounded from below. We may assume there are infinitely many distinct Q_k, because otherwise Λ will contain a hyperbolic periodic orbit of index i of f, which is a trivial i-homoclinic class anyway. Then there are infinitely many distinct q_k. By taking subsequence if necessary, we assume $q_k \to q \in \Lambda$. Then there is k_0 such that for all $k \ge k_0$, Q_k are mutually H-related. Let $H(P)$ be the i-homoclinic class that contains these Q_k. Then $H(P) \cap \Lambda \ne \emptyset$. Likewise for Δ^u respecting f^{-1}. This proves Lemma 3.8.

The following general fact gives a situation when a point of bad rate occurs.

Lemma 3.9. *Let Λ be a compact invariant set of f, and E be a Df-invariant subbundle of $T_\Lambda M$. If E is not contracting, then there is $b \in \Lambda$ such that $\|Df^n|_{E(b)}\| \geq 1$ for all $n \geq 1$. Likewise, if E is not expanding, then there is $b' \in \Lambda$ such that $\|Df^{-n}|_{E(b')}\| \geq 1$ for all $n \geq 1$.*

Proof. The proof is straightforward by compactness of Λ, hence omitted.

For the rest of the paper we will always assume that f can not be C^1 approximated by systems with homoclinic tangencies (to remind ourselves we will state this assumption each time). In particular, we fix the C^1 neighborhood $\mathcal{V}$ of f and the four constants

$$K \geq 2,\ \delta > 0,\ 0 < \lambda < 1,\ \iota \in IN$$

provided by Lemma 3.4. The next lemma ensures some points of good rate when we have a fundamental limit.

Lemma 3.10. *Assume f can not be C^1 approximated by systems with homoclinic tangencies. Let $\Lambda = \lim\,(P_k, g_k)$ be a fundamental i-limit of f, $1 \leq i \leq d-1$, and $\Delta^s \oplus \Delta^u$ be the (ι, λ)-dominated splitting of index i on Λ, guaranteed by item (4) of Lemma 3.4.*

(1) *If there is $p_k \in P_k$ for arbitrarily large k such that*

$$\prod_{j=0}^{[\pi(p_k)/\iota]-1} \|Dg_k^\iota|_{E^s(g_k^{j\iota}(p_k))}\| \leq K\lambda^{[\pi(p_k)/\iota]},$$

where E^s is the stable subbundle over P_k, then for any $\mu \in (\lambda, 1)$, there is $c \in \Lambda$ such that

$$\prod_{j=0}^{n-1} \|Df^\iota|_{\Delta^s(f^{j\iota}(c))}\| \leq \mu^n$$

for all $n \geq 1$.

(2) *If the assumption in item (1) is not satisfied, then Δ^s splits into a dominated splitting $V^s \oplus V^c$ on Λ with $\dim V^c = 1$ such that for any $\mu \in (\lambda, 1)$, there is $c' \in \Lambda$ such that*

$$\prod_{j=0}^{n-1} \|Df^\iota|_{V^s(f^{j\iota}(c'))}\| \leq \mu^n$$

for all $n \geq 1$.

Likewise for Δ^u respecting f^{-1}.

Proof. Note that $\Delta^s \oplus \Delta^u$ is just the limit of $E^s(P_k, g_k) \oplus E^u(P_k, g_k)$, as $k \to \infty$. We assume the periods $\pi(p_k)$ are unbounded, for otherwise the conclusion holds automatically. Note that each P_k is by definition hyperbolic.

For item (1), there is $p_k \in P_k$ for arbitrarily large k such that

$$\prod_{j=0}^{[\pi(p_k)/\iota]-1} \|Dg_k^\iota|_{E^s(g_k^{j\iota}(p_k))}\| \le K\lambda^{[\pi(p_k)/\iota]},$$

where E^s is the stable subbundle over P_k. Take any $\mu \in (\lambda, 1)$. We adopt the argument of Liao [L2]. For large integers q, let $i_{k,q}$ be the maximal non-negative integer such that

$$\prod_{j=0}^{i_{k,q}-1} \|Dg_k^\iota|_{E^s({g_k}^{j\iota}(p_k))}\| \ge (\lambda + 1/q)^{i_{k,q}}.$$

Denote $p_{i_{k,q}} = (g_k^\iota)^{i_{k,q}}(p_k)$. Then

$$\prod_{j=i_{k,q}}^{[\pi(p_k)/\iota]-1} \|Dg_k^\iota|_{E^s({g_k}^{j\iota}(p_{i_{k,q}}))}\| \le (\lambda + 1/q)^{[\pi(p_k)/\iota]-i_{k,q}}$$

by the definition of $i_{k,q}$. Clearly

$$\lim_{k\to\infty}([\pi(p_k)/\iota] - i_{k,q}) = \infty$$

for fixed q. Letting $k \to \infty$, and taking subsequence if necessary, we assume $p_{i_{k,q}} \to c_q \in \Lambda$. Letting $q \to \infty$ and taking subsequence we may assume $c_q \to c \in \Lambda$. Then

$$\prod_{j=0}^{n-1} \|Df^\iota|_{\Delta^s(f^{j\iota}(c))}\| \le \mu^n$$

for all $n \ge 1$.

For item (2), the assumption of item (1) does not hold. By (2) of Lemma 3.4, p_k must have a (unique) δ-neutral eigenvalue (though P_k is hyperbolic) and, by (3) of Lemma 3.4, $E^s(p_k, g_k)$ contains the 1-dimensional δ-neutral direction $G^c(p_k, g_k)$. Being the stable subspace, $E^s(p_k, g_k)$ must contain the contracting part $G^s(p_k, g_k)$, but can not intersect the expanding part $G^u(p_k, g_k)$. Thus $E^s(p_k, g_k)$ equals exactly $G^s(p_k, g_k) \oplus G^c(p_k, g_k)$ in the partially hyperbolic splitting given in item (3) of Lemma 3.4. Taking limit again gives a three ways dominated splitting $V^s \oplus V^c \oplus \Delta^u$ on Λ, as well as a point $c' \in \Lambda$, such that

$$\prod_{j=0}^{n-1} \|Df^\iota|_{V^s(f^{j\iota}(c'))}\| \leq \mu^n$$

for all $n \geq 1$. This proves Lemma 3.10.

A compact f-invariant set Λ is called (ι, λ)-*sole-neutral dominated* of index i, $0 \leq i \leq d-1$, if there is a three ways continuous Df-invariant splitting $T_\Lambda M = \Delta^s \oplus \Delta^c \oplus \Delta^u$ on Λ with $\dim \Delta^s = i$ and $\dim \Delta^c = 1$ such that Δ^c is neither contracting nor expanding, and such that Δ^s is (ι, λ)-dominated by Δ^c, and Δ^c is (ι, λ)-dominated by Δ^u. If Λ is a periodic orbit, then Λ is sole-neutral dominated if and only if it is sole-neutral partially hyperbolic. Hence Δ^s and Δ^u are in this case contracting and expanding respectively, and hence there can be no more than one sole-neutral dominated splitting on the periodic orbit Λ. For a general compact invariant set Λ, being sole-neutral dominated is not equivalent to being sole-neutral partially hyperbolic and, generally, Λ may admit more than one sole-neutral dominated splittings. Nevertheless for fixed i, sole-neutral dominated splitting of index i is unique.

Proposition 3.11. *Assume f can be C^1 approximated neither by systems that exhibit a homoclinic tangency nor by systems that exhibit a heterodimensional cycle. Also, assume f satisfies the C^1 generic conditions stated in Propositions 2.3, 2.4 and Lemma 3.6. Then every non-simple type minimally non-hyperbolic set Λ of f is the Hausdorff limit of a sequence of non-hyperbolic periodic orbits (P_k, g_k), where $g_k \to f$. In particular, Λ has a sole-neutral (ι, λ)-dominated splitting $\Delta^s \oplus \Delta^c \oplus \Delta^u$ of some index $0 \leq i \leq d-1$, respecting f.*

Proof. Let Λ be a non-simple type minimally non-hyperbolic set of f. We prove for any C^1 neighborhood $\mathcal{W}$ of f in $\mathrm{Diff}(M)$ and any neighborhood W of Λ in M, there is $g \in \mathcal{W}$ with a non-hyperbolic periodic orbit P contained entirely in W.

Suppose for the contrary there is a C^1 neighborhood $\mathcal{W}$ of f in $\mathrm{Diff}(M)$ and a neighborhood W of Λ in M, such that every periodic orbit P of every $g \in \mathcal{W}$ contained entirely in W is hyperbolic. Let $\mathcal{U}_0$, $\mathcal{V}$, and the constants K, λ, δ and $\iota \in IN$ be determined as in Lemma 3.2 through 3.4 above. The only thing to notice is that we can, and we do, assume $\mathcal{U}_0 \subset \mathcal{W}$. Thus, in particular, with δ-stretching or depressing along any single periodic orbit P of $g \in \mathcal{V}$, one will never get out of $\mathcal{W}$. Then, any hyperbolic periodic point p of any $g \in \mathcal{V}$ whose orbit is entirely contained in W will have no δ-neutral eigenvalue. By item (2) of Lemma 3.4,

$$\prod_{j=0}^{m-1} \|Dg^{\iota}|_{E^s(g^{j\iota}(p))}\| \le K\lambda^m, \tag{*}$$

$$\prod_{j=0}^{m-1} \|Dg^{-\iota}|_{E^u(g^{-j\iota}(p))}\| \le K\lambda^m, \tag{**}$$

where $m = [\pi(p)/\iota]$. In particular, using the standard arguments of Pliss [Pl], there are only finitely many (hyperbolic) periodic sinks or sources of g that are contained entirely in W.

Claim 1. Λ is not minimal. Indeed, Λ contains a hyperbolic subset, say H_1 of index m_1.

We follow the argument of Liao in [L2]. Note that being a non-simple type minimally non-hyperbolic set, Λ is transitive. By Lemma 3.6, Λ is the Hausdorff limit of a sequence of hyperbolic periodic orbits P_k of f. Taking subsequence if necessary, we assume all P_k have the same index, say l. Note that $1 \le l \le d-1$ because, otherwise, either every P_k would be a hyperbolic periodic sink, or every P_k would be a hyperbolic periodic source. Since f has only finitely many periodic sinks (or sources) contained entirely in W, Λ itself would be a hyperbolic periodic sink or source, contradicting that Λ is non-hyperbolic. Thus $1 \le l \le d-1$. By item (4) of Lemma 3.4, Λ has (ι, λ)-dominated splitting $\Delta_l^s \oplus \Delta_l^u$ of index l, respecting f. (We will concern more than one dominated splittings, to avoid confusion we put on the indices. Note that here by definition $\dim \Delta_l^u = d - l$.) Since Λ is non-hyperbolic, either Δ_l^s is not contracting, or Δ_l^u is not expanding. We assume Δ_l^s is not contracting. The other case can be proved similarly. Then by lemma 3.9, there is $b \in \Lambda$ with

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta_l^s(f^{j\iota}(b))}\| \ge 1$$

for all $n \ge 1$. If the tilda condition for Δ_l^s is satisfied, by the selecting lemma, there will be a periodic orbit Q of f of index l contained entirely in W with E^s-rate arbitrarily mild, contradicting the inequality (*) in the previous paragraph. Thus the tilda condition for Δ_l^s is not satisfied. Hence, for any $\mu \in (\lambda, 1)$, which we fix in the proof of this proposition, there is $x_1 \in \Lambda$ such that $H_1 = \omega(x_1)$ contains no point $c \in \Lambda$ satisfying

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta_l^s(f^{j\iota}(c))}\| \le \mu^n$$

for all $n \geq 1$. On the other hand, since Λ is the Hausdorff limit of a sequence of hyperbolic periodic orbits P_k of f of index l, by the above inequality $(*)$ and item (1) of Lemma 3.10, there do exist such a point $c \in \Lambda$. Thus H_1 is a proper subset of Λ, hence hyperbolic, say of index m_1. This proves Claim 1.

Note that $1 \leq m_1 \leq d-1$ because, otherwise, Λ would reduce to a hyperbolic periodic sink (or source), contradicting that Λ is non-hyperbolic. Also note that Λ is non-trivial because, otherwise, Λ would reduce to a hyperbolic periodic saddle, contradicting that Λ is non-hyperbolic. By Proposition 2.4, Λ is contained in a homoclinic class $H(P)$ for a hyperbolic periodic orbit P of f of index m_1, $1 \leq m_1 \leq d-1$, hence contained in $\overline{P^{m_1}(f)}$. By item (4) of Lemma 3.4, there is an (ι, λ)-dominated splitting $\Delta^s_{m_1} \oplus \Delta^u_{m_1}$ of index m_1 on Λ. (That is, the hyperbolic splitting of index m_1 on H_1 spreads out to a dominated splitting of index m_1 on the whole Λ.)

We analyze this splitting similarly as in Claim 1, with some refinement. Since Λ is non-hyperbolic, either $\Delta^s_{m_1}$ is not contracting, or $\Delta^u_{m_1}$ is not expanding. We assume $\Delta^s_{m_1}$ is not contracting. The other case is proved similarly. Then by lemma 3.9 there is $b' \in \Lambda$ with

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta^s_{m_1}(f^{j\iota}(b'))}\| \geq 1$$

for all $n \geq 1$. If the tilda condition for $\Delta^s_{m_1}$ is satisfied, the same contradiction as discussed in Claim 1 for Δ^s_l would occur. Therefore the tilda condition for $\Delta^s_{m_1}$ is not satisfied. Then there is $x_2 \in \Lambda$ such that $H_2 = \omega(x_2)$ contains no point $c' \in \Lambda$ satisfying

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta^s_{m_1}(f^{j\iota}(c'))}\| \leq \mu^n$$

for all $n \geq 1$. On the other hand, since H_1 is a hyperbolic set of f of index m_1, there is always a sequence of hyperbolic periodic orbits Q_k of f of index m_1 that converges to a compact invariant set $\Gamma_1 \subset H_1$ in the Hausdorff metric (For instance, taking Γ_1 to be any minimal set in H_1 and using the usual shadowing lemma). This gives by Lemma 3.4 an (ι, λ)-dominated splitting $\Delta^s_{m_1} \oplus \Delta^u_{m_1}$ of index m_1 on Γ_1. (Of course it coincides with the m_1-hyperbolic splitting of H_1 restricted to Γ_1, of which the hyperbolic rates are however unknown. We will hence use Lemma 3.10 as follows.) By the above inequality $(*)$ and item (1) of Lemma 3.10, there is $c' \in \Gamma_1 \subset \Lambda$ with

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta^s_{m_1}(f^{j\iota}(c'))}\| \leq \mu^n$$

for all $n \geq 1$. Thus $H_2 = \omega(x_2)$ is a proper subset of Λ, hence hyperbolic of f, say of index m_2.

Claim 2. $m_2 < m_1$.

Since H_2 is a hyperbolic set of f of index m_2, there is always a sequence of hyperbolic periodic orbits Q'_k of f of index m_2 that converges to a compact invariant set $\Gamma_2 \subset H_2$ in the Hausdorff metric. This gives an (ι, λ)-dominated splitting $\Delta^s_{m_2} \oplus \Delta^u_{m_2}$ of index m_2 on Γ_2. By the above inequality $(*)$ and item (1) of Lemma 3.10, there is $z \in \Gamma_2 \subset \Lambda$ with

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta^s_{m_2}(f^{j\iota}(z))}\| \leq \mu^n$$

for all $n \geq 1$. Note that the splitting $\Delta^s_{m_1} \oplus \Delta^u_{m_1}$ is on the whole Λ. If $m_2 \geq m_1$, then $\Delta^s_{m_1}(z) \subset \Delta^s_{m_2}(z)$ by Lemma 2.6, hence

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta^s_{m_1}(f^{j\iota}(z))}\| \leq \mu^n$$

for all $n \geq 1$, contradicting that H_2 does not contain any such point z. Thus $m_2 < m_1$, proving Claim 2.

Thus Λ contains two hyperbolic subsets H_1 and H_2 of different indices m_1 and m_2. By Proposition 2.3, a heterodimensional cycle can be created, a contradiction. This proves that for any C^1 neighborhood $\mathcal{W}$ of f in $\mathrm{Diff}(M)$ and any neighborhood W of Λ in M, there is $g \in \mathcal{W}$ with a non-hyperbolic periodic orbit P contained entirely in W.

Then there is a sequence of non-hyperbolic periodic orbits (R_k, h_k) that converge in the Hausdorff metric to a compact f-invariant set $\Gamma \subset \Lambda$ as $h_k \to f$. Clearly Γ will be non-hyperbolic of f. Since Λ is minimally non-hyperbolic, $\Gamma = \Lambda$. By (3) of Lemma 3.4, every R_k is (ι, λ)-sole-neutral partially hyperbolic of h_k for large k. Taking subsequences if necessary we assume every R_k is (ι, λ)-sole-neutral partially hyperbolic of index i, for some $0 \leq i \leq d-1$. This gives an (ι, λ)-sole-neutral dominated splitting $\Delta^s \oplus \Delta^c \oplus \Delta^u$ on Λ of index i, and proves Proposition 3.11.

In the proof of Proposition 3.11, we did not really verify the tilda condition of the selecting lemma. We treated the tilda condition just as one logical case. We also considered the opposite case when the tilda condition is not satisfied. The next lemma concerns the second case without assuming simultaneously the two inequalities $(*)$ and $(**)$ in the proof of Proposition 3.11. We state the result for minimally non-hyperbolic sets only.

Lemma 3.12. *Assume f can not be C^1 approximated by systems with homoclinic tangencies. Let Λ be a minimally non-hyperbolic set of f with (ι, λ)-dominated splitting $\Delta^s \oplus \Delta^u$ of index i, $1 \le i \le d-1$. Assume*

(1) *There is a constant $\mu \in (\lambda, 1)$ and a point $c \in \Lambda$ satisfying*

$$\prod_{j=0}^{n-1} \|Df^\iota|_{\Delta^s(f^{j\iota}(c))}\| \le \mu^n,$$

for all $n \ge 1$.

(2) *The tilda condition of the selecting lemma is not satisfied.*

Then Λ contains a hyperbolic subset of index $\le i$. Likewise for Δ^u respecting f^{-1}.

Proof. Since the tilda condition is not satisfied, for the number $\mu \in (\lambda, 1)$ given in condition (1), there is $x \in \Lambda$ such that $H = \omega(x)$ does not contain any point $c \in \Lambda$ satisfying

$$\prod_{j=0}^{n-1} \|Df^\iota|_{\Delta^s(f^{j\iota}(c))}\| \le \mu^n$$

for all $n \ge 1$. Since Λ contains such a point by condition (1), H is a proper subset of Λ. Since Λ is minimally non-hyperbolic, H is hyperbolic, say of index m, $1 \le m \le d-1$. We prove $m \le i$. Note that, as in the proof of Proposition 3.11, there is a sequence of hyperbolic periodic orbits P_n of f of index m that converge in the Hausdorff metric to a compact f-invariant set $\Gamma \subset H$. This gives an m-dominated splitting $\Delta_m^s \oplus \Delta_m^u$ on Γ that coincides with the m-hyperbolic splitting on H. By Lemma 3.10, either there is $c' \in \Gamma$ such that

$$\prod_{j=0}^{n-1} \|Df^\iota|_{\Delta_m^s(f^{j\iota}(c'))}\| \le \mu^n$$

for all $n \ge 1$, or Δ_m^s splits into a dominated splitting $V^s \oplus V^c$ on Γ with $\dim V^c = 1$, and there is $c'' \in \Gamma$ such that

$$\prod_{j=0}^{n-1} \|Df^\iota|_{V^s(f^{j\iota}(c''))}\| \le \mu^n$$

for all $n \geq 1$. If $m > i$, then $m - 1 \geq i$. By Lemma 2.6, in the first case we will have $\Delta^s_m(c') \supset \Delta^s(c')$, and in the second case we will have $V^s(c'') \supset \Delta^s(c'')$. In both cases Δ^s would have contraction rate no greater than μ at some point of $\Gamma \subset H$, contradicting that H does not contain any such point. This proves $m \leq i$. Likewise for Δ^u respecting f^{-1}. This proves Lemma 3.12.

Proposition 3.13. *Assume f can be C^1 approximated neither by systems that exhibit a homoclinic tangency nor by systems that exhibit a heterodimensional cycle. Also, assume f satisfies the C^1 generic conditions stated in Propositions* 2.2 *and* 2.3*. Let Λ be a non-simple type minimally non-hyperbolic set f with a three ways (ι, λ)-dominated splitting $\Delta^s \oplus \Delta^c \oplus \Delta^u$ such that each of the three summands is nontrivial and that Δ^c is neither contracting nor expanding. Then either Δ^s is contracting, or Δ^u is expanding.*

We remark that by a three ways (ι, λ)-dominated splitting $A \oplus B \oplus C$ we mean A is (ι, λ)-dominated by B, and B is (ι, λ)-dominated by C. Also note that in this proposition Δ^c is not necessarily 1-dimensional.

Proof. Denote $\dim \Delta^s = i$. Note that $1 \leq i \leq d - 2$, since each of the three subbundles is non-trivial. Suppose for the contrary neither Δ^s is contracting, nor Δ^u is expanding.

Since Δ^s is not contracting, by Lemma 3.9, there is $y_1 \in \Lambda$ such that

$$\prod_{j=0}^{n-1} \|Df^\iota|_{\Delta^s(f^{j\iota}(y_1))}\| \geq 1$$

for all $n \geq 1$. On the other hand, since Δ^c is not expanding, by Lemma 3.9, there is $y_2 \in \Lambda$ such that

$$\prod_{j=0}^{n-1} \|Df^{-\iota}|_{\Delta^c(f^{-j\iota}(y_2))}\| \geq 1$$

for all $n \geq 1$. That is,

$$\prod_{j=0}^{n-1} m(Df^\iota|_{\Delta^c(f^{(n-j)\iota}(y_2))}) \leq 1$$

for all $n \geq 1$. Taking a limit point of $\{f^{n\iota}(y_2)\}$ gives a point $y_3 \in \Lambda$ such that

$$\prod_{j=0}^{n-1} m(Df^\iota|_{\Delta^c(f^{-j\iota}(y_3))}) \leq 1$$

for all $n \geq 1$. Since Δ^s is (ι, λ)-dominated by Δ^c,

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta^s(f^{j\iota}(y_3))}\| \leq \lambda^n$$

for all $n \geq 1$. Take $\mu \in (\lambda, 1)$. Then condition (1) of the selecting Lemma and condition (1) of Lemma 3.12 both hold for (Δ^s, f), respecting the dominated splitting $\Delta^s \oplus (\Delta^c \oplus \Delta^u)$. Likewise, these two conditions both hold for (Δ^u, f^{-1}), respecting the dominated splitting $(\Delta^s \oplus \Delta^c) \oplus \Delta^u$. There are three cases to consider, each of which will lead to a contradiction.

Case 1. The tilda condition for (Δ^s, f) and (Δ^u, f^{-1}) are both satisfied.

In this case, by Lemma 3.8, Λ intersects an i-homoclinic class Γ_1 which, by definition, contains hyperbolic periodic orbits of index i. Likewise, by Lemma 3.8, Λ intersects an $(i + i_c)$-homoclinic class Γ_2 which, by definition, contains hyperbolic periodic orbits of index $i + i_c$, where $i_c = \dim(\Delta^c) \geq 1$. Since all the three sets Λ, Γ_1 and Γ_2 are transitive and the first intersects both of the other two, they are contained by Proposition 2.2 in the same weakly transitive component C. By Proposition 2.3, a heterodimensional cycle can be created via C^1 perturbation, a contradiction.

Case 2. The tilda condition for (Δ^s, f) and (Δ^u, f^{-1}) are both not satisfied.

In this case, by Lemma 3.12, Λ contains two hyperbolic subsets of different indices (one is of index $\leq i$, and the other one is of index $\geq i + i_c$). By Proposition 2.3, a heterodimensional cycle can be created via C^1 perturbation, a contradiction.

Case 3. The tilda condition is satisfied for either (Δ^s, f) or (Δ^u, f^{-1}), but not both.

We may assume the tilda condition is satisfied for (Δ^s, f), but not for (Δ^u, f^{-1}). In this case, as discussed in case 1 and 2, Λ intersects an i-homoclinic class Γ which, by definition, contains hyperbolic periodic orbits of index i, and Λ contains a hyperbolic set $H \subset \Lambda$ of index $\geq i + i_c$. Since Λ and Γ are both transitive, they are contained by Proposition 2.2 in the same weakly transitive component C. By Proposition 2.3, a heterodimensional cycle can be created via C^1 perturbation, a contradiction. This proves Proposition 3.13.

Theorem B. *There is a C^1 residual subset $\mathcal{R}$ in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle, such that any non-simple type minimally non-hyperbolic set Λ of any $f \in \mathcal{R}$ has the following feature:*

(1) *Λ is the common Hausdorff limit of two sequences of hyperbolic periodic orbits of different indices. More precisely, there are two sequences of hyperbolic periodic orbits $\{P_k\}$ and $\{Q_k\}$ of f that both converge to Λ in the Hausdorff metric, such that* $\operatorname{ind}(Q_k) = \operatorname{ind}(P_k) + 1$ *and $W^u(P)$ intersects $W^s(Q)$ transversely. In particular, Λ can not be contained in any normally hyperbolic arc or circle of f.*

(2) *Λ is partially hyperbolic with central bundle at most 2-dimensional. More precisely, either there is a three-ways Df-invariant splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$, where E^s is dominated by E^c, and E^c is dominated by E^u, such that E^s is contracting, E^u is expanding, and E^c is 1-dimensional and is neither contracting nor expanding, or, there is a four-ways Df-invariant splitting $T_\Lambda M = E^s \oplus E^{cs} \oplus E^{cu} \oplus E^u$, where E^s is dominated by E^{cs}, E^{cs} is dominated by E^{cu}, and E^{cu} is dominated by E^u, such that E^s is contracting, E^u is expanding, and E^{cs} and E^{cu} are each* 1*-dimensional and neither contracting nor expanding.*

Proof. Let $\mathcal{R}$ be the set of diffeomorphisms in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle that are Kupka-Smale and satisfy the C^1 generic conditions stated in Propositions 2.2 through 2.4, as well as Lemmas 3.5 and 3.6. Let $f \in \mathcal{R}$. We prove f satisfies Theorem B. By Proposition 3.11, Λ is the Hausdorff limit of a sequence of non-hyperbolic periodic orbits R_k of g_k, where $g_k \to f$ in the C^1 topology. By item (3) of Lemma 3.3, R_k has exactly one eigenvalue of absolute value 1, for k large. It is then easy to see through a saddle-node bifurcation that Λ is the Hausdorff limit of two sequences of hyperbolic periodic orbits A_k and B_k of g'_k, where $g'_k \to f$ in the C^1 topology, such that $\operatorname{ind}(B_k) = \operatorname{ind}(A_k) + 1$, and $W^u(A_k, g'_k)$ intersects $W^s(B_k, g'_k)$ transversely. By Lemma 3.5, there are two sequences of hyperbolic periodic orbits P_k and Q_k of f itself, both converging to Λ in the Hausdorff metric, such that $\operatorname{ind}(Q_k) = \operatorname{ind}(P_k) + 1$. In fact, a slight refinement of the proof of lemma 3.5 shows that, in this case, $W^u(P, f)$ intersects $W^s(Q, f)$ transversely.

It is easy to see Λ can not be contained in any normally hyperbolic arc or circle of f. We take the case of a circle. Suppose Λ is contained in a normally hyperbolic circle C of f. There is $k_0 \geq 1$ such that $P_k, Q_k \subset C$ for all $k \geq k_0$. By the well known properties of circle diffeomorphisms, all periodic orbits of $f|_C$ have the same period. Being the Hausdorff limit of P_k (of the same period), Λ will be a periodic orbit. Since f is Kupka-Smale, Λ is hyperbolic, contradicting that Λ is a (minimally) non-hyperbolic set.

To prove (2), note that by Proposition 3.11 Λ has at least one sole-neutral (ι, λ)-dominated splitting. We prove Λ has at most two different sole-neutral (ι, λ)-

dominated splittings. Suppose for the contrary Λ has three different sole-neutral (ι, λ)-dominated splittings of indices $0 \le i_1 < i < i_2 \le d-1$. For explicity we put on the index by writing the second sole-neutral (ι, λ)-dominated splitting on Λ as $\Delta_i^s \oplus \Delta_i^c \oplus \Delta_i^u$, where $\dim \Delta_i^s = i$, $\dim \Delta_i^c = 1$ and $\dim \Delta_i^u = d - i - 1$. Note that Δ_i^c is neither contracting nor expanding, and that Δ_i^s and Δ_i^u are both nontrivial. By Proposition 3.13, either Δ_i^s is contracting, or Δ_i^u is expanding. But this is impossible because Δ_i^s contains $\Delta_{i_1}^c$ hence is not contracting, and Δ_i^u contains $\Delta_{i_2}^c$ hence is not expanding. This proves that Λ has at most two different sole-neutral (ι, λ)-dominated splittings.

Case 1. Λ has exactly two different sole-neutral (ι, λ)-dominated splittings, say $\Delta_i^s \oplus \Delta_i^c \oplus \Delta_i^u$ and $\Delta_j^s \oplus \Delta_j^c \oplus \Delta_j^u$ with $0 \le i < j \le d-1$.

This gives a five ways (ι, λ)-dominated splitting

$$\Delta_i^s \oplus \Delta_i^c \oplus (\Delta_i^u \cap \Delta_j^s) \oplus \Delta_j^c \oplus \Delta_j^u$$

such that Δ_i^c and Δ_j^c are each neither contracting nor expanding, and $\dim(\Delta_i^c) = 1, \dim(\Delta_j^c) = 1$. It is easy to see $\Delta_i^u \cap \Delta_j^s = \{0\}$. In fact, if $\Delta_i^u \cap \Delta_j^s$ is non-trivial, it would be neither contracting nor expanding because it dominates Δ_i^c and is dominated by Δ_j^c. Treating $\Delta_i^s \oplus \Delta_i^c$, $\Delta_i^u \cap \Delta_j^s$ and $\Delta_j^c \oplus \Delta_j^u$ as Δ^s, Δ^c and Δ^u in Proposition 3.13, respectively, this would contradict Proposition 3.13. Thus $\Delta_i^u \cap \Delta_j^s = \{0\}$. That is, it is actually a four ways dominated splitting

$$\Delta_i^s \oplus \Delta_i^c \oplus \Delta_j^c \oplus \Delta_j^u,$$

and actually $j = i + 1$.

It is easy to see that Δ_i^s is contracting and Δ_j^u is expanding because, suppose for instance Δ_i^s is not contracting (hence non-trivial), then treating Δ_i^s as Δ^s, Δ_i^c as Δ^s, and $\Delta_j^c \oplus \Delta_j^u$ as Δ^u would contradict Proposition 3.13. Rewriting Δ_i^s as E^s, Δ_i^c as Δ^{cs}, Δ_j^c as Δ^{cu}, and Δ_j^u as E^u then gives the required splitting

$$E^s \oplus \Delta^{cs} \oplus \Delta^{cu} \oplus E^u,$$

hence completes the proof in Case 1.

Case 2. Λ has exactly one sole-neutral dominated splitting $\Delta_i^s \oplus \Delta_i^c \oplus \Delta_i^u$.

It remains to prove that Δ_i^s is contracting and Δ_i^u is expanding. By Lemma 3.12, either Δ_i^s is contracting, or Δ_i^u is expanding. We may assume Δ_i^u is expanding and proceed to prove Δ_i^s is contracting.

Suppose Δ_i^s is not contracting of f. By Lemma 3.9, there is $y_1 \in \Lambda$ such that

$$\prod_{j=0}^{n-1} \|Df^\iota|_{\Delta_i^s(f^{j\iota}(y_1))}\| \ge 1$$

for all $n \geq 1$. Since Δ_i^c is not expanding, there is $y_2 \in \Lambda$ such that

$$\prod_{j=0}^{n-1} \|Df^{-\iota}|_{\Delta_i^c(f^{-j\iota}(y_2))}\| \geq 1$$

for all $n \geq 1$. As shown in the proof of Proposition 3.13, this gives a point $y_3 \in \Lambda$ such that

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta_i^s(f^{j\iota}(y_3))}\| \leq \lambda^n$$

for all $n \geq 1$. Fix $\mu \in (\lambda, 1)$. Then condition (1) of the selecting lemma and condition (1) of Lemma 3.12 both hold for Δ_i^s, respecting the dominated splitting $\Delta_i^s \oplus (\Delta_i^c \oplus \Delta_i^u)$. There are two subcases. We prove that each leads to a contradiction.

Subcase 2.1. The tilda condition for Δ_i^s is satisfied.

By the selecting lemma, there is a periodic orbit Q of f of index i contained in an arbitrarily small neighborhood of Λ with E^s-rate arbitrarily mild. By item (2) of Lemma 3.4, $E^s(Q)$ contains a δ-neutral eigenvalue. Since $\delta \to 0$ when the size of perturbation $\to 0$, taking limit gives a compact invariant set $\Gamma \subset \Lambda$ with a 1-dimensional neutral subbundle $\Delta^c \subset \Delta_i^s$ on Γ. Hence Γ is non-hyperbolic and hence coincides with Λ. Thus Λ would have two different neutral directions, contradicting the hypothesis of Case 2.

Subcase 2.2. The tilda condition for Δ_i^s is not satisfied.

Take a sequence $\lambda_k \in (\lambda, 1)$ with $\lambda_k \to 1$. Since the tilda condition for Δ_i^s is not satisfied, there is a sequence $x_k \in \Lambda$ such that $H_k = \omega(x_k)$ contains no point $c \in \Lambda$ with

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta_i^s(f^{j\iota}(c))}\| \leq \lambda_k^n$$

for all $n \geq 1$. That is, for any $x \in H_k$,

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta_i^s(f^{j\iota}(x))}\| > \lambda_k^n \tag{\#}$$

for all $n \geq 1$. Due to the existence of the point $y_3 \in \Lambda$, H_k is a proper subset of Λ, hence hyperbolic, say of index m_k. By (the proof of) Lemma 3.12, $m_k \leq i$

for all k. We claim that $m_k = i$ for all k. To this end we look at the same splitting $\Delta_i^s \oplus (\Delta_i^c \oplus \Delta_i^u)$, but respecting f^{-1}. Note that the splitting $\Delta_i^s \oplus (\Delta_i^c \oplus \Delta_i^u)$ is (ι, λ)-dominated for f^{-1} as well. We hence argue in the same way as we just did for Δ_i^s respecting f before subcase 2.1: Since the subbundle $(\Delta_i^c \oplus \Delta_i^u)$ is clearly not contracting respecting f^{-1}, by Lemma 3.9, there is $z_1 \in \Lambda$ such that

$$\prod_{j=0}^{n-1} \|Df^{-\iota}|_{(\Delta_i^c \oplus \Delta_i^u)(f^{-j\iota}(z_1))}\| \geq 1$$

for all $n \geq 1$. Since Δ_i^s is assumed not contracting respecting f, there is $z_2 \in \Lambda$ such that

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta_i^s(f^{j\iota}(z_2))}\| \geq 1$$

for all $n \geq 1$. As shown in the proof of Proposition 3.13, this gives a point $z_3 \in \Lambda$ such that

$$\prod_{j=0}^{n-1} \|Df^{-\iota}|_{(\Delta_i^c \oplus \Delta_i^u)(f^{-j\iota}(z_3))}\| \leq \lambda^n$$

for all $n \geq 1$. Thus condition (1) of the selecting lemma and condition (1) of Lemma 3.12 both hold for the subbundle $(\Delta_i^c \oplus \Delta_i^u)$, respecting the dominated splitting $\Delta_i^s \oplus (\Delta_i^c \oplus \Delta_i^u)$ for f^{-1}. If the tilda condition for $(\Delta_i^c \oplus \Delta_i^u)$ respecting f^{-1} is satisfied, then Λ intersects an i-homoclinic class (here the index i is respecting f). If the tilda condition for $(\Delta_i^c \oplus \Delta_i^u)$ respecting f^{-1} is not satisfied, then Λ contains a hyperbolic set with index $\geq i$ (here the index is respecting f). In either case, if $m_k < i$ for some k, a heterodimensional cycle would be created, giving a contradiction. This proves the claim that $m_k = i$ for all k.

Now we adopt an argument of Mañé [M3], with a slight modification (we will be using the shadowing lemma, rather than the ergodic closing lemma). Let μ_k be a limit point of the sequence of measures

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x_k)}.$$

Then μ_k is f-invariant. The above inequality (#) for the Δ_i^s-rates of points of H_k then gives

$$\int_{H_k} \phi(x) d\mu_k \geq \lambda_k,$$

where

$$\phi(x) = \log \|Df^{\iota}|_{\Delta_i^s(x)}\|.$$

Since the set $\mathrm{Rec}(f)$ of recurrent points of f has the full measure,

$$\int_{H_k \cap \mathrm{Rec}(f)} \phi(x) d\mu_k \geq \lambda_k.$$

By Birkhoff theorem,

$$\int_{H_k \cap \mathrm{Rec}(f)} \overline{\phi}(x) d\mu_k \geq \lambda_k,$$

where

$$\overline{\phi}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)).$$

Thus there is $z_k \in H_k \cap \mathrm{Rec}(f)$ such that

$$\overline{\phi}(z) \geq \lambda_k.$$

That is,

$$\prod_{j=0}^{n-1} \|Df^{\iota}|_{\Delta_i^s(f^{j\iota}(z_k))}\| \geq ((1-\delta)\lambda_k)^n$$

for all $n \geq 1$ large, where $\delta > 0$ can be arbitrarily small. Since $z_k \in H_k$ is recurrent, for any $\varepsilon > 0$, there is $n_k \geq 1$ with the distance $\mathrm{d}(z_k, f^{n_k}(z_k))$ small enough so that the periodic pseudo-orbit $z_k, f(z_k), \ldots, f^{n_k}(z_k), z_k, f(z_k), \ldots$ fits the ε-shadowing condition of the hyperbolic set H_k. By the shadowing lemma, there is a periodic orbit Q_k of f of index i, ε-close to H_k, with E^s-rate close to 1. Then similar argument as in Subcase 2.1 shows Λ would have two different neutral directions, contradicting the hypothesis of Case 2. This completes the proof for Case 2 and proves Theorem B.

References

[Ab1] Flavio Abdenur, Generic robustness of spectral decompositions, to appear in *Annales Scientifiques de l'Ecole Normale Superieure*.

[Ab2] Flavio Abdenur, Attractors of generic diffeomorphisms are persistent, Preprint.

[Ar] M-C Arnaud, Création de connexions en topologie C^1, *Ergodic Theory and Dynamical Systems,* **31** (2001), 339-381.

[AS] R. Abraham and S. Smale, Nongenericity of Ω-stability, *Proc. Symp. in Pure Math., Amer. Math. Soc.*, **14** (1970), 5-8.

[BD] Ch. Bonatti and L. J. Díaz, Persistence of transitive diffeomorphisms, *Ann. Math.*, **143** (1996), 357-396.

[BV] Ch. Bonatti and M. Viana, SRB measures for partially hyperbolic attractors: the contracting case, *Israel J. Math.*, **115** (2000), 157-193.

[CMP] C. M. Carballo, C. A. Morales and M. J. Pacifico, Maximal transitive sets with singularities for generic C^1 vector fields. *Bol. Soc. Brasil. Mat.* **31** (2000), no. 3, 287–303.

[F] J. Franks, Necessary conditions for stability of diffeomorphisms, *Trans. Amer. Math. Soc.*, **158** (1977), 301-308.

[G] S. Gan, A generalized shadowing lemma. *Discrete and Continuous Dynamical Systems*, **8** (2002), 627-632.

[GW] S. Gan and L. Wen, Heteroclinic cycles and homoclinic closures for generic diffeomorphisms, *J. Dynam. Diff. Eq.*, **15** (2003), 451-471.

[L1] Shantao Liao, Obstruction sets I, *Acta Math. Sinica*, **23** (1980), 411-453.

[L2] Shantao Liao, Obstruction sets II, *Acta Sci. Natur. Univ. Pekinensis*, **2** (1981), 1-36.

[L3] Shantao Liao, On the stability conjecture, *Chinese Annals of Math.*, **1** (1980), 9-30. (in English)

[L4] Shantao Liao, An existence theorem for periodic orbits, *Acta Sci. Natur. Univ. Pekinensis*, **1** (1979), 1-20.

[L5] Shantao Liao, *Qualitative Theory of Differentiable Dynamical Systems*, China Science Press, (1996). (in English)

[M1] R. Mañé, Contributions to the stability conjecture, *Topology,* **17** (1978), 397-405.

[M2] R. Mañé, Quasi-Anosov diffeomorphisims and hyperbolic manifolds, *Trans. Amer. Math. Soc.*, **229** (1977), 351-370.

[M3] R. Mañé, An ergodic closing lemma, *Ann. Math.*, **116**(1982), 503-540.

[N1] S. Newhouse, Nondensity of Axiom A(a) on S^2, *Proc. Symp. in Pure Math., Amer. Math. Soc.* **14** (1970), 191-202.

[P] J. Palis, *A global view of dynamics and a conjecture of the denseness of finitude of attractors, Astérisque*, **261** (2000), 335-347.

[PT] J. Palis and F. Takens, *Hyperbilicity and sensitive chaotic dynamics at homoclinic bifurcations,* Cambridge University Press, 1993.

[Pl] V. Pliss, A hypothesis due to Smale, Diff. Eq. **8** (1972), 203-214.

[Pu] C. Pugh, The closing lemma, *Amer. J. Math*, **89** (1967), 956-1009.

[PR] C. Pugh and C. Robinson, The C^1 closing lemma, including Hamiltonians, *Ergod. Th. and Dynam. Sys.*, **3** (1983), 261-313.

[PS] E. Pujals and M. Sambarino, Homoclinic tangencies and hyperbolicity for surface diffeomorphisms, *Ann. Math.*, **151** (2000), 961-1023.

[Rs] C. Robinson, *Dynamical Systems, Stability, Symbolic Dynamics, and Chaos*, CRC Press, 1994.

[S] J. Selgrade, Isolated invariant sets for flows on vector bundles, *Trans. Amer. Math. Soc.*, **203** (1975), 359-390.

[Sh1] M. Shub, Topological transitive diffeomorphisms on T^4, *Lecture Notes in Math.*, **206** (1971), Springer-Verlag.

[Sh2] M. Shub, *Global Stability of Dynamical Systems*, Springer-Verlag, 1987.

[Si] C. P. Simon, A 3-dimensional Abraham-Smale example, *Proc. Amer. Math. Soc.*, **34** (1972), 629-630.

[Sm1] S. Smale, Structurally stable systems are not dense, *Amer. J. Math.,* **88** (1966), 491-496.

[Sm2] S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Sci.,* **73** (1967), 747-817.

[SS] R. Sacker and G. Sell, Existence of dichotomies and invariant splittings for linear differential systems, *J. Diff. Eq*, **22** (1976), 478-496.

[W1] L. Wen, On the preperiodic sets, *Discrete and Continuous Dynamical Systems,* **1** (2000), 237-241.

[W2] L. Wen, Homoclinic tangencies and dominated splittings, *Nonlinearity*, **15** (2002), 1445-1469.

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