

c -Graded filiform Lie algebras

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Abstract. In this paper we generalize naturally graded filiform Lie algebras as well as filiform Lie algebras admitting a connected gradation of maximal length, by introducing the concept of c -graded complex filiform Lie algebras. We deal with the particular case of 3-graded filiform Lie algebras and we obtain their classification in arbitrary dimension. We finally show a link among derived algebras, graded filiform and rigid solvable Lie algebras.

Keywords: characteristically nilpotent Lie algebras, graded filiform Lie algebras, rigid solvable Lie algebras.

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Introduction

Vergne introduced in [12] the concept of naturally graded filiform Lie algebras as those admitting a gradation associated with the lower central series. In that paper, she also classified them, up to isomorphism. Apart from that, several authors have studied algebras which admit a connected gradation of maximal length, this is, whose length is exactly the dimension of the algebra. So, Y. Hakimjanov started this study in [8], Reyes, in this Ph. D. Thesis [11] (later published in [3]), continued this research by giving an induction classification method and finally, Millionschikov in [10] gave the full list of these algebras (over an arbitrary field of zero characteristic).

The main goal of this paper is to introduce and study a kind of graded filiform Lie algebras, the c -graded ones, which generalizes both concepts recalled above. So, 1-graded filiform Lie algebras are naturally graded filiform Lie algebras and 2-graded filiform Lie algebras are the algebras by Hakimjanov. Although general results related c -graded filiform Lie algebras are obtained, these are

particularized to establish the classification of 3-graded filiform Lie algebras in arbitrary dimension.

For several reasons, we think that this classification can suppose a step forward in getting the classification of Lie algebras in general. Indeed, by considering the usual gradation of the second space of Chevalley-Eilenberg's cohomology of the model filiform Lie algebra P_n , given by $H^2(P_n, P_n) = \bigoplus_{c \geq 0} H_{c+1}^2(P_n, P_n)$ and by taking into account that every filiform Lie algebra is isomorphic to $(P_n)_\psi$, where $\psi \in H^2(P_n, P_n)$ [12], it is possible to consider c -graded Lie algebras as algebras isomorphic to $(P_n)_{\psi_c}$, with $\psi_c \in H_{c+1}^2(P_n, P_n)$ (see [7]).

Therefore, the classification of c -graded Lie algebras allows, for one thing, to settle the classification of filiform Lie algebras of the type $(P_n)_{\psi_c + \psi_k}$ and, for another, to progress in the knowledge of the structure of $(P_n)_\psi$, because by considering the gradation, $\psi = \psi_c + \psi_{c+1} + \dots + \psi_k$ then ψ_c and ψ_k have to be cocycles such that $(P_n)_{\psi_c}$ and $(P_n)_{\psi_k}$ are c -graded Lie algebras (see Remark 4.1).

Note also that c -graded Lie algebras are related with sill algebras introduced by Goze and Hakimjanov in [6], although they do not give any gradation for them.

Finally, by virtue of a result seen in [1], we obtain that for fixed $c \geq 2$, there exists $m = m(c) \in \mathbb{N}$ such that every c -graded Lie algebra of dimension $n \geq m$ is a derived algebra of a rigid solvable Lie algebra of dimension $n + 1$.

1 Definitions and notations

In this paper, we will consider complex Lie algebras of finite dimension with laws denoted by $[\ , \]$.

In a Lie algebra $\mathcal{A}$, one can consider the lower central series: $C^1(\mathcal{A}) = \mathcal{A}$, $C^2(\mathcal{A}) = [\mathcal{A}, \mathcal{A}]$, $\dots$, $C^k(\mathcal{A}) = [C^{k-1}(\mathcal{A}), \mathcal{A}]$, $\dots$. It is said that $\mathcal{A}$ is filiform if $\dim C^k(\mathcal{A}) = n - k$, for $k \geq 2$, with $n = \dim \mathcal{A}$. Note that filiform Lie algebras are a subset of nilpotent Lie algebras.

As a consequence of Engel's Theorem, it is possible to obtain a basis $\{e_1, \dots, e_n\}$ of every filiform Lie algebra such that,

$$[e_1, e_n] = 0, \quad [e_1, e_h] = e_{h+1} \quad (h = 2, \dots, n-1), \quad [e_2, e_{n-1}] = 0.$$

Such a basis is called *adapted basis* and with respect to it, it is verified that $C^k(\mathcal{A}) = \langle e_{k+1}, \dots, e_n \rangle$, $2 \leq k \leq n-1$.

According to a result by M. Vergne ([12]) there are two filiform Lie algebras

$\mathcal{A} = P_n$ whatever n is, and $\mathcal{A} = Q_n$ for $n \geq 4$ even, defined, respectively, by:

$$P_n : [e_1, e_k] = e_{k+1} \quad 2 \leq k \leq n-1$$

$$\begin{aligned} Q_n : [e_1, e_k] &= e_{k+1} & 2 \leq k \leq n-1 \\ [e_k, e_{n+1-k}] &= (-1)^{k+1} e_n & 3 \leq k \leq \frac{n}{2} \\ [e_2, e_k] &= e_{k+1} & 3 \leq k \leq n-2 \end{aligned}$$

which are both isomorphic to the graded algebra obtained when considering the lower central series, this is the gradation defined with respect to an adapted basis $\{e_1, \dots, e_n\}$:

$$\begin{cases} \mathcal{H}_1 &= \langle e_1, e_2 \rangle \\ \mathcal{H}_k &= \langle e_{k+1} \rangle \quad 2 \leq k \leq n-1. \end{cases}$$

So, $\mathcal{A} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \oplus \mathcal{H}_{n-1}$.

A Lie algebra $\mathcal{A}$ is said to be derived if there exists a Lie algebra $\mathcal{L}$ such that $\mathcal{A} = C^2(\mathcal{L})$.

2 c-graded Lie algebras

Definition 2.1. A gradation $\{\mathcal{G}_i\}_{i \geq 1}$ of a Lie algebra $\mathcal{A}$ of dimension n is called a *c-gradation* if

$$\mathcal{A} = \mathcal{G}_1 \oplus 0 \oplus \dots \oplus 0 \oplus \mathcal{G}_c \oplus \dots \oplus \mathcal{G}_{n+c-2},$$

where there are n one-dimensional nonzero homogeneous spaces, if $c > 2$ or

$$\mathcal{A} = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \dots \oplus \mathcal{G}_{n+c-2},$$

if $c = 2$.

Definition 2.2. A filiform Lie algebra $\mathcal{A}$ is called *c-graded* for $c \geq 2$ if it admits a *c-gradation*.

Note that for $c = 2$, we obtain a connected gradation, this is, a gradation without zero homogeneous elements. However if $c > 2$, we obtain a disconnected gradation with $c - 2$ zero homogeneous elements in it.

Proposition 2.1. The filiform Lie algebra $\mathcal{A}$ of dimension n is *c-graded* if and only if there exists an adapted basis $B = \{e_1, \dots, e_n\}$ of $\mathcal{A}$ such that:

$$[e_h, e_k] = a_{h,k} e_{h+k+c-2}$$

where $a_{h,k} \in \mathbb{C}$, $h + k + c - 2 \leq n$, for all $e_h, e_k \in B$, with $1 < h, k < n$. In this case, the basis B is called a *c-graded basis* of $\mathcal{A}$.

Proof. Let consider the following c -gradation

$$\mathcal{A} = \mathcal{G}_1 \oplus 0 \oplus \cdots \oplus 0 \oplus \mathcal{G}_c \oplus \cdots \oplus \mathcal{G}_{n+c-2},$$

then we can take $z_1 \in \mathcal{G}_1$, $z_k \in \mathcal{G}_{k+c-2}$ for $2 \leq k \leq n$. As $\mathcal{A}$ is filiform, we have $[z_1, z_k] \neq 0$, for all k . So, by a suitable choice of $\alpha_l \in \mathbb{C}$, $1 \leq l \leq n$, we can consider $e_l = \alpha_l z_l$ and thus $\{e_1, \dots, e_n\}$ is a c -graded basis satisfying brackets above.

Conversely, let $\mathcal{A}$ be a filiform Lie algebra whose brackets with respect to an adapted basis $\{e_1, \dots, e_n\}$ are $[e_h, e_k] = a_{h,k} e_{h+k+c-2}$. Then we can consider the gradation in $\mathcal{A}$ defined by:

$$\begin{cases} \mathcal{G}_1 &= \langle e_1 \rangle \\ \mathcal{G}_k &= \langle e_{k-c+2} \rangle \quad c \leq k \leq n+c-2. \end{cases}$$

We have $\mathcal{A} = \mathcal{G}_1 \oplus 0 \oplus \cdots \oplus 0 \oplus \mathcal{G}_c \oplus \cdots \oplus \mathcal{G}_{n+c-2}$. It completes the proof. $\square$

Lemma 2.2. *If $\mathcal{A}$ is a c -graded Lie algebra of dimension n , then*

$$2 \leq c \leq n-3.$$

Proof. Since $\mathcal{A}$ is c -graded filiform, if $\{e_1, \dots, e_n\}$ is a c -graded basis, we have that:

$$[e_h, e_k] \in [\mathcal{A}, C^2(\mathcal{A})] = C^3(\mathcal{A})$$

for $1 < h, k \leq n$. So,

$$e_{h+k+c-2} \in \langle e_4, \dots, e_n \rangle.$$

Therefore, $h+k+c-2 \leq n$. As $1 \neq h \neq k \neq 1$ then c will be maximum when $h=2$, $k=3$. $\square$

Remark 2.1. *According to previous definitions, the Lie algebras P_n , one for each dimension, are c -graded for all c . This is immediate if we consider $a_{h,k} = 0$, $\forall h, k > 1$. From now on, these algebras will be called model Lie algebras.*

Moreover, by taking into account the following lemma, this condition characterizes the model algebras.

Lemma 2.3. *If a non-model Lie algebra is c_1 -graded and c_2 -graded, then $c_1 = c_2$.*

Proof. Let suppose $c_1 < c_2$. Let $B = \{e_1, \dots, e_n\}$ and $B' = \{e'_1, \dots, e'_n\}$ be c_1 -graded and c_2 -graded bases of $\mathcal{A}$ respectively. Since $\mathcal{A}$ is a non-model algebra, there exist $i, j \in \{2, \dots, n\}$, $i < j$ such that $[e_i, e_j] \neq 0$, and then $[e_i, e_j] = a_{i,j}e_{i+j+c_1-2}$.

By considering the adapted basis change between both base:

$$e_h = \sum_{k=1}^n C_{h,k}e'_k, \quad 1 \leq h \leq n,$$

we have by filiformity that $[e_1, e_h] = e_{h+1}$ for $2 \leq h \leq n-1$, therefore $C_{h,k} = 0$ for $k < h$, and $3 \leq h \leq n$. So, $e_n = C_{n,n}e'_n$ which implies $C_{n,n} \neq 0$. By a recurrence way it is easy to check that $C_{h,h} \neq 0$ for $3 \leq h \leq n$. So we have:

$$e_h = \sum_{k=1}^n C_{h,k}e'_k, \quad h = 1, 2$$

$$e_h = \sum_{k \geq h} C_{h,k}e'_k, \quad C_{h,h} \neq 0, \quad h \geq 3$$

As B and B' are adapted, by considering $[e_2, e_{n-1}] = 0$, we deduce:

$$e_h = \sum_{k \geq h} C_{h,k}e'_k, \quad C_{h,h} \neq 0, \quad 1 \leq h \leq n.$$

As a consequence, we obtain that the matrix $(C_{h,k})$ of the change between two c -graded base is upper-triangular.

Note that:

$$[e_i, e_j] = a_{i,j}e_{i+j+c_1-2} = a_{i,j} \sum_{k \geq i+j+c_1-2} C_{i+j+c_1-2,k}e'_k =$$

$$= \left[\sum_{k \geq i} C_{i,k}e'_k, \sum_{h \geq j} C_{j,h}e'_h \right] = C_{i,i} C_{j,j} a'_{i,j} e'_{i+j+c_2-2} + \sum_{p > i+j+c_2-2} \lambda_p e'_p$$

where $\lambda_p \in \mathbb{C}$ and $a'_{i,j}$ are structure constants of $\mathcal{A}$ with respect to B' .

Therefore $a_{i,j} \neq 0$ and $i+j+c_1-2 < i+j+c_2-2$, which is contradictory. Finally we get a similar conclusion if we suppose that $c_2 < c_1$. $\square$

3 Rank of c -graded filiform Lie algebras

Let us recall that the rank of a nilpotent Lie algebra is the dimension of a maximal exterior torus of derivations. If the nilpotent Lie algebra is filiform, then its rank is strictly smaller than 3 (see [4]). Let $\mathcal{G}$ be a filiform Lie algebra. In [4] is also proved that $r(\mathcal{G}) = 2$ (where $r(\mathcal{G})$ denotes the rank of $\mathcal{G}$) if and only if $\mathcal{G} = P_n$ or $\mathcal{G} = Q_n$. Moreover, it can be checked that the torus of derivations if $\mathcal{G} = P_n$ is generated by:

$$g_1 = Id_{P_n} \quad g_2(e_1) = e_1, \quad g_2(e_i) = i e_i, \quad 2 \leq i \leq n$$

and if $\mathcal{G} = Q_n$, the torus is generated by:

$$h_1(z_1) = 0, \quad h_1(z_i) = z_i, \quad 2 \leq i \leq n-1 \quad h_1(z_n) = 2 z_n$$

$$h_2(z_1) = z_1, \quad h_2(z_i) = (i-2) z_i, \quad 2 \leq i \leq n-1 \quad h_2(z_n) = (n-3) z_n$$

where $\{z_1, \dots, z_n\}$ is a basis of Q_n such that $z_1 = e_1 + e_2$, $z_i = e_i$, $i > 1$ and brackets are defined by $[z_1, z_i] = z_{i+1}$ $2 \leq i \leq n-2$, $[z_1, z_{n-i+1}] = (-1)^{i+1} z_n$.

If $\mathcal{A}$ is a non model c -graded filiform Lie algebra with $c \geq 2$, then we can consider the following non nilpotent derivation:

$$d(e_1) = e_1, \quad d(e_2) = c e_2, \quad \dots, \quad d(e_n) = (n-2+c) e_n$$

where $\{e_1, \dots, e_n\}$ is a c -graded basis of $\mathcal{A}$. Moreover, as $\mathcal{A}$ is not isomorphic to Q_n , then $r(\mathcal{A}) = 1$.

Recall that characteristically nilpotent Lie algebras are those in which every derivation is nilpotent. For a general overview of these algebras, the reader can consult [5]. In [9] we have also proved that c -graded filiform Lie algebras are not characteristically nilpotent. So, as a consequence of these results, the following is true:

Proposition 3.1. *For $c \geq 2$, every c -graded filiform Lie algebra is derived from any solvable Lie algebra $\mathcal{L}$, of the form $\mathcal{L} = \mathcal{A} \oplus \langle U \rangle$, of dimension $n+1$, with $ad U = d$ ($ad U$ is the adjoint mapping $x \mapsto [U, x]$). $\square$*

4 Structure of c -graded filiform Lie algebras

Let $\mathcal{A}$ be a c -graded filiform Lie algebra, with $c \geq 2$, and $B = \{e_1, \dots, e_n\}$ be a c -graded basis of $\mathcal{A}$. We will denote by $t_{q-1} \in \mathbb{C}$ the coefficient of e_{2q-1+c} in

the brackets $[e_q, e_{q+1}]$, with $2 \leq q \leq [\frac{n-c+1}{2}]$, ($[x]$ denotes the integer part of x), that is:

$$\begin{aligned} [e_2, e_3] &= t_1 e_{3+c} \\ [e_3, e_4] &= t_2 e_{5+c} \\ [e_4, e_5] &= t_3 e_{6+c} \\ &\vdots \\ [e_{[\frac{n-c+1}{2}]}, e_{[\frac{n-c+1}{2}]+1}] &= t_{[\frac{n-c+1}{2}]-1} e_{2[\frac{n-c+1}{2}]+c-1}. \end{aligned}$$

If we know these brackets, it is possible to determine the rest of brackets in $\mathcal{A}$ by considering, in a recurrent way, Jacobi identities associated with the elements e_1, e_q, e_p , for $q = [\frac{n-c+1}{2}], \dots, 2$ and $p = q + 1, \dots, n + 2 - c - q$. These identities will be denoted by $J(e_1, e_q, e_p) = 0$.

Then, according to this notation the following result is verified:

Theorem 4.1. *The structure of $\mathcal{A}$ is the following:*

$$\begin{aligned} [e_1, e_q] &= e_{q+1} & 2 \leq q \leq n-1 \\ [e_q, e_{q+1}] &= t_{q-1} e_{2q+c-1} & 2 \leq q \leq [\frac{n-c+1}{2}] \\ [e_q, e_{q+2}] &= t_{q-1} e_{2q+c} & 2 \leq q \leq [\frac{n-c+1}{2}] \\ & & 2q+c \leq n \\ [e_q, e_p] &= \left(\sum_{l=0}^{[\frac{p-q-1}{2}]} (-1)^l \binom{p-1-q-l}{l} t_{q-1+l} \right) e_{q+p+c-2} & q+2 < p \leq n-q+2-c \\ & & 2 \leq q \leq [\frac{n-c+1}{2}] \end{aligned}$$

Moreover, as $\mathcal{A}$ is a Lie algebra, the rest of Jacobi identities are verified, that is:

$$J(e_q, e_p, e_r) = 0$$

with $2 \leq q < p < r \leq n - 2c - 1$ and $p + q + r - 4 + 2c \leq n$. □

Remark 4.1. By considering the second space $H^2(P_n, P_n)$ of Chevalley-Eilenberg's cohomology of the model filiform Lie algebra, it is easy to observe that the c -graded filiform Lie algebras are Lie algebras $(P_n)_{\psi_c}$ (see [7]), where $\psi_c \in H_{c+1}^2(P_n, P_n)$, this is, ψ_c belongs to one of the elements of the usual gradation $H^2(P_n, P_n) = \bigoplus_{c \geq 0} H_{c+1}^2(P_n, P_n)$.

In order to classify c -graded filiform Lie algebras, $c \geq 2$, we will consider the general c -graded algebra of dimension n defined by Theorem 4.1, where t_i

are parameters in $\mathbb{C}$. The solution of the polynomial equations $P_{q,p,r} = 0$ (in $\mathbb{C}[t_1, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1}]$), associated with Jacobi identities $J(e_q, e_p, e_r) = 0$, with $2 \leq q < p < r \leq n - 2c - 1$ and $p + q + r - 4 + 2c \leq n$, allows in the first place to determine the c -graded filiform Lie algebras and secondly to get their classification.

Taking it into account, we will denote by $\mathcal{A}_n^c(t_1, t_2, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$ a Lie algebra whose law is defined as in Theorem 4.1, where t_i are the corresponding coefficients in $[e_{i+1}, e_{i+2}]$.

Proposition 4.2.

$$P_{q,p,r} = P_{q,p,r-1} - P_{q,p+1,r-1} - P_{q+1,p,r-1}$$

for $q < p < r$ and $p + q + r - 4 + 2c = n$.

Proof. Denoting the structure constants in $\mathcal{A}_n^c(t_1, t_2, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$ by $a_{q,p}$, it is easily checked from Theorem 4.1, that:

$$a_{q,p} = a_{q+1,p} + a_{q,p+1}$$

for $q < p$, $q + p + c - 2 \leq n - 1$. As

$$P_{q,p,r} = a_{q,p}a_{q+p+c-2,r} - a_{q,r}a_{q+r+c-2,p} + a_{p,r}a_{p+r+c-2,q},$$

it completes the proof. $\square$

As a consequence of these relations among polynomial equations associated with Jacobi equations, it is possible (as the following theorem affirms) to obtain a smaller number of such polynomial equations, which constitute a generator system of the rest of equations.

Remark 4.2. *It is clear that if $\{e_1, \dots, e_n\}$ is a c -graded basis of a c -graded Lie algebra, $c \geq 2$, $\mathcal{A}_n^c(t_1, t_2, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$, then the ideal center is:*

$$Z(\mathcal{A}_n^c(t_1, t_2, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1})) = \langle e_n \rangle.$$

According to Theorem 4.1, it can be proved that:

$$\mathcal{A}_n^c(t_1, t_2, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1}) / \langle e_n \rangle$$

is a c -graded filiform Lie algebra of dimension $n - 1$ and $\{e'_1 = e_1 + \langle e_n \rangle, e'_i = e_i + \langle e_n \rangle, 2 \leq i \leq n - 1\}$ is a c -graded basis of it.

Moreover,

$$\mathcal{A}_n^c(t_1, t_2, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1}) / \langle e_n \rangle = \mathcal{A}_{n-1}^c(t_1, t_2, \dots, t_{\lfloor \frac{n-c}{2} \rfloor - 1}).$$

This comes from the following fact: if we denote by $a_{q,p}$ the structure constants in the algebra $\mathcal{A}_n^c(t_1, t_2, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$ and $\bar{a}_{q,p}$ denote the structure constants in the algebra $\mathcal{A}_{n-1}^c(t_1, t_2, \dots, t_{\lfloor \frac{n-c}{2} \rfloor - 1})$, it is verified that:

$$a_{q,p} = \bar{a}_{q,p}$$

for $p + q + c - 2 \leq n - 1$. Then, if we denote by $\bar{P}_{q,p,r}$ the Jacobi polynomial equations in $\mathcal{A}_{n-1}^c(t_1, t_2, \dots, t_{\lfloor \frac{n-c}{2} \rfloor - 1})$, we have that:

$$P_{q,p,r} = \bar{P}_{q,p,r}$$

for $q < p < r$ and $q + p + r - 4 + 2c \leq n - 1$.

Recall that to obtain c -graded filiform Lie algebras, $c \geq 2$, of dimension n we need to solve the polynomial equations system $P_{q,p,r} = 0$, for $2 \leq q < p < r \leq n - 2c - 1$, $p + q + r - 4 + 2c \leq n$. By using Remark 4.2, it is possible to determine these algebras by recurrence on the dimension. So, if we know the c -graded filiform Lie algebras of dimension $n - 1$, $\mathcal{A}_{n-1}^c(\lambda_1, \dots, \lambda_{\lfloor \frac{n-c}{2} \rfloor - 1})$, then λ_i verify $P_{q,p,r} = 0$, for $q + p + r - 4 - 2c \leq n - 1$. Hence the c -graded filiform Lie algebras $\mathcal{A} = \mathcal{A}_n^c(t_1, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$, of dimension n , are such that $t_i = \lambda_i$ for $1 \leq i \leq \lfloor \frac{n-c}{2} \rfloor - 1$ and t_i only must verify $P_{q,p,r} = 0$, for $q + p + r - 4 + 2c = n$.

Under these conditions, fixed the coefficients $\lambda_1, \dots, \lambda_{\lfloor \frac{n-c}{2} \rfloor - 1}$ such that $\mathcal{A}_{n-1}^c(\lambda_1, \dots, \lambda_{\lfloor \frac{n-c}{2} \rfloor - 1})$ is a c -graded filiform Lie algebra of dimension $n - 1$, the two following results are verified:

Theorem 4.3. *If $n - c$ is even, $\mathcal{A}_n^c(\lambda_1, \dots, \lambda_{\lfloor \frac{n-c}{2} \rfloor - 1})$ is a c -graded filiform Lie algebra if and only if $\lambda_1, \dots, \lambda_{\lfloor \frac{n-c}{2} \rfloor - 1}$ satisfy the following polynomial equations:*

- $J(e_{2+2k}, e_{\frac{n+1-2c}{2}-k}, e_{\frac{n+3-2c}{2}-k}) = 0$, if n is odd, where $0 \leq 3k \leq \frac{n-2c-5}{2}$.
- $J(e_{3+2k}, e_{\frac{n-2c}{2}-k}, e_{\frac{n-2c+2}{2}-k}) = 0$, if n is even, where $0 \leq 3k \leq \frac{n-2c-8}{2}$.

Theorem 4.4. *If $n - c$ is odd, $\mathcal{A}_n^c(\lambda_1, \dots, \lambda_{\lfloor \frac{n-c}{2} \rfloor - 1}, t_{\lfloor \frac{n-c}{2} \rfloor})$ is a c -graded filiform Lie algebra if and only if $\lambda_1, \dots, \lambda_{\lfloor \frac{n-c}{2} \rfloor - 1}, t_{\lfloor \frac{n-c}{2} \rfloor}$, satisfy the following polynomial equations:*

- $J(e_{2+2k}, e_{\frac{n+1-2c}{2}-k}, e_{\frac{n+3-2c}{2}-k}) = 0$, if n is odd, where $0 \leq 3k \leq \frac{n-2c-5}{2}$.

- $J(e_{3+2k}, e_{\frac{n-2c}{2}-k}, e_{\frac{n-2c+2}{2}-k}) = 0$, if n is even, where $0 \leq 3k \leq \frac{n-2c-8}{2}$.

To prove both results, according to Proposition 4.2, we have:

$$P_{q,p,r} = -P_{q,p+1,r} - P_{q+1,p,r}$$

for $q < p < r$, and $q + p + r - 4 + 2c = n$, since $P_{q,p,r-1} = 0$, because $q + p + r - 1 - 4 + 2c = n - 1$. Then, by recurrence on q, p, r , we obtain that any Jacobi polynomial equation is a linear combination of $P_{q,p,p+1}$ with $q + 2p + 1 - 4 + 2c = n$. $\square$

Theorem 4.5. *Two c -graded filiform Lie algebras of dimension n , $c \geq 2$, $\mathcal{A}_n^c(t_1, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$ and $\mathcal{A}_n^c(t'_1, \dots, t'_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$, are isomorphic if and only if there exists a complex $\lambda \neq 0$ such that $t'_h = \lambda t_h$ for $1 \leq h \leq \lfloor \frac{n-c+1}{2} \rfloor - 1$.*

Proof. Let $\mathcal{A}_n^c(t_1, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$ and $\mathcal{A}_n^c(\lambda t_1, \dots, \lambda t_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$, $\lambda \neq 0$, be two filiform Lie algebras and $B = \{e_1, \dots, e_n\}$, $B' = \{e'_1, \dots, e'_n\}$, c -graded bases of them, respectively. The basis change $e'_1 = e_1$, $e'_i = \lambda e_i$, $2 \leq i \leq n$ proves that these two algebras are isomorphic.

Conversely, if we consider two c -graded bases B and B' of the algebras $\mathcal{A}_n^c(t_1, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$ and $\mathcal{A}_n^c(t'_1, \dots, t'_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$ respectively, satisfying the hypothesis of the theorem, we assume that there exists a basis change:

$$e'_h = \sum_{k \geq h} C_{h,k} e_k, \quad C_{k,k} \neq 0, \quad 1 \leq h \leq n.$$

Starting from $[e'_1, e'_k] = e'_{k+1}$, $2 \leq k \leq n - 1$, we obtain that:

$$C_{k,k} = (C_{1,1})^{n-k} C_{2,2}, \quad 3 \leq k \leq n.$$

Moreover, if we denote by $a'_{q,p}$ the structure constants in $\mathcal{A}_n^c(t'_1, \dots, t'_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$ with respect to B' and by considering $[e'_q, e'_p]$ the ones with respect to B , we conclude that:

$$a_{q,p} C_{p,p} C_{q,q} = a'_{q,p} C_{p+q-2+c, p+q+c-2}$$

for $1 \leq q < p \leq n$, $p + q + c - 2 \leq n$. Consequently,

$$t'_{q-1} = a'_{q,q+1} = \frac{C_{q,q} C_{q+1,q+1}}{C_{2q+c-1, 2q+c-1}} t_{q-1} = C_{2,2} (C_{1,1})^{n+c-1} t_{q-1}$$

for $2 \leq q \leq \lfloor \frac{n-c+1}{2} \rfloor$. Hence $\lambda = C_{2,2} (C_{1,1})^{n+c-1} \neq 0$ verifies required conditions. $\square$

Remark 4.3. *Note that, if we have two c -graded filiform Lie algebras $\mathcal{A}_n^c$, $\mathcal{A}_n'^c$ such that $\mathcal{A}_n^c/Z(\mathcal{A}_n^c)$ is not isomorphic to $\mathcal{A}_n'^c/Z(\mathcal{A}_n'^c)$, then $\mathcal{A}_n^c$ is not isomorphic to $\mathcal{A}_n'^c$.*

5 Classification of 3-graded filiform Lie algebras

By using the notation of Theorem 4.1 and by taking into consideration the previous theorem, we have that every 3-graded filiform Lie algebra is isomorphic to one of the following Lie algebras. Note that for dimension less than 6, the unique 3-graded Lie algebra in each dimension is the model algebra.

- In $\dim \mathcal{A} = 6$: $\mathbf{g}_6^1 := \mathcal{A}_6^3(0)$, $\mathbf{g}_6^2 := \mathcal{A}_6^3(1)$.
- In $\dim \mathcal{A} = 7$: $\mathbf{g}_7^1 := \mathcal{A}_7^3(0)$, $\mathbf{g}_7^2 := \mathcal{A}_7^3(1)$.
- In $\dim \mathcal{A} = 8$: $\mathbf{g}_8^1 := \mathcal{A}_8^3(0, 0)$, $\mathbf{g}_8^2 := \mathcal{A}_8^3(\lambda, 1)$, $\mathbf{g}_8^3 := \mathcal{A}_8^3(1, 0)$.
- In $\dim \mathcal{A} = 9$: $\mathbf{g}_9^1 := \mathcal{A}_9^3(0, 0)$, $\mathbf{g}_9^2 := \mathcal{A}_9^3(\lambda, 1)$, $\mathbf{g}_9^3 := \mathcal{A}_9^3(1, 0)$.
- In $\dim \mathcal{A} = 10$: $\mathbf{g}_{10}^1 := \mathcal{A}_{10}^3(0, 0, 0)$, $\mathbf{g}_{10}^2 := \mathcal{A}_{10}^3(\lambda, 1, 0)$,
 $\mathbf{g}_{10}^3 := \mathcal{A}_{10}^3(1, 0, 0)$, $\mathbf{g}_{10}^4 := \mathcal{A}_{10}^3(\alpha, \beta, 1)$ with $\lambda, \alpha, \beta \in \mathbb{C}$.
- In $\dim \mathcal{A} = 11$: $\mathbf{g}_{11}^1 := \mathcal{A}_{11}^3(0, 0, 0)$, $\mathbf{g}_{11}^2 := \mathcal{A}_{11}^3(1, 0, 0)$,
 $\mathbf{g}_{11}^3 := \mathcal{A}_{11}^3(\frac{4\lambda^2-3\lambda}{3}, \lambda, 1)$ with $\lambda \in \mathbb{C}$.
- In $\dim \mathcal{A} = 12$: $\mathbf{g}_{12}^1 := \mathcal{A}_{12}^3(0, 0, 0, 0)$, $\mathbf{g}_{12}^2 := \mathcal{A}_{12}^3(1, 0, 0, \lambda)$,
 $\mathbf{g}_{12}^3 := \mathcal{A}_{12}^3(0, 0, 0, 1)$, $\mathbf{g}_{12}^4 := \mathcal{A}_{12}^3(\frac{4\lambda^2-3\lambda}{3}, \lambda, 1, \mu)$ with $\lambda, \mu \in \mathbb{C}$.
- In $\dim \mathcal{A} = 13$: $\mathbf{g}_{13}^1 := \mathcal{A}_{13}^3(0, 0, 0, 0)$, $\mathbf{g}_{13}^2 := \mathcal{A}_{13}^3(1, 0, 0, 0)$,
 $\mathbf{g}_{13}^3 := \mathcal{A}_{13}^3(0, 0, 0, 1)$, $\mathbf{g}_{13}^4 := \mathcal{A}_{13}^3(\frac{4\lambda^2-3\lambda}{3}, \lambda, 1, \frac{5\lambda-10}{4\lambda^2-5\lambda-4})$ for $\lambda \in \mathbb{C}$ such that $4\lambda^2 - 5\lambda - 4 \neq 0$.
- In $\dim \mathcal{A} = 14$: $\mathbf{g}_{14}^1 := \mathcal{A}_{14}^3(0, 0, 0, 0, 0)$,
 $\mathbf{g}_{14}^2 := \mathcal{A}_{14}^3(0, 0, 0, 0, 1)$, $\mathbf{g}_{14}^3 := \mathcal{A}_{14}^3(1, 0, 0, 0, \alpha)$,
 $\mathbf{g}_{14}^4 := \mathcal{A}_{14}^3(0, 0, 0, 1, \alpha)$ $\alpha \in \mathbb{C}$
 $\mathbf{g}_{14}^5 := \mathcal{A}_{14}^3(\frac{4\lambda^2-3\lambda}{3}, \lambda, 1, \frac{5\lambda-10}{4\lambda^2-5\lambda-4}, \frac{15\lambda-40}{8\lambda^3-6\lambda^2-13\lambda-4})$ for $\lambda \in \mathbb{C}$ such that $4\lambda^2 - 5\lambda - 4 \neq 0$ and $8\lambda^3 - 6\lambda^2 - 13\lambda - 4 \neq 0$.

- In $\dim \mathcal{A} = 15$: $\mathbf{g}_{15}^1 := \mathcal{A}_{15}^3(0, 0, 0, 0, 0)$,
 $\mathbf{g}_{15}^2 := \mathcal{A}_{15}^3(0, 0, 0, 0, 1)$, $\mathbf{g}_{15}^3 := \mathcal{A}_{15}^3(1, 0, 0, 0, 0)$,
 $\mathbf{g}_{15}^4 := \mathcal{A}_{15}^3(0, 0, 0, 1, 4)$,
 $\mathbf{g}_{15}^5 := \mathcal{A}_{15}^3(42, 6, 1, \frac{2}{11}, \frac{5}{11 \cdot 13})$, $\mathbf{g}_{15}^6 := \mathcal{A}_{15}^3(\frac{10}{3}, 2, 1, 0, -1)$,
 $\mathbf{g}_{15}^7 := \mathcal{A}_{15}^3(\frac{21+7\sqrt{11}}{12}, \frac{5+\sqrt{11}}{4}, 1, \frac{-3+\sqrt{11}}{-1+\sqrt{11}}, \frac{-17+3\sqrt{11}}{2+3\sqrt{11}})$,
 $\mathbf{g}_{15}^8 := \mathcal{A}_{15}^3(\frac{21-7\sqrt{11}}{12}, \frac{5-\sqrt{11}}{4}, 1, \frac{3+\sqrt{11}}{1+\sqrt{11}}, \frac{17+3\sqrt{11}}{-2+3\sqrt{11}})$.
- In $\dim \mathcal{A} = 16$: $\mathbf{g}_{16}^1 := \mathcal{A}_{16}^3(0, 0, 0, 0, 0, 0)$,
 $\mathbf{g}_{16}^2 := \mathcal{A}_{16}^3(0, 0, 0, 0, 0, 1)$, $\mathbf{g}_{16}^3 := \mathcal{A}_{16}^3(0, 0, 0, 0, 1, \lambda)$,
 $\mathbf{g}_{16}^4 := \mathcal{A}_{16}^3(42, 6, 1, \frac{2}{11}, \frac{5}{11 \cdot 13}, \frac{1}{11 \cdot 13})$, $\mathbf{g}_{16}^5 := \mathcal{A}_{16}^3(1, 0, 0, 0, 0, \lambda)$,
 $\mathbf{g}_{16}^6 := \mathcal{A}_{16}^3(0, 0, 0, 1, 4, 25)$, $\mathbf{g}_{16}^7 := \mathcal{A}_{16}^3(\frac{10}{3}, 2, 1, 0, -1, \frac{-5}{3})$,
 $\mathbf{g}_{16}^8 := \mathcal{A}_{16}^3(\frac{21+7\sqrt{11}}{12}, \frac{5+\sqrt{11}}{4}, 1, \frac{-3+\sqrt{11}}{-1+\sqrt{11}}, \frac{-17+3\sqrt{11}}{2+3\sqrt{11}}, 6\frac{-52+17\sqrt{11}}{-31+\sqrt{11}})$,
 $\mathbf{g}_{16}^9 := \mathcal{A}_{16}^3(\frac{21-7\sqrt{11}}{12}, \frac{5-\sqrt{11}}{4}, 1, \frac{3+\sqrt{11}}{1+\sqrt{11}}, \frac{17+3\sqrt{11}}{-2+3\sqrt{11}}, 6\frac{52+17\sqrt{11}}{31-\sqrt{11}})$, $\lambda \in \mathbb{C}$.

We show now in detail some examples of c -graded Lie algebras which allow to check the classification above indicated. We use dimension 6 and 13 due to in each of these cases the computations are different. Later, we will continue with the classification for $\dim \mathcal{A} \geq 17$.

Example 5.1. Let $\mathcal{A}$ be a 3-graded Lie algebra of dimension 6.

In this case, according to Theorem 4.1 we have an one-parametric family of algebras:

$$\begin{aligned} \mathcal{A}_6^3(t_1) : \quad [e_1, e_k] &= e_{k+1} \quad \text{for } 2 \leq k \leq 5 \\ [e_2, e_3] &= t_1 e_6 \end{aligned}$$

where the parameter $t_1 \in \mathbb{C}$. So, according to Theorem 4.5, we distinguish:

- If $t_1 = 0$, we obtain the model algebra L_6 ,

$$\mathcal{A}_6^3(0) : \quad [e_1, e_k] = e_{k+1} \quad \text{for } 2 \leq k \leq 5.$$

- If $t_1 \neq 0$, all the algebras $\mathcal{A}_6^3(t_1)$ with $t_1 \neq 0$ are isomorphic to:

$$\begin{aligned} \mathcal{A}_6^3(1) : \quad [e_1, e_k] &= e_{k+1} \quad \text{for } 2 \leq k \leq 5 \\ [e_2, e_3] &= e_6. \end{aligned}$$

Then, a 3-graded filiform Lie algebra of dimension 6 is isomorphic to $\mathcal{A}_6^3(0)$ or $\mathcal{A}_6^3(1)$.

The classification of 3-graded Lie algebras up dimension 12 can be obtained in a similar way.

To classify 3-graded Lie algebras of dimension greater or equal than 13 we use an inductive method based on Theorems 4.1, 4.3, 4.4 and 4.5 and Remarks 4.2 and 4.3.

Example 5.2. We start from the four-parametric family $\mathcal{A}_{12}^3(t_1, t_2, t_3, t_4)$ of algebras in dimension 12, (see Theorem 4.1), where the parameters t_1, t_2, t_3 and t_4 have to verify the condition $p1 = P_{2,3,4}$:

$$p1 = 3t_2t_3 + 3t_1t_3 - 4t_2^2 = 0,$$

and we obtain that a 3-graded filiform Lie algebra of dimension 12 is isomorphic to one of the following: $\mathcal{A}_{12}^3(0, 0, 0, 0)$, $\mathcal{A}_{12}^3(0, 0, 0, 1)$, $\mathcal{A}_{12}^3(1, 0, 0, \alpha)$ or $\mathcal{A}_{12}^3(\frac{4\lambda^2-3\lambda}{3}, \lambda, 1, \mu)$, where $\alpha, \lambda, \mu \in \mathbb{C}$.

Now, for dimension 13, starting from the four-parametric family of algebras:

$$\begin{aligned} \mathcal{A}_{13}^3(t_1, t_2, t_3, t_4) : \quad [e_1, e_k] &= e_{k+1} \quad \text{for } 2 \leq k \leq 12 \\ [e_2, e_{10}] &= (t_1 - 6t_2 + 10t_3 - 4t_4) e_{13} \\ [e_2, e_9] &= (t_1 - 5t_2 + 6t_3 - t_4) e_{12} \\ [e_2, e_8] &= (t_1 - 4t_2 + 3t_3) e_{11} \\ [e_2, e_7] &= (t_1 - 3t_2 + t_3) e_{10} \\ [e_2, e_6] &= (t_1 - 2t_2) e_9 \\ [e_2, e_5] &= (t_1 - t_2) e_8 \\ [e_2, e_4] &= t_1 e_7 \\ [e_2, e_3] &= t_1 e_6 \\ [e_3, e_9] &= (t_2 - 4t_3 + 3t_4) e_{13} \\ [e_3, e_8] &= (t_2 - 3t_3 + t_4) e_{12} \\ [e_3, e_7] &= (t_2 - 2t_3) e_{11} \\ [e_3, e_6] &= (t_2 - t_3) e_{10} \\ [e_3, e_5] &= t_2 e_9 \\ [e_3, e_4] &= t_2 e_8 \\ [e_4, e_8] &= (t_3 - 2t_4) e_{13} \\ [e_4, e_7] &= (t_3 - t_4) e_{12} \\ [e_4, e_6] &= t_3 e_{11} \\ [e_4, e_5] &= t_3 e_{10} \\ [e_5, e_7] &= t_4 e_{13} \\ [e_5, e_6] &= t_4 e_{12} \end{aligned}$$

where the parameters t_1, t_2, t_3 and t_4 have to verify conditions $p_1 = 0$ (see Remark 4.2) and $p_2 = P_{2,4,5}^{13=0}$, which is reduced to:

$$p_2 = -5t_2t_3 + 10t_3^2 - 4t_3t_4 + 3t_1t_4 - 2t_2t_4 = 0 \quad (1)$$

by taking into account Theorem 4.3.

Hence, from classification for dimension 12 and Remark 4.3, we conclude:

- From $\langle (0, 0, 0, 0) \rangle$ we obtain the model algebra $\mathcal{A}_{13}^3(0, 0, 0, 0)$, because $(0, 0, 0, 0)$ verifies (1).
- From $\cup_{\lambda \in \mathbb{C}} \langle (1, 0, 0, \lambda) \rangle$ we obtain $\mathcal{A}_{13}^3(1, 0, 0, 0)$, since $\lambda = 0$ is necessary for $(1, 0, 0, \lambda)$ to verify (1).
- From $\langle (0, 0, 0, 1) \rangle$ we have $\mathcal{A}_{13}^3(0, 0, 0, 1)$, because $(0, 0, 0, 1)$ verifies (1).
- From $\cup_{\lambda, \mu \in \mathbb{C}} \langle (\frac{4\lambda^2-3\lambda}{3}, \lambda, 1, \mu) \rangle$ we obtain $\mathcal{A}_{13}^3(\frac{4\lambda^2-3\lambda}{3}, \lambda, 1, \frac{5\lambda-10}{4\lambda^2-5\lambda-4})$, since $\mu = \frac{5\lambda-10}{4\lambda^2-5\lambda-4}$, for $\lambda \in \mathbb{C}$ with $4\lambda^2 - 5\lambda - 4 \neq 0$ is necessary for $(\frac{4\lambda^2-3\lambda}{3}, \lambda, 1, \mu)$ to verify (1).

Next, we prove that for $\dim \mathcal{A} \geq 17$, only five 3-graded filiform Lie algebras (up to isomorphism) are obtained in the case of even dimension, whereas, four algebras and two one-parametric families of 3-graded filiform Lie algebras (up to isomorphism) are obtained in the case of odd dimension.

Theorem 5.1. *Let $\mathcal{A}$ be a 3-graded filiform Lie algebra of dimension n , with $n \geq 17$. Then they are verified:*

- If n is odd, the Lie algebra $\mathcal{A}_n^3(t_1, \dots, t_{\frac{n-5}{2}})$ is isomorphic to one of the following algebras:
 - $\mathcal{A}_n^3(0, 0, \dots, 0)$
 - $\mathcal{A}_n^3(1, 0, \dots, 0)$
 - $\mathcal{A}_n^3(0, \dots, 0, 1)$
 - $\mathcal{A}_n^3(0, \dots, 0, 1, \frac{(n-3)(n-7)}{24})$
 - $\mathcal{A}_n^3(\lambda_1, \dots, \lambda_{\frac{n-5}{2}})$, where $\lambda_{k+1} = \frac{k+1}{2(2k+5)}\lambda_k$ for $k \geq 1$ and $\lambda_1 = 42$.
- If n is even, the Lie algebra $\mathcal{A}_n^3(t_1, \dots, t_{\frac{n-4}{2}})$ is isomorphic to one of the following algebras:

- $\mathcal{A}_n^3(0, 0, \dots, 0)$
- $\mathcal{A}_n^3(1, 0, \dots, 0, \lambda)$ with $\lambda \in \mathbb{C}$
- $\mathcal{A}_n^3(0, \dots, 1, \lambda)$ with $\lambda \in \mathbb{C}$
- $\mathcal{A}_n^3(0, \dots, 0, 1)$
- $\mathcal{A}_n^3(0, \dots, 0, 1, \frac{(n-4)(n-8)}{24}, \frac{(n-6)^2(n-8)(n-4)}{3 \cdot 32 \cdot 4})$
- $\mathcal{A}_n^3(\lambda_1, \dots, \lambda_{\frac{n-4}{2}})$, where $\lambda_{k+1} = \frac{k+1}{2(2k+5)} \lambda_k$ for $k \geq 1$ and $\lambda_1 = 42$.

Proof. We proceed by induction on $n = \dim \mathcal{A}$. Note that some details of the induction will be omitted, due to the proof could be lengthy.

- For $\dim \mathcal{A} = 17$ we obtain:

$$\begin{aligned} \mathbf{g}_{17}^1 &:= \mathcal{A}_{17}^3(0, 0, 0, 0, 0, 0), \quad \mathbf{g}_{17}^2 := \mathcal{A}_{17}^3(0, 0, 0, 0, 0, 1), \\ \mathbf{g}_{17}^3 &:= \mathcal{A}_{17}^3(0, 0, 0, 0, 1, \frac{35}{6}), \quad \mathbf{g}_{17}^4 := \mathcal{A}_{17}^3(1, 0, 0, 0, 0, 0), \\ \mathbf{g}_{17}^5 &:= \mathcal{A}_{17}^3(42, 6, 1, \frac{2}{11}, \frac{5}{11 \cdot 13}, \frac{1}{11 \cdot 13}). \end{aligned}$$

- For $\dim \mathcal{A} = 18$:

$$\begin{aligned} \mathbf{g}_{18}^1 &:= \mathcal{A}_{18}^3(0, 0, 0, 0, 0, 0, 0), \quad \mathbf{g}_{18}^2 := \mathcal{A}_{18}^3(0, 0, 0, 0, 0, 0, 1), \\ \mathbf{g}_{18}^3 &:= \mathcal{A}_{18}^3(0, 0, 0, 0, 0, 1, \lambda), \quad \mathbf{g}_{18}^4 := \mathcal{A}_{18}^3(0, 0, 0, 0, 1, \frac{35}{6}, \frac{105}{2}), \\ \mathbf{g}_{18}^5 &:= \mathcal{A}_{18}^3(42, 6, 1, \frac{2}{11}, \frac{5}{11 \cdot 13}, \frac{1}{11 \cdot 13}, \frac{7}{11 \cdot 13 \cdot 17 \cdot 2}), \\ \mathbf{g}_{18}^6 &:= \mathcal{A}_{18}^3(1, 0, 0, 0, 0, 0, \lambda), \quad \lambda \in \mathbb{C}. \end{aligned}$$

Let suppose the result proved for $\dim \mathcal{A} = n - 1$. We will show that it is also true for $\dim \mathcal{A} = n$. Indeed:

- If n es odd. By using Theorem 4.3 we have that the parameters $t_1, \dots, t_{\frac{n-5}{2}}$ of any 3-graded filiform Lie algebra $\mathcal{A}_n^3(t_1, \dots, t_{\frac{n-5}{2}})$ have to satisfy:

$$J(e_{2+2r}, e_{\frac{n-5}{2}-r}, e_{\frac{n-3}{2}-r}) = 0$$

where $0 \leq 3r \leq \frac{n-11}{2}$. From Theorem 4.1, it is equivalent to:

$$\begin{aligned} & - t_{\frac{n-7-2r}{2}} [e_{2r+2}, e_{n-2r-3}] - \\ & - \sum_{l=0}^{\lfloor \frac{n-6r-9}{4} \rfloor} (-1)^l \binom{\frac{n-6r-9}{2} - l}{l} t_{2r+1+l} [e_{\frac{n+2r+3}{2}}, e_{\frac{n-5-2r}{2}}] + \\ & + \sum_{l=0}^{\lfloor \frac{n-6r-11}{4} \rfloor} (-1)^l \binom{\frac{n-6r-11}{2} - l}{l} t_{2r+1+l} [e_{\frac{n+2r+1}{2}}, e_{\frac{n-3-2r}{2}}] = 0. \end{aligned} \tag{2}$$

By hypothesis, the algebra $\mathcal{A}_{n-1}^3(t_1, \dots, t_{\frac{n-5}{2}})$ is isomorphic to one of the following: $\mathcal{A}_{n-1}^3(0, \dots, 0)$, $\mathcal{A}_{n-1}^3(1, \dots, 0, \lambda)$, $\mathcal{A}_{n-1}^3(0, \dots, 0, 1)$, $\mathcal{A}_{n-1}^3(0, \dots, 0, 1, \lambda)$, $\mathcal{A}_{n-1}^3(0, \dots, 0, 1, \frac{(n-5)(n-9)}{24}, \frac{(n-7)^2(n-9)(n-5)}{32 \cdot 4 \cdot 3})$ or $\mathcal{A}_{n-1}^3(\lambda_1, \dots, \lambda_{\frac{n-5}{2}})$, where $\lambda_{k+1} = \frac{k+1}{4k+10} \lambda_k$, for $k \geq 1$ and $\lambda_1 = 42$. Then:

- From the first four algebras previously mentioned, we obtain, respectively: $\mathcal{A}_n^3(0, \dots, 0)$, $\mathcal{A}_n^3(0, \dots, 0, 1)$, $\mathcal{A}_n^3(1, 0, \dots, 0)$ and $\mathcal{A}_n^3(0, \dots, 0, 1, \frac{(n-3)(n-7)}{24})$.
- From $\mathcal{A}_{n-1}^3(0, \dots, 0, 1, \frac{(n-5)(n-9)}{24}, \frac{(n-7)^2(n-9)(n-5)}{32 \cdot 4 \cdot 3})$ we do not obtain any Lie algebra, because

$$\left(0, \dots, 0, 1, \frac{(n-5)(n-9)}{24}, \frac{(n-7)^2(n-9)(n-5)}{32 \cdot 4 \cdot 3}\right)$$

does not satisfy $J(e_4, e_{\frac{n-7}{2}}, e_{\frac{n-5}{2}}) = 0$.

- From $\mathcal{A}_{n-1}^3(\lambda_1, \dots, \lambda_{\frac{n-5}{2}})$, where $\lambda_{k+1} = \frac{k+1}{2(2k+5)} \lambda_k$ for $k \geq 1$ and $\lambda_1 = 42$, we obtain $\mathcal{A}_n^3(\lambda_1, \dots, \lambda_{\frac{n-5}{2}})$, since it is isomorphic to a finite quotient algebra of the infinite dimensional Witt Lie algebra:

$$W_\infty : [x_i, x_j] = (j-i)x_{i+j} \text{ for } 1 \leq i, j$$

defined by:

$$W_\infty / \langle x_2, x_{n+2}, \dots \rangle.$$

So, with respect to the basis $\{x_1, x_3, x_4, \dots, x_{n+1}\}$, the quotient algebra is defined by $[x_i, x_j] = (j-i)x_{i+j}$ for $i+j \leq n+1$. And the isomorphism is defined by $e_1 = x_1$, $e_i = 6(i-1)! 420 x_{i+1}$ for $i \geq 2$.

- If n is even, by taking into account the Theorem 4.4 we have that the parameters $t_1, \dots, t_{\frac{n-4}{2}}$ of any 3-graded filiform Lie algebra $\mathcal{A}_n^3(t_1, \dots, t_{\frac{n-4}{2}})$ have to satisfy:

$$J(e_{3+2r}, e_{\frac{n-6}{2}-r}, e_{\frac{n-4}{2}-r}) = 0$$

where $0 \leq 3r \leq \frac{n-14}{2}$. Then,

$$\begin{aligned} & t_{n-\frac{n-8-2r}{2}} [e_{n-4-2r}, e_{3+2r}] - \\ & - \sum_{l=0}^{\lfloor \frac{n-6r-12}{4} \rfloor} (-1)^l \binom{\frac{n-6r-12}{2} - l}{l} t_{2r+2+l} [e_{\frac{n+}{2}+r}, e_{\frac{n-6}{2}-r}] + \\ & + \sum_{l=0}^{\lfloor \frac{n-6r-14}{4} \rfloor} (-1)^l \binom{\frac{n-6r-14}{2} - l}{l} t_{2r+2+l} [e_{\frac{n+2}{2}+r}, e_{\frac{n-4}{2}-r}] = 0. \end{aligned} \quad (3)$$

By hypothesis, the algebra $\mathcal{A}_{n-1}^3(t_1, \dots, t_{\frac{n-6}{2}})$ is isomorphic to one of the following algebras:

$$\begin{aligned} & \mathcal{A}_{n-1}^3(0, \dots, 0), \quad \mathcal{A}_{n-1}^3(1, \dots, 0), \quad \mathcal{A}_{n-1}^3(0, \dots, 0, 1), \\ & \mathcal{A}_{n-1}^3(0, \dots, 0, 1, \frac{(n-4)(n-8)}{24}) \quad \text{or} \quad \mathcal{A}_{n-1}^2(\lambda_1, \dots, \lambda_{\frac{n-5}{2}}), \end{aligned}$$

where $\lambda_{k+1} = \frac{k+1}{4k+10} \lambda_k$, for $k \geq 1$ and $\lambda_1 = 42$. Then:

- From $\mathcal{A}_{n-1}^3(0, \dots, 0)$ we obtain $\mathcal{A}_n^3(0, \dots, 0, t_{\frac{n-4}{2}}), t_{\frac{n-4}{2}} \in \mathbb{C}$.

Hence, we can distinguish:

- * If $t_{\frac{n-4}{2}} = 0$, then

$$\mathcal{A}_n^3(0, \dots, 0).$$

- * If $t_{\frac{n-4}{2}} \neq 0$, then all algebras $\mathcal{A}_n^3(0, \dots, 0, t_{\frac{n-4}{2}})$ are isomorphic to:

$$\mathcal{A}_n^3(0, \dots, 0, 1).$$

And $(0, \dots, 0), (0, \dots, 1)$ satisfy Jacobi equations (3).

- From the three algebras $\mathcal{A}_{n-1}^3(1, 0, \dots, 0), \mathcal{A}_{n-1}^3(0, \dots, 0, 1)$ and $\mathcal{A}_{n-1}^3(0, \dots, 0, 1, \frac{(n-4)(n-8)}{24})$ we now obtain

$$\mathcal{A}_n^3(1, 0, \dots, 0, \lambda), \quad \mathcal{A}_n^3(0, \dots, 0, 1, \lambda),$$

$$\text{and } \mathcal{A}_n^3\left(0, \dots, 0, 1, \frac{(n-4)(n-8)}{24}, \frac{(n-6)^2(n-4)(n-8)}{32 \cdot 4 \cdot 3}\right),$$

respectively.

- From $\mathcal{A}_{n-1}^3(\lambda_1, \dots, \lambda_{\frac{n-6}{2}})$, where $\lambda_{k+1} = \frac{k+1}{2(2k+5)}\lambda_k$ for $k \geq 1$ and $\lambda_1 = 42$, we obtain $\mathcal{A}_n^3(\lambda_1, \dots, \lambda_{\frac{n-4}{2}})$, where $\lambda_{k+1} = \frac{k+1}{2(2k+5)}\lambda_k$ for $k \geq 1$ and $\lambda_1 = 42$, since similarly to the case n odd, it is a quotient algebra of the infinite Witt Lie algebra. $\square$

6 Rigid solvable Lie algebras whose nil-radical is 3-graded filiform

Let M^n be the algebraic variety of Lie algebras of dimension n imbedded in $\mathbb{C}^{\frac{n^3-n^2}{2}}$. A Lie algebra $\mathcal{L}$ of dimension n is called rigid if its orbit is a Zariski open set of M^n . For a general overview of these algebras, the reader can consult [5].

According to a result by Carles [2], it follows that every solvable rigid Lie algebra $\mathcal{L}$ is decomposable in the sense:

$$\mathcal{L} = \mathcal{N} \oplus \mathcal{T}$$

where $\mathcal{N}$ is the nil-radical of $\mathcal{L}$ and $\mathcal{T}$ is an exterior torus of derivations.

In [1] authors study solvable rigid Lie algebras whose nil-radical is filiform. Particularly, they study algebras whose nil-radical is the c -graded Lie algebra $\mathcal{A}_n^c(t_1, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$.

Then, by using Theorem 3.13 of [1] and our 3-graded filiform Lie algebras classification, we deduce the following:

Theorem 6.1. *Let $\mathcal{L} = \mathcal{T} \oplus \mathcal{A}_n^3(t_1, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$ be a decomposable Lie algebra of dimension $n + 1$, where $\mathcal{A}_n^3(t_1, \dots, t_{\lfloor \frac{n-c+1}{2} \rfloor - 1})$ is a 3-graded filiform Lie algebra. Then:*

- $\mathcal{L}$ is not rigid if n is odd and $11 \leq n \leq 13$ or if n is even and $12 \leq n \leq 14$.
- $\mathcal{L}$ is rigid if $n \geq 15$ with n odd or $n \geq 16$ with n even, and $\mathcal{A}_n^3$ is neither the model algebra nor an algebra belonging to the families $\mathcal{A}_n^3(1, 0, \dots, 0, \lambda)$ and $\mathcal{A}_n^3(0, \dots, 1, \lambda)$. $\square$

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