BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

*Bull Braz Math Soc, New Series 36(1), 99-114*


# Stable constant mean curvature hypersurfaces in $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}(-1)$

Leung-fu Cheung* and Detang Zhou**

**Abstract.** In this paper, we show that all complete stable hypersurfaces in $\mathbb{R}^{n+1}$(or $\mathbb{H}^{n+1}(-1)$) ($n = 3, 4, 5$) with constant mean curvature $H > 0$ (or $H > 1$, respectively) and finite $L^2$ norm of traceless second fundamental form are compact geodesic spheres.

**Keywords:** stable hypersurface, constant mean curvature, isometric immersion, Bernstein theorem.

**Mathematical subject classification:** 53C42, 53C20.

## 1 Introduction

The celebrated Bernstein theorem on minimal surface states that a two dimensional entire minimal graph must be planar. In the last forty years much work has been devoted to generalize it. From the works of Fleming [15], de Giorgi [12], Almgren [3] and Simons [17], one knows that all complete area-minimizing graphs in the Euclidean space $\mathbb{R}^{n+1}$ are hyperplanes when $n \leq 7$. Counterexamples found by Bombieri-de Giorgi-Giusti [6] show that there are non-planar entire minimal graphs in $\mathbb{R}^{n+1}$ for $n > 7$.

In two dimensional case, Fischer-Colbrie–Schoen [14] and do Carmo-Peng [8] independently proved that all complete, oriented and immersed stable minimal surfaces in $\mathbb{R}^3$ are planes. Here a complete oriented minimal hypersurface $M$ in $\mathbb{R}^{n+1}$ is stable means that the second variation of the volume is non-negative on any compact subset of $M$. It is still an open question whether there exist non-planar complete stable minimal hypersurfaces in $\mathbb{R}^{n+1}$ for $n \leq 7$. Due to the counterexamples by Bombieri-de Giorgi-Giusti [6], one should expect

Received 17 March 2004.

*Supported by PolyU grant G-T575.

**Partially supported by CNPq of Brazil.

difficulties in proving higher dimensional versions of the generalized Bernstein problem. In [9], do Carmo and Peng proved that

**Theorem 1.** *Let $M^n$ be a complete stable minimal hypersurfaces in $\mathbb{R}^{n+1}$ with*

$$\int_M |A|^2 dv < +\infty,$$

*where $|A|$ is the second fundamental form of $M$. Then $M^n$ is a hyperplane in $\mathbb{R}^{n+1}$.*

In this paper, we study the Bernstein problem for constant mean curvature hypersurfaces in space forms $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}(-1)$. To state our results, we first recall some notations.

Let $x : M^n \to N^{n+1}$ be an oriented isometric immersion of a connected manifold $M^n$ in $N^{n+1}$. Denote by $\overline{\nabla}$ and $\nabla$ the Levi-Civita connection of $N^{n+1}$ and $M^n$ respectively. Fix a point $p \in M^n$ and a local orthonormal frame field $\{e_1, e_2, \cdots, e_n, \nu\}$ at $p$ such that $\{e_1, e_2, \cdots, e_n\}$ are tangent fields and $\nu$ is a unit normal vector field at $p$. Define

$$\langle AX, Y\rangle = \left\langle \overline{\nabla}_X Y, \nu \right\rangle,$$

where $X, Y$ are tangent vector fields. The mean curvature of $M^n$ is defined as

$$H = \frac{1}{n}\mathrm{tr}A.$$

**Definition 1.** *A immersion $x : M^n \to N^{n+1}$ with constant mean curvature $H$ ($H \neq 0$) is called* **strongly stable** *if for all $f \in C_o^\infty(M)$,*

$$\int_M \left\{|\nabla f|^2 - \left(\overline{Ric}\,(\nu, \nu) + |A|^2\right) f^2\right\} dv \geq 0, \tag{1}$$

*where $\nabla f$ is the gradient of $f$ in the induced metric and $dv$ is the volume form.*

*It is called* **stable** *if (1) holds only for $f \in C_o^\infty(M)$ satisfying the condition*

$$\int_M f dv = 0. \tag{2}$$

*A minimal immersion $x : M^n \to N^{n+1}$ is called* **stable** *if (1) holds for all $f \in C_o^\infty(M)$.*

In the theory of constant mean curvature hypersurfaces, the latter stability is more natural, as this class of test functions defines deformation preserving volumes (see Barbosa and Berard [4] for more discussions). In the rest of the paper, we will omit the "$dv$" in integral formulas when it is obvious in the text.

In studying hypersurfaces with constant mean curvature $H$, it is convenient to modify $A$ by introducing a new linear map $\phi : T_pM \to T_pM$ defined by

$$\langle \phi X, Y \rangle = -\langle AX, Y \rangle + H \langle X, Y \rangle .$$

This map $\phi$ is traceless and satisfies $|A|^2 = |\phi|^2 + nH^2$. So we can write (1) as

$$\int_M |\nabla f|^2 \, dv \geq \int_M \left( \overline{\mathrm{Ric}} \, (\nu, \nu) + nH^2 + |\phi|^2 \right) f^2 dv. \tag{3}$$

Using $\phi$, it is proved by Alencar-do Carmo [2] ($n \leq 5$) and do Carmo-Zhou [10] ($n \leq 6$) that any complete strongly stable hypersurfaces with constant mean curvature in $\mathbb{R}^{n+1}$ with $\int_M |\phi|^2 < \infty$ must be hyperplanes. In this paper we can prove a result in stable cases.

**Theorem 2.** *Let $M^n$ be a complete stable hypersurface in $\mathbb{R}^{n+1}$ ($n = 3, 4, 5$) with constant mean curvature $H > 0$ and $\int_M |\phi|^2 < \infty$. Then $M$ must be a round sphere.*

When the ambient space is $\mathbb{H}^{n+1}(-1)$, all compact stable hypersurfaces with constant mean curvature $H$ have been characterized by Barbosa-do Carmo-Eschenberg in [5]. It is proved by da Silveira [16] that there exist complete noncompact surfaces with constant mean curvature $H$ in $\mathbb{H}^3(-1)$ if $H < 1$. In this paper we prove the following

**Theorem 3.** *All complete stable hypersurfaces in $\mathbb{H}^{n+1}(-1)$ ($n = 3, 4, 5$) with constant mean curvature $H > 1$ and $\int_M |\phi|^2 < \infty$ are compact geodesic spheres.*

We see in the above that Theorem 1 of do Carmo-Peng holds without the restriction on dimensions. But we do not know now whether the dimension condition in Theorem 3 is essential. In fact we show in Theorem 4 that the finiteness of $\int_M |\phi|^2$ implies the finiteness of $\int_M |\phi|^5$ if $M^n$ is a complete minimal (or constant mean curvature $H \geq 1$) hypersurfaces of finite index in $\mathbb{R}^{n+1}$ (or $\mathbb{H}^{n+1}(-1)$, respectively) when $n \leq 7$. Theorem 3 follows by combining the compactness result of do Carmo-Cheung-Santos [7] which says that complete finite index hypersurfaces in $\mathbb{H}^{n+1}(-1)$ with constant mean curvature $H > 1$ and $\int_M |\phi|^n < +\infty$ are compact. So we need $3 \leq n \leq 5$.

This paper is organized as follows. In Section 2 we study some properties of the solutions to a certain class of differential inequalities. Some special cases of such inequalities are the Simons' inequalities for constant mean curvature hypersurfaces when the coefficients are suitably chosen. We study them in generality, since they may be of independent interests. In Section 3, we recall some standard tools for studying constant mean curvature hypersurfaces, such as Simons' inequality for the traceless second fundamental form. After collecting the facts, we apply the results proved in the Section 2 to complete hypersurfaces with constant mean curvature in space forms to obtain Theorem 2 and Theorem 3.

## 2 Some results on differential inequalities

In this section, we consider properties of solutions of a class of Simons' type inequality which will be used in Section 3.

Let $M$ be a complete noncompact Riemannian manifold. We consider nonnegative solution of the following inequality

$$u\Delta u \geq k|\nabla u|^2 + au^4 + bu^3 + cu^2, \tag{4}$$

where $k, a, b, c$ are real constants.

**Proposition 1.** *Let u be a nonnegative locally Lipschitz function satisfying (4). Suppose that for some compact domain $D \subset M$*

$$\int_{M\setminus D} |\nabla f|^2 \geq \int_{M\setminus D} (u^2 + d) f^2, \tag{5}$$

*(where d is a non-negative number) holds for compactly supported function $f \in C_o^\infty(M\setminus D)$ and $\frac{k}{2} + a + 1 > 0$. Then*

$$\int_{M\setminus D} u^2 < +\infty$$

*implies*

$$\int_{M\setminus D} u^4 < +\infty.$$

**Proof.** Multiplying (1) on both sides by a smooth function $\varphi^2$ compactly supported in $M\setminus D$ and integrating by parts, we obtain

$$\begin{aligned} &-\int_{M\setminus D} \varphi^2 |\nabla u|^2 - 2\int_{M\setminus D} u\varphi \langle \nabla u, \nabla\varphi\rangle \\ &\geq k\int_{M\setminus D} \varphi^2 |\nabla u|^2 + a\int_{M\setminus D} \varphi^2 u^4 + b\int_{M\setminus D} \varphi^2 u^3 + c\int_{M\setminus D} \varphi^2 u^2. \end{aligned} \tag{6}$$

Next we choose $f = u\varphi$ as test function in (5),

$$\int_{M\setminus D} |\nabla (u\varphi)|^2 \geq \int_{M\setminus D} \left(u^4\varphi^2 + du^2\varphi^2\right),$$

which is the same as

$$\int_{M\setminus D} \varphi^2 |\nabla u|^2 + \int_{M\setminus D} u^2 |\nabla \varphi|^2 + 2\int_{M\setminus D} \varphi u \langle \nabla\varphi, \nabla u\rangle$$
$$\geq \int_{M\setminus D} u^4\varphi^2 + d\int_{M\setminus D} u^2\varphi^2. \tag{7}$$

Adding to (6), we have

$$\int_{M\setminus D} u^2 |\nabla\varphi|^2 \geq k\int_{M\setminus D} \varphi^2 |\nabla u|^2$$
$$+ (a+1)\int_{M\setminus D} u^4\varphi^2 + b\int_{M\setminus D} u^3\varphi^2 + (d+c)\int_{M\setminus D} u^2\varphi^2.$$

Rearranging terms gives

$$-b\int_{M\setminus D} u^3\varphi^2 + \int_{M\setminus D} u^2 |\nabla\varphi|^2$$
$$\geq k\int_{M\setminus D} \varphi^2 |\nabla u|^2 + (a+1)\int_{M\setminus D} u^4\varphi^2 + (d+c)\int_{M\setminus D} u^2\varphi^2. \tag{8}$$

Using the inequality $2xy \leq x^2 + y^2$ we get from (7)

$$2\int_{M\setminus D} \varphi^2 |\nabla u|^2 + 2\int_{M\setminus D} u^2 |\nabla\varphi|^2 \geq \int_{M\setminus D} u^4\varphi^2 + d\int_{M\setminus D} u^2\varphi^2 \tag{9}$$

Then it follows from (8) and (9) that

$$k\int_{M\setminus D} \varphi^2|\nabla u|^2 + (a+1)\int_{M\setminus D} u^4\varphi^2 + (d+c)\int_{M} u^2\varphi^2$$
$$\geq \left(\frac{k}{2} + a + 1\right)\int_{M\setminus D} u^4\varphi^2 + \left(\frac{kd}{2} + d + c\right)\int_{M} u^2\varphi^2 - k\int_{M\setminus D} u^2|\nabla\varphi|^2.$$

Now (8) implies

$$-b\int_{M\setminus D} u^3\varphi^2 + \int_{M\setminus D} u^2|\nabla\varphi|^2$$
$$\geq \left(\frac{k}{2} + a + 1\right)\int_{M\setminus D} u^4\varphi^2 + \left(\frac{kd}{2} + d + c\right)\int_{M\setminus D} u^2\varphi^2 - k\int_{M\setminus D} u^2|\nabla\varphi|^2.$$

We now apply again Young's inequality: $xy \le \frac{\epsilon x^2}{2} + \frac{y^2}{2\epsilon}$ to the left-hand side to get

$$\begin{aligned} &\frac{|b|}{2\epsilon}\int_{M\setminus D} u^2\varphi^2 + \frac{\epsilon\,|b|}{2}\int_{M\setminus D} u^4\varphi^2 + (1+k)\int_{M\setminus D} u^2\,|\nabla\varphi|^2 \\ &\quad \ge \left(\frac{k}{2}+a+1\right)\int_{M\setminus D} u^4\varphi^2 + \left(\frac{kd}{2}+d+c\right)\int_{M\setminus D} u^2\varphi^2, \end{aligned} \tag{10}$$

where $\epsilon$ is an arbitrary positive constant. (10) gives

$$\begin{aligned} &\left[\frac{|b|}{2\epsilon} - \left(\frac{kd}{2}+d+c\right)\right]\int_{M\setminus D} u^2\varphi^2 + (1+k)\int_{M} u^2\,|\nabla\varphi|^2 \\ &\quad \ge \left(\frac{k}{2}+a+1-\frac{\epsilon\,|b|}{2}\right)\int_{M\setminus D} u^4\varphi^2. \end{aligned}$$

Since $\dfrac{k}{2}+a+1 > 0$ therefore we can choose $\epsilon$ sufficiently small such that $\dfrac{k}{2}+a+1 \ge 2\,|b|\,\epsilon$, then we have a positive constant $C_1$ such that

$$\int_{M\setminus D} u^4\varphi^2 \le C_1\left(\int_{M\setminus D} u^2\varphi^2 + \int_{M\setminus D} u^2\,|\nabla\varphi|^2\right), \tag{11}$$

for any smooth function $\varphi$ compactly supported in $M\setminus D$. We can choose $R_0$ such that $D$ is contained in some geodesic ball $B_{R_0}(p)$. For any positive number $R > 1$ we can choose function $\varphi(x) \in [0, 1]$ such that

$$\varphi(x) = \begin{cases} 0, \text{ on } & B_{R_0}(p); \\ 1, \text{ on } & B_{R_0+R+1}(p)\setminus B_{R_0+1}(p); \\ 0, \text{ on } & M\setminus B_{R_0+2R+1}(p); \end{cases}$$

and $|\nabla\varphi| \le C_2$, where $C_2$ is a constant. From (11) we have

$$\int_{B_{R_0+R+1}(p)\setminus B_{R_0+1}(p)} u^4 \le C_1\left(\int_{M\setminus D} u^2 + C_2^2\int_{M\setminus D} u^2\right).$$

Since $R$ can be arbitrarily large we conclude that $\int_{M\setminus D} u^4 < \infty$. □

If $k > \frac{1}{4},\ a \ge -1$, we can improve Proposition 1 to

**Proposition 2.** *Let $u$ be a nonnegative locally Lipschitz function satisfying (4). Suppose that for some compact domain $D \subset M$, (5) holds for any compactly supported function $f \in C_o^\infty(M\backslash D)$ and $k > \frac{1}{4}$, $a \geq -1$. Then*

$$\int_{M\backslash D} u^2 < +\infty$$

*implies*

$$\int_{M\backslash D} u^5 < +\infty.$$

**Proof.** Under the conditions of the proposition $\frac{k}{2}+a+1 > 0$, from our previous arguments, we can assume that

$$\int_{M\backslash D} u^4 < \infty,$$

and then we have

$$\int_{M\backslash D} u^3 \leq \left(\int_{M\backslash D} u^2\right)^{\frac{1}{2}} \cdot \left(\int_{M\backslash D} u^4\right)^{\frac{1}{2}} < \infty.$$

Now we multiply (4) on both sides by $uf^2$ where $f$ is a smooth function compactly supported in $M\backslash D$. Then we obtain upon integration

$$\int_{M\backslash D} u^2 f^2 \Delta u \geq k \int_{M\backslash D} |\nabla u|^2 uf^2 + a \int_{M\backslash D} u^5 f^2 + b \int_{M\backslash D} u^4 f^2 + c \int_{M\backslash D} u^3 f^2.$$

Integration by parts yields

$$-2 \int_{M\backslash D} uf^2 |\nabla u|^2 - 2 \int_{M\backslash D} u^2 f \langle \nabla u, \nabla f\rangle$$
$$\geq k \int_{M\backslash D} |\nabla u|^2 uf^2 + a \int_{M\backslash D} u^5 f^2 + b \int_{M\backslash D} u^4 f^2 + c \int_{M\backslash D} u^3 f^2. \quad (12)$$

Next we apply the stability inequality (5) to the test function $u^{3/2} f$. Since

$$\nabla \left(u^{3/2} f\right) = \frac{3}{2} u^{1/2} f \nabla u + u^{3/2} \nabla f.$$

therefore

$$\left|\nabla\left(u^{3/2}f\right)\right|^2 = \left(\frac{3}{2}u^{1/2}f\nabla u + u^{3/2}\nabla f\right)^2$$
$$= \frac{9}{4}uf^2\left|\nabla u\right|^2 + u^3\left|\nabla f\right|^2 + 3u^2 f\left\langle\nabla u, \nabla f\right\rangle.$$

Putting this into the stability inequality (5), we have

$$\begin{aligned}\frac{9}{4}\int_{M\setminus D} uf^2\left|\nabla u\right|^2 + \int_{M\setminus D} u^3\left|\nabla f\right|^2 + 3\int_{M\setminus D} u^2 f\left\langle\nabla u, \nabla f\right\rangle \\ \geq \int_{M\setminus D} u^5 f^2 + d\int_{M\setminus D} u^3 f^2.\end{aligned} \tag{13}$$

Adding (12) and (13) gives

$$\int_{M\setminus D} u^3\left|\nabla f\right|^2 + \int_{M\setminus D} u^2 f\left\langle\nabla u, \nabla f\right\rangle \geq \left(k - \frac{1}{4}\right)\int_{M\setminus D}\left|\nabla u\right|^2 uf^2$$
$$+ (a+1)\int_{M\setminus D} u^5 f^2 + b\int_{M\setminus D} u^4 f^2 + (c+d)\int_{M\setminus D} u^3 f^2.$$

Next, we estimate the second term on the left-hand side from above by Cauchy's inequality,

$$\begin{aligned}\int_{M\setminus D} u^3\left|\nabla f\right|^2 + \frac{\varepsilon}{2}\int_{M\setminus D} u\left|\nabla u\right|^2 f^2 + \frac{1}{2\varepsilon}\int_{M\setminus D} u^3\left|\nabla f\right|^2 \\ \geq \left(k - \frac{1}{4}\right)\int_{M\setminus D}\left|\nabla u\right|^2 uf^2 + (a+1)\int_{M\setminus D} u^5 f^2 \\ + b\int_{M\setminus D} u^4 f^2 + (c+d)\int_{M\setminus D} u^3 f^2.\end{aligned}$$

Choosing $\varepsilon = \left(k - \frac{1}{4}\right)/2$, then

$$\left(\frac{1}{k - \frac{1}{4}} + 2\right)\int_{M\setminus D} u^3\left|\nabla f\right|^2 \geq \left(\frac{k - \frac{1}{4}}{2}\right)\int_{M\setminus D}\left|\nabla u\right|^2 uf^2$$
$$+ (a+1)\int_{M\setminus D} u^5 f^2 + b\int u^4 f^2 + (c+d)\int_{M\setminus D} u^3 f^2,$$

which implies

$$-b\int_{M\setminus D} u^4 f^2 - (c+d)\int_{M\setminus D} u^3 f^2 + \left(\frac{1}{k-\frac{1}{4}}+2\right)\int_{M\setminus D} u^3\,|\nabla f|^2$$
$$\geq \left(\frac{k-\frac{1}{4}}{2}\right)\int_{M\setminus D} |\nabla u|^2\, u f^2 + (a+1)\int_{M\setminus D} u^5 f^2.$$

Since all terms on the left-hand side of this inequality are bounded, we have

$$\int_{M\setminus D} |\nabla u|^2\, u f^2 < \infty.$$

Now, we put this term back in the stability inequality (13) with the left-hand side estimated from above by Schwarz's inequality, i.e.

$$\frac{9}{4}\int_{M\setminus D} u f^2\,|\nabla u|^2 + \int_{M\setminus D} u^3\,|\nabla f|^2 + \frac{3}{2}\int_{M\setminus D} u^3\,|\nabla f|^2$$
$$+\frac{3}{2}\int_{M\setminus D} u\,|\nabla u|^2\, f^2 \geq \int_{M\setminus D} u^5 f^2.$$

Using the same test function as in the proof of Proposition 1, we get

$$\int_{M\setminus D} u^5 < \infty.$$

This completes our proof. □

Now we prove another type of result which will be applied in proving Bernstein type theorems.

**Proposition 3.** *Let $u$ be a nonnegative locally Lipschitz function satisfying (4) with $k = 2/n$ for $n \geq 3$. If the first eigenvalue of the operator $-\Delta - u^2 - d$ is nonnegative, i.e. (5) holds for any compactly supported function $f \in C_0^\infty(M)$ and*

$$[a(n-2)+n]t^2 + b(n-2)t + c(n-2) + nd$$

*is bounded below by some positive constant, assume*

$$\int_M u^{\frac{2(n-2)}{n}} < +\infty,$$

*then $u \equiv 0$.*

**Proof.** Let $g = u^{\frac{n-2}{n}}$, then $\nabla g = \frac{n-2}{n} u^{-\frac{2}{n}} \nabla u$ and

$$\begin{aligned}\Delta g &= -\frac{2(n-2)}{n^2} u^{-\frac{n+2}{n}} |\nabla u|^2 + \frac{n-2}{n} u^{-\frac{2}{n}} \Delta u \\ &= \frac{n-2}{n} u^{-\frac{n+2}{n}} [u \Delta u - \frac{2}{n} |\nabla u|^2] \\ &\geq \frac{n-2}{n} u^{-\frac{n+2}{n}} (au^4 + bu^3 + cu^2) \\ &= \frac{n-2}{n} g(ag^{\frac{2n}{n-2}} + bg^{\frac{n}{n-2}} + c).\end{aligned}$$

This inequality is valid whenever $u \neq 0$. Otherwise, since $g$ is locally Lipschitz, we can multiply by $g$ and interpret the inequality

$$g \Delta g \geq \frac{n-2}{n} g^2 (ag^{\frac{2n}{n-2}} + bg^{\frac{n}{n-2}} + c)$$

in the distributional sense. Multiplying by $\varphi^2$ on both sides of this inequality, we obtain

$$\int_M \varphi^2 g \Delta g \geq \int_M \frac{n-2}{n} \varphi^2 g^2 (ag^{\frac{2n}{n-2}} + bg^{\frac{n}{n-2}} + c)$$

where the right-hand side makes sense since $n \geq 3$.

Since

$$\int_M \varphi^2 g \Delta g = -\int_M 2\varphi g \langle \nabla \varphi, \nabla g \rangle - \int_M \varphi^2 |\nabla g|^2 ,$$

we get

$$\int_M \frac{n-2}{n} \varphi^2 g^2 \left( ag^{\frac{2n}{n-2}} + bg^{\frac{n}{n-2}} + c \right) \leq -\int_M 2\varphi g \langle \nabla \varphi, \nabla g \rangle - \int_M \varphi^2 |\nabla g|^2 .$$

The stability condition (5) implies

$$\begin{aligned}\int_M \left( g^{\frac{2n}{n-2}} + d \right) g^2 \varphi^2 &\leq \int_M |\nabla (g\varphi)|^2 \\ &= \int_M g^2 |\nabla \varphi|^2 + 2 \int_M g\varphi \langle \nabla \varphi, \nabla g \rangle + \int_M \varphi^2 |\nabla g|^2.\end{aligned}$$

Adding this to the preceding inequality, we get

$$\int_M \varphi^2 g^2 \left[ \left( \frac{n-2}{n} a + 1 \right) g^{\frac{2n}{n-2}} + \frac{n-2}{n} bg^{\frac{n}{n-2}} + \frac{n-2}{n} c + d \right] \leq \int_M g^2 |\nabla \varphi|^2.$$

We choose the $C^\infty$ function $\varphi$ satisfying:

(1) $0 \leq \varphi \leq 1$,

(2) $\varphi \equiv 1$, on $B(R)$, and $\varphi \equiv 0$, outside $B(2R)$,

(3) $|\nabla \varphi| \leq \frac{2}{R}$.

Then when

$$\left(\frac{n-2}{n}a+1\right)g^{\frac{2n}{n-2}} + \frac{n-2}{n}bg^{\frac{n}{n-2}} + \frac{n-2}{n}c+d$$

is bounded below by a positive constant, there exists a constant $C$ such that

$$\int_{B(R)} g^2 \leq \frac{C}{R^2}\int_{B(2R)} g^2 = \frac{C}{R^2}\int_{B(2R)} u^{\frac{2(n-2)}{n}},$$

Letting $R \to +\infty$ yields $g \equiv 0$. □

## 3 Bernstein type theorems

In this section, we give the applications of Proposition 2 and Proposition 3 to stable and strongly stable constant mean curvature hypersurfaces in space forms.

First, we recall Simons' inequality for traceless second fundamental form. Let $\mathbb{Q}^{n+1}(\kappa)$ be the space form of constant sectional curvature $\kappa$ and $M^n$ a hypersurface in $\mathbb{Q}^{n+1}(\kappa)$ with constant mean curvature $H$. The following is computed in numerous works (see for example, [11] and [1]). Choose an orthonormal frame $\{e_1, \cdots, e_n\}$ which diagonalizes $\phi$ at each fixed point on $M^n$, i.e. $\phi e_i = \mu_i e_i$, and let $\nabla$ be the induced connection on $M^n$. Then we can write [11, p.198],

$$\frac{1}{2}\Delta |\phi|^2 = \sum_{i,j,l} \phi_{ijl}^2 + \sum_i \mu_i \left(\mathrm{tr}\phi\right)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}\left(\mu_i - \mu_j\right)^2,$$

where $\phi_{ijl}$ are components of the covariant derivative of the tensor $\phi$, and $R_{ijij}$ is the sectional curvature of the plane spanned by $\left\{e_i, e_j\right\}$. By Gauss formula, we conclude that

$$\begin{aligned}\frac{1}{2}\sum_{i,j} R_{ijij}\left(\mu_i - \mu_j\right)^2 &= \frac{1}{2}\sum_{i,j}\left(\kappa + \mu_i\mu_j\right)\left(\mu_i - \mu_j\right)^2 \\ &\quad - \frac{H}{2}\sum_{i,j}\left(\mu_i + \mu_j\right)\left(\mu_i - \mu_j\right) + \frac{H^2}{2}\sum_{i,j}\left(\mu_i - \mu_j\right)^2.\end{aligned}$$

Since $\sum \mu_i = 0$, it is easy to check that

$$\sum_{i,j} \left(\mu_i - \mu_j\right)^2 = 2n\,|\phi|^2\,,$$

$$\sum_{i,j} \left(\mu_i + \mu_j\right)\left(\mu_i - \mu_j\right)^2 = 2n \sum_i \mu_i^3,$$

$$\sum_{i,j} \mu_i \mu_j \left(\mu_i - \mu_j\right)^2 = -2\,|\phi|^4\,.$$

From the above, it follows that

$$\begin{aligned}\frac{1}{2}\Delta\,|\phi|^2 &= |\phi|\,\Delta\,|\phi| + |\nabla\,|\phi||^2 = \\ &= \sum_{i,j,l} \phi_{ijl}^2 - |\phi|^4 - nH \sum_i \mu_i^3 + n\left(H^2 + \kappa\right)|\phi|^2\,.\end{aligned}$$

In this case, it follows from ([9] (2.3), (2.4)) that

$$\sum_{i,j,l} \phi_{ijl}^2 \geq \frac{2}{n}\,|\nabla\,|\phi||^2 + |\nabla\,|\phi||^2\,.$$

By using a lemma of Okumura (see [1] for a proof), we have

$$\sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n\,(n-1)}}\,|\phi|^3\,.$$

So we have finally the following Simons' inequality:

$$|\phi|\,\Delta\,|\phi| \geq \frac{2}{n}\,|\nabla\,|\phi||^2 - |\phi|^4 - \frac{n\,(n-2)}{\sqrt{n\,(n-1)}} H\,|\phi|^3 + n\left(H^2 + \kappa\right)|\phi|^2\,. \quad (14)$$

Now we prove our theorems.

If we write $q\,(x) = \overline{\mathrm{Ric}}\,(\nu, \nu) + nH^2 + |\phi|^2$, then the fact that $M^n$ is strongly stable is equivalent to $\lambda_1\,(L) \geq 0$ where $L = -\Delta - q\,(x)$. Note that when $N^{n+1} = \mathbb{Q}^{n+1}(\kappa)$, $q\,(x) = n\left(H^2 + \kappa\right) + |\phi|^2$. From Proposition 2 we have the following higher integrability result.

**Theorem 4.** *Let $M^n$ $(n \leq 7)$ be a complete noncompact stable hypersurface in $\mathbb{Q}^{n+1}(\kappa)$ with constant mean curvature $H \geq -\kappa$. If*

$$\int_M |\phi|^2 < \infty$$

*then*

$$\int_M |\phi|^5 < \infty.$$

**Proof.** From the definition of stability, the index of $M$ is at most 1. We know from a result of Fischer-Colbrie [13] that if $M$ has finite index, then it is strongly stable outside a compact set, i.e. we have a compact set $D \subset M$ such that

$$\int_{M\setminus D} |\nabla f|^2 \, dv \geq \int_{M\setminus D} \left(n(H^2+\kappa)+|\phi|^2\right) f^2 dv$$

for all smooth functions $f$ compactly supported in $M\setminus D$. Since $H^2 \geq -\kappa$ and $|\phi|$ satisfies Simon's inequality (4), it is easy to see that the conditions of Proposition 2 is satisfied when $n \leq 7$. Thus the conclusion follows. $\square$

**Remark 1.** It should be remarked that we have actually proved that this theorem is true when $M$ has finite index.

Now we use Theorem 4 to prove Theorem 2 and 3.

**Proof of Theorem 3.** When $n = 3, 4, 5$, we know from Theorem 4 that $\int |\phi|^2 < +\infty$, $\int |\phi|^5 < \infty$, which also implies $\int |\phi|^3 < \infty$ and $\int |\phi|^4 < \infty$ by the Cauchy-Schwarz inequality. Next we apply Theorem 1.1 in do Carmo-Cheung-Santos [7] for the dimensions 3, 4, 5 to show that $M$ is compact. Finally a theorem in Barabosa-do Carmo-Eschenburg [5] shows that the hypersurfaces are indeed geodesic spheres. $\square$

Similarly we can give

**Proof of Theorem 2.** When $H > 0$ from Theorem 1.1 in do Carmo-Cheung-Santos [7] for the dimensions 3, 4, 5 we know that $M$ is compact. The same theorem in Barabosa-do Carmo-Eschenburg [5] says that the hypersurfaces are round spheres. $\square$

**Remark 2.** When $n = 2$ the conclusion is contained in Silveira's theorem [16]. In this case, the assumption $\int |\phi|^2 < \infty$ is not necessary as it follows from the stability assumption thanks to Huber's theorem. On the other hand, this assumption indeed implies that either $H = 1$ or the constant mean curvature hypersurface is compact. In higher dimensional cases, it is not known whether the stability assumption implies finiteness of $\int |\phi|^2$ .

The following theorem is a straightforward application of Proposition 3.

**Theorem 5.** $\mathbb{H}^{n+1}(-1)$ *does not admit any complete strongly stable hypersurfaces $M$ with constant mean curvature $H$, satisfying*

$$n^2\left(8n-8-n^2\right)H^2-16(n-1)^2>0$$

*and*

$$\int_M |\phi|^{2\left(1-\frac{2}{n}\right)}<\infty.$$

**Proof.** Since there is no compact strongly stable hypersurface with constant mean curvature $H$ in $\mathbb{H}^{n+1}(-1)$. We suppose on the contrary that there exists a complete noncompact strongly stable hypersurface $M$ with constant mean curvature $H$ in $\mathbb{H}^{n+1}(-1)$ satisfying $n^2\left(8n-8-n^2\right)H^2-16(n-1)^2>0$ and

$$\int_M |\phi|^{2\left(1-\frac{2}{n}\right)}<\infty.$$

Now we verify the assumptions in Proposition 3. Let

$$a=-1,\quad b=-\frac{n(n-2)}{\sqrt{n(n-1)}}H,\quad k=\frac{2}{n}$$

and $c=d=n(H^2-1)$ then the polynomial in Proposition 3 reads

$$2t^2+b(n-2)t+2n(n-1)\left(H^2-1\right).$$

To guarantee that this polynomial is bounded from below by some positive constant, we only need to consider the term $16n(n-1)(H^2-1)-b^2(n-2)^2$. Since

$$\begin{aligned}16n(n-1)(H^2-1)-b^2(n-2)^2&=\frac{n}{n-1}[16(n-1)^2(H^2-1)-(n-2)^4H^2]\\&=\frac{n}{n-1}[n^2(8n-8-n^2)H^2-16(n-1)^2]>0,\end{aligned}$$

we have that $M$ is totally umbilic. Since there is no totally umbilic hypersurfaces satisfying the condition in the theorem, the proof is complete. □

**Acknowledgments.** This work was partially done while the second author was visiting the Department of Applied Mathematics, The Hong Kong Polytechnic University. He wishes to thank this institution for the financial support and hospitality. We want to thank the kind anonymous referee for many suggestions which improved our paper.

## References


[1] H. Alencar and M. do Carmo, Hypersurfaces with constant mean curvature in spheres, Proc. Amer. Math. Soc. **120** (1994), no. 4, 1223–1229.

[2] H. Alencar and M. do Carmo, *Hypersurfaces with constant mean curvature in space forms,* An. Acad. Bras. Ci. **66** (1994), 265–274.

[3] F. Almgren Jr., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. Math. **84** (1966), 277–292.

[4] J. L. Barbosa and P. Berard, *Eigenvalue and "twisted" eigenvalue problems. Applications to CMC surfaces,* J. Math. Pures Appl. (9) **79** (2000), 427–450.

[5] J. L. Barbosa, M. do Carmo and J. Eschenburg, *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds,* Math. Z. **197** (1988), no. 1, 123–138.

[6] E. Bombieri, E. de Giorgi and E. Giusti, *Minimal cones and the Bernstein problem,* Invent. Math. **7** (1969), 243–268.

[7] M. do Carmo, L.F. Cheung and W. Santos, *On the compactness of constant mean curvature hypersurfaces with finite total curvature,* Arch. Math. **73** (1999), no. 3, 216–222.

[8] M. do Carmo and C. K. Peng, *Stable complete minimal surfaces in $R^3$ are planes,* Bull. Amer. Math. Soc. **1** (1979), 903–906.

[9] M. do Carmo and C. K. Peng, *Stable complete minimal hypersurfaces,* Proc. 1980 Beijing Symp. (eds.) S. S. Chern, W. Wu, Gordon and Breach Science Pub. 1349–1358.

[10] M. do Carmo and D. Zhou, *Bernstein–type theorems in hypersurfaces with constant mean curvature,* An. Acad. Brasil Ci. **72** (2000), no. 3, 301–310.

[11] S. Y. Cheng and S. T. Yau, *Hypersurfaces with constant scalar curvature,* Math. Ann. **225** (1977), no. 3, 195–204.

[12] E. De Giorgi, *Una estensione del teorema di Bernstein*, Ann. Scuola Nor. Sup. pisa **19** (1965), 79–85.

[13] D. Fischer-Colbrie, *On complete minimal surfaces with finite Morse index in three manifolds,* Invent. Math. **82** (1985), 121–132.

[14] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in* 3*-manifolds of nonnegative scalar curvature,* Comm. Pure Appl. Math. **33** (1980), no. 2, 199–211.

[15] W. Fleming, *On the oriented Plateau problem,* Rend. Circ. Mat. Palerino **11** (1962), 69–90.

[16] A. da Silveira, *Stability of complete noncompact surfaces with constant mean curvature,* Math. Ann. **277** (1987), no. 4, 629–638.

[17] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. Math.(2) **88**(1968), 62–105.

**Leung-fu Cheung**
Department of Mathematics
The Chinese University of Hong Kong
Shatin, New Territories
HONG KONG

E-mail: lfcheung@math.cuhk.edu.hk

**Detang Zhou**
Departamento de Geometria
Insitituto de Matematica
Universidade Federal Fluminense – UFF
Centro, 24020-140 Niterói, RJ
BRAZIL

E-mail: zhou@mat.uff.br