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The Weierstrass semigroup of a pair of Galois Weierstrass points with prime degree on a curve

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Abstract. We describe the Weierstrass semigroup of a Galois Weierstrass point with prime degree and the Weierstrass semigroup of a pair of Galois Weierstrass points with prime degree, where a *Galois Weierstrass point with degree n* means a total ramification point of a cyclic covering of the projective line of degree n.

Keywords: Galois Weierstrass point, Weierstrass semigroup of a point, Weierstrass semigroup of a pair of points.

Mathematical subject classification: Primary: 14H55; Secondary: 14H30, 14H45.

1 Introduction

Let $\mathbb{N}_0$ be the additive semigroup of non-negative integers. A subsemigroup H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \backslash H$ of H in $\mathbb{N}_0$ is finite. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H. A numerical semigroup H is called an *n-semigroup* if the least positive integer in H is n. Let C be a complete nonsingular irreducible curve of genus $g \geq 2$ over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. Let $\mathbb{K}(C)$ be the field of rational functions on C. For a point P of C, we set

$$H(P) := \{\alpha \in \mathbb{N}_0 | \text{ there exists } f \in \mathbb{K}(C) \text{ with } (f)_\infty = \alpha P\},$$

which is called the *Weierstrass semigroup of the point P*. We note that $H(P)$ is a numerical semigroup of genus g. An integer n is called the *first non-gap* of P

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if $H(P)$ is an n-semigroup. For distinct points P and Q of C, we set

$$H(P,Q) := \{(\alpha,\beta) \in \mathbb{N}_0 \times \mathbb{N}_0 | \quad \text{there exists} \quad f \in \mathbb{K}(C) \text{ with } (f)_\infty = \alpha P + \beta Q\},$$

which is called the *Weierstrass semigroup of the pair* (P, Q) *of points*. If C is a hyperelliptic curve of genus $g \geq 2$ and P is its point, then the semigroup $H(P)$ is well-known. Moreover, if P and Q are distinct points of the hyperelliptic curve C, Kim [4] determined the semigroup $H(P,Q)$. If C is a curve of genus $g \leq 7$, then every candidate, i.e., every numerical semigroup of genus $g \leq 7$, appears as the Weierstrass semigroup of a point (for the case $g = 4$ see Lax [3], and for the cases $g = 5, 6, 7$ see Komeda [10]). In the case where C is a non-hyperelliptic curve of genus 3, for all distinct points P and Q of C the semigroup $H(P,Q)$ is determined by Kim-Komeda [6]. If P is a point of a curve with first non-gap $a \leq 5$, then every candidate, i.e., every numerical semigroup with first non-gap $a \leq 5$, appears as the Weierstrass semigroup of a point (for the case $a = 3$ see Maclachlan [11] and for the case $a = 4$ (resp. 5) see Komeda [8] (resp. [9])). If P and Q are distinct points whose first non-gaps are 3, then the semigroup $H(P,Q)$ is determined by Kim-Komeda [7].

In Section 2 we give a necessary and sufficient computable condition for a p-semigroup to be the Weierstrass semigroup of a Galois Weierstrass point with degree p where p is a prime number. In Section 3 we determine the Weierstrass semigroup of a pair of Galois Weierstrass points with degree p.

2 The semigroup of a Galois Weierstrass point with prime degree

First we give the notation which we will use in this section. For an n-semigroup H we set $s_i = \mathrm{Min}\{h \in H | h \equiv i \bmod n\}$ for $i = 1, \ldots, n-1$. The set $S(H) = \{n, s_1, \ldots, s_{n-1}\}$ is called the *standard basis for* H. An n-semigroup H is said to be *cyclic* if there is a Galois Weierstrass point P with degree n such that $H(P) = H$. The following result is classical.

Remark 2.1. Any 3-semigroup is cyclic.

Cyclic p-semigroups have the following property:

Remark 2.2. (Morrison-Pinkham [12]). Let p be a prime number. If H is a cyclic p-semigroup, then we have

$$s_i + s_{p-i} = s_j + s_{p-j}, \text{ all } 1 \leq i, j \leq p-1,$$

which are called the *M-P equalities*.

The above condition is a necessary and sufficient condition in the case $p = 5, 7$.

Remark 2.3. If $p = 5$ or 7, then any p-semigroup satisfying the M-P equalities is cyclic (for example, see Morrison-Pinkham [12]).

For an arbitrary prime number p, Theorem 2.1 in Kim-Komeda [5] gives a necessary and sufficient condition for a p-semigroup to be cyclic. Using the theorem we can show that the condition satisfying the M-P equalities is not sufficient for every $p \geq 11$.

Remark 2.4. (Kim-Komeda [5]). If $p \geq 11$, then there exists a non-cyclic p-semigroup satisfying the M-P equalities.

We want to find a strictly additional *computable* condition for a p-semigroup satisfying the M-P equalities to be cyclic. From now on, let p be an odd prime number. We assume that H is a p-semigroup satisfying the M-P equalities. We set

$$S(H) = \{p, pa_l + l \ (l = 1, \dots, p-1)\}.$$

We call

$$\text{(I)} \begin{cases} j_1 + \cdots + j_{\frac{p+1}{2}} = a_1 + a_{p-1} + 1 \\ \displaystyle\sum_{q=1}^{\frac{p+1}{2}} \pi(lq) j_q = pa_l + l \qquad\qquad (l = 1, \dots, \tfrac{p-1}{2}) \end{cases}$$

the *system of linear equations associated to* H, where

$$\pi(x) = x - \left[\frac{x}{p}\right] p$$

for any integer x and [ ] denotes the Gauss symbol. Here $j_1, \dots, j_{\frac{p+1}{2}}$ are the variables. Using Carliz-Olsen [1] we can see that the determinant of the coefficients of (I) is non-zero. Hence (I) has a unique solution. If we can find the solution, we get the necessary and sufficient condition for a p-semigroup satisfying the M-P equalities to be cyclic which will be described in Theorem 2.7.

Proposition 2.5. *Let H be a p-semigroup. Then the following conditions are equivalent.*

i) *H is cyclic.*

ii) $S(H) = \{p\} \cup \left\{ \sum_{q=1}^{p-1} \pi(lq) i_q \;\middle|\; l = 1, 2, \ldots, p-1 \right\}$

for some non-negative integers $i_1, i_2, \ldots, i_{p-1}$ with $\sum_{q=1}^{p-1} q i_q \equiv 1 \bmod p$.

Proof. ii) implies i) by Theorem 2.1 in [5]. We assume that i) holds. Then there is a Galois Weierstrass point P on a curve C such that $H(P) = H$. We may assume that the C is defined by an equation of the form

$$z^p = \prod_{q=1}^{p-1} \prod_{j=1}^{\mu_q} (x - c_{qj})^q \tag{1}$$

where

$$\sum_{q=1}^{p-1} q\mu_q \not\equiv 0 \bmod p$$

and c_{qj}'s are distinct elements of k. Let $f : C \longrightarrow \mathbb{P}^1$ be the morphism corresponding to the inclusion

$$\mathbb{K}(\mathbb{P}^1) = k(x) \subset k(x, z) = \mathbb{K}(C), \text{ i.e., } f(R) = (1 : x(R)).$$

In this case, we may take the point P as $f^{-1}((0 : 1)) = \{P\}$. There exists an integer m with $1 \leq m \leq p - 1$ such that

$$m \sum_{q=1}^{p-1} q\mu_q \equiv 1 \bmod p.$$

For any q with $1 \leq q \leq p - 1$ we have $mq = n_q p + r_q$ for some integers n_q and r_q with $1 \leq r_q \leq p - 1$. Then the m-th power of the equation (1) becomes

$$z^{pm} = \prod_{q=1}^{p-1} \prod_{j=1}^{\mu_q} ((x - c_{qj})^{n_q})^p (x - c_{qj})^{r_q}.$$

Hence, if we set

$$Z = \frac{z^m}{\prod_{q=1}^{p-1} \prod_{j=1}^{\mu_q} (x - c_{qj})^{n_q}},$$

we get

$$Z^p = \prod_{q=1}^{p-1} \prod_{j=1}^{\mu_q} (x - c_{qj})^{r_q}$$

with $\sum_{q=1}^{p-1} r_q \mu_q \equiv 1 \bmod p$. Moreover, we have $\mathbb{K}(C) = k(x, z) = k(x, Z)$, because p is prime. By the proof of Theorem 2.1 in Kim-Komeda [5] we have

$$S(H(P)) = \left\{ p, \sum_{q=1}^{p-1} r_q \mu_q, \ldots, \sum_{q=1}^{p-1} \pi(tr_q)\mu_q, \ldots, \sum_{q=1}^{p-1} \pi((p-1)r_q)\mu_q \right\}$$
$$= S(H).$$

For any $q = 1, 2, \ldots, p-1$ we set $i_{r_q} = \mu_q$. Then we have

$$\sum_{q=1}^{p-1} \pi(tr_q)\mu_q = \sum_{q=1}^{p-1} \pi(tr_q) i_{r_q} = \sum_{q=1}^{p-1} \pi(tq) i_q.$$

Moreover, we get $\sum_{q=1}^{p-1} q i_q \equiv 1 \bmod p$, because

$$\sum_{q=1}^{p-1} r_q \mu_q = \sum_{q=1}^{p-1} r_q i_{r_q} = \sum_{q=1}^{p-1} q i_q. \qquad \square$$

Proposition 2.6. *Let H be a p-semigroup satisfying the M-P equalities. The semigroup H is cyclic if and only if the system of linear equations*

$$\text{(II)} \begin{cases} i_1 + \cdots + i_{p-1} = a_1 + a_{p-1} + 1 \\ \sum_{q=1}^{p-1} \pi(lq) i_q = pa_l + l \qquad (l = 1, \ldots, \frac{p-1}{2}), \end{cases}$$

has a solution $(i_1, \ldots, i_{p-1}) = (i_1^0, \ldots, i_{p-1}^{(0)})$ consisting of non-negative integers.

Proof. Assume that H is cyclic. By Proposition 2.5 we have

$$S(H) = \{p\} \cup \left\{ \sum_{q=1}^{p-1} \pi(lq) i_q^{(0)} \;\middle|\; l = 1, 2, \ldots, p-1 \right\}$$

for some non-negative integers $i_1^{(0)}, i_2^{(0)}, \ldots, i_{p-1}^{(0)}$ with $\sum_{q=1}^{p-1} q i_q^{(0)} \equiv 1 \bmod p$. Hence, we get

$$\sum_{q=1}^{p-1} \pi(lq) i_q^{(0)} \equiv l \bmod p,$$

which implies that

$$\sum_{q=1}^{p-1} \pi(lq) i_q^{(0)} = pa_l + l \bmod p$$

for all l. Since $q + \pi((p-1)q) = p$ for all q, we have

$$\sum_{q=1}^{p-1} \pi((p-1)q) i_q^{(0)} = \sum_{q=1}^{p-1} (p-q) i_q^{(0)}.$$

Thus, we obtain

$$i_1^{(0)} + \cdots + i_{p-1}^{(0)} = a_1 + a_{p-1} + 1.$$

Therefore, the system (II) has a solution consisting of the non-negative integers $i_1^{(0)}, i_2^{(0)}, \ldots, i_{p-1}^{(0)}$.

Assume that (II) has a solution $(i_1 \ldots, i_{p-1}) = (i_1^{(0)} \ldots, i_{p-1}^{(0)})$ consisting of non-negative integers. Since H satisfies the M-P equalities and we have

$$\pi(lq) + \pi((p-l)q) = p \text{ for all } q = 1, \ldots, \frac{p-1}{2},$$

we see that

$$\sum_{q=1}^{p-1} \pi(lq) i_q^{(0)} = pa_l + l \ (l = 1, \ldots, p-1).$$

Thus, we get

$$S(H) = \{p\} \cup \left\{ \sum_{q=1}^{p-1} \pi(lq) i_q^{(0)} \;\middle|\; l = 1, 2, \dots, p-1 \right\}.$$

By Proposition 2.5 H must be cyclic. □

Theorem 2.7. *Let H be a p-semigroup satisfying the M-P equalities. Let*

$$(j_1, \dots, j_{\frac{p+1}{2}}) = (A_1, \dots, A_{\frac{p+1}{2}})$$

be the unique solution of the system (I) *of linear equations associated to H.*

(1) *If there is $t \in \left\{1, \dots, \frac{p+1}{2}\right\}$ such that A_t is not an integer, then H is non-cyclic.*

(2) *If all A_t's are integers, then the following conditions are equivalent:*

(i) *H is cyclic, i.e., there is a Galois Weierstrass point P with degree p such that $H(P) = H$.*

(ii) $\sum_{r \in \mathcal{R}_H} A_r + A_{\frac{p-1}{2}} \geq 0$ *and* $\sum_{r \in \mathcal{R}_H} A_r + A_{\frac{p+1}{2}} \geq 0$ *where*

$$\mathcal{R}_H := \left\{ r \in \left\{1, \dots, \frac{p-3}{2}\right\} \;\middle|\; A_r < 0 \right\}.$$

Proof. (1) Consider the system of linear equations

$$\text{(II)} \begin{cases} i_1 + \cdots + i_{p-1} = a_1 + a_{p-1} + 1 \\ \displaystyle\sum_{q=1}^{p-1} \pi(lq) i_q = p a_l + l & (l = 1, \dots, \frac{p-1}{2}), \end{cases}$$

where $S(H) = \{p, pa_l + l \ (l = 1, \dots, p-1)\}$. By the assumption we get the solutions of (II)

$$\begin{cases} i_1 = A_1 + i_{p-1} \\ i_2 = A_2 + i_{p-2} \\ \cdots\cdots\cdots\cdots \\ i_{\frac{p-3}{2}} = A_{\frac{p-3}{2}} + i_{\frac{p+3}{2}} \\ i_{\frac{p-1}{2}} = A_{\frac{p-1}{2}} - i_{\frac{p+3}{2}} - \cdots - i_{p-2} - i_{p-1} \\ i_{\frac{p+1}{2}} = A_{\frac{p+1}{2}} - i_{\frac{p+3}{2}} - \cdots - i_{p-2} - i_{p-1}. \end{cases}$$

Assume that there exists $t \in \left\{1, \ldots, \frac{p+1}{2}\right\}$ such that A_t is not an integer. If H were cyclic, then some solution $(i_1, \ldots, i_{p-1})$ must consist of integers by Proposition 2.6. But by the expression of the solutions, i_t is not an integer. This is a contradiction.

(2) Assume that all A_t's are integers. First we prove that (i) implies (ii). Assume that we had

$$\sum_{r \in \mathcal{R}_H} A_r + A_{\frac{p-1}{2}} < 0 \quad \text{or} \quad \sum_{r \in \mathcal{R}_H} A_r + A_{\frac{p+1}{2}} < 0.$$

Since H is cyclic, we get a solution $(i_1, \ldots, i_{p-1})$ of (II) consisting of non-negative integers. For any $r \in \left\{1, \ldots, \frac{p-3}{2}\right\}$ we have $i_r = A_r + i_{p-r} \geq 0$, which implies that $i_{p-r} \geq -A_r$. If $\sum_{r \in \mathcal{R}_H} A_r + A_{\frac{p-1}{2}} < 0$, then we get

$$\begin{aligned} 0 \leq i_{\frac{p-1}{2}} &= A_{\frac{p-1}{2}} - i_{\frac{p+3}{2}} - \cdots - i_{p-2} - i_{p-1} \\ &\leq A_{\frac{p-1}{2}} - \sum_{r \in \mathcal{R}_H} i_{p-r} \leq A_{\frac{p-1}{2}} + \sum_{r \in \mathcal{R}_H} A_r < 0. \end{aligned}$$

This is a contradiction. If $\sum_{r \in \mathcal{R}_H} A_r + A_{\frac{p+1}{2}} < 0$, the same proof works well.

Next we prove that (ii) implies (i). Let

$$s \in \left\{1, \ldots, \frac{p-3}{2}\right\}$$

such that $s \notin \mathcal{R}_H$. We set $i_{p-s} = 0$, which implies that $i_s = A_s + i_{p-s} = A_s \geq 0$. Let $r \in \mathcal{R}_H$. We set $i_{p-r} = -A_r > 0$. Then $i_r = A_r + i_{p-r} = A_r - A_r = 0$. Moreover, we have

$$\begin{aligned} i_{\frac{p-1}{2}} &= A_{\frac{p-1}{2}} - i_{\frac{p+3}{2}} - \cdots - i_{p-2} - i_{p-1} \\ &= A_{\frac{p-1}{2}} - \sum_{r \in \mathcal{R}_H} i_{p-r} \\ &= A_{\frac{p-1}{2}} + \sum_{r \in \mathcal{R}_H} A_r \geq 0. \end{aligned}$$

Similarly we get $i_{\frac{p+1}{2}} = A_{\frac{p+1}{2}} + \sum_{r \in \mathcal{R}_H} A_r \geq 0$. Hence we get $i_q \geq 0$ for all $q = 1, \ldots, p-1$, which implies that H is cyclic. □

Using Theorem 2.7 we can give an example of a cyclic (resp. non-cyclic) semigroup satisfying the M-P equalities.

Example 2.8. Let H be the 11-semigroup with

$$S(H) = \{11, 23, 24, 25, 26, 27, 39, 40, 41, 42, 43\}$$

$$(\text{resp. } \{11, 12, 16, 18, 19, 20, 24, 25, 26, 28, 32\}).$$

Then H satisfies the M-P equalities. Moreover,

$$\begin{cases} j_1 + j_2 + j_3 + j_4 + j_5 + j_6 = 6 \text{ (resp. 4)} \\ j_1 + 2j_2 + 3j_3 + 4j_4 + 5j_5 + 6j_6 = 23 \text{ (resp. 12)} \\ 2j_1 + 4j_2 + 6j_3 + 8j_4 + 10j_5 + j_6 = 24 \\ 3j_1 + 6j_2 + 9j_3 + j_4 + 4j_5 + 7j_6 = 25 \\ 4j_1 + 8j_2 + j_3 + 5j_4 + 9j_5 + 2j_6 = 26 \\ 5j_1 + 10j_2 + 4j_3 + 9j_4 + 3j_5 + 8j_6 = 27 \text{ (resp. 16)} \end{cases}$$

is the system (I) of linear equations associated to H. The unique solution is $(3, -1, 0, 0, 2, 2)$ (resp. $(1, 1, 1, -1, 2, 0)$), which implies that $\mathcal{R}_H = \{2\}$ (resp. $\{4\}$). Hence we get

$$-1 + 2 \geq 0 \quad \text{and } -1 + 2 \geq 0 \text{ (resp. } -1 + 2 \geq 0 \quad \text{and } -1 + 0 < 0),$$

which implies that H is cyclic (resp. non-cyclic) by Theorem 2.7 (2).

3 The semigroup of a pair of Galois Weierstrass points with prime degree

Throughout this section let C be a curve of genus g. We determine the Weierstrass semigroup at a pair of Galois Weierstrass points P, Q with prime degree. First we review the properties of the semigroup $H(P, Q)$.

Remark 3.1. (Kim [4] and Homma [2]). Let P and Q be distinct points of C. Then we have the following:

i) For each $l \in G(P) = \mathbb{N}_0 \backslash H(P)$, the integer $\text{Min}\{\beta \mid (l, \beta) \in H(P, Q)\}$ must be equal to some element in $G(Q) = \mathbb{N}_0 \backslash H(Q)$, say $\sigma(l)$, and this correspondence σ gives a bijection between the sets $G(P)$ and $G(Q)$.

ii) The semigroup $H(P, Q)$ is completely determined by the bijective correspondence σ, i.e.,

$$G(P, Q) = \bigcup_{l \in G(P)} \Big(\{(l, \beta) | \beta = 0, 1, \ldots, \sigma(l) - 1\} \cup \{(\alpha, \sigma(l)) | \alpha = 0, 1, \ldots, l - 1\} \Big)$$

where we set $G(P, Q) = (\mathbb{N}_0 \times \mathbb{N}_0) \setminus H(P, Q)$. Thus, it suffices to determine the graph $\Gamma(P, Q)$ of σ, i.e.,

$$\Gamma(P, Q) = \{(l, \sigma(l)) \mid l \in G(P)\},$$

for describing the semigroup $H(P, Q)$. We call $\Gamma(P, Q)$ the *generating set* for $H(P, Q)$.

Remark 3.2. We can describe the semigroup of a pair of points whose first non-gaps a are 2 (resp. 3) using the generating set (see Kim [4] (resp. Kim-Komeda [7]) for $a = 2$ (resp. 3)).

Let P be a Galois Weierstrass point of degree p on a curve C. By the proof of Proposition 2.5 the curve C can be defined by an equation of the form

$$z^p = \prod_{q=1}^{p-1} \prod_{j=1}^{i_q} (x - c_{qj})^q \tag{2}$$

where

$$\sum_{q=1}^{p-1} q i_q \equiv 1 \bmod p$$

and c_{qj}'s are distinct elements of k. Let $f : C \longrightarrow \mathbb{P}^1$ be the morphism corresponding to the inclusion

$$\mathbb{K}(\mathbb{P}^1) = k(x) \subset k(x, z) = \mathbb{K}(C), \text{ i.e., } f(R) = (1 : x(R)).$$

In this case, we may take the point P as $f^{-1}((0 : 1)) = \{P\}$. By Theorem 2.1 in Kim-Komeda [5] we have

$$S(H(P)) = \left\{ p, \sum_{q=1}^{p-1} q i_q, \ldots, \sum_{q=1}^{p-1} \pi(tq) i_q, \ldots, \sum_{q=1}^{p-1} \pi((p-1)q) i_q \right\}.$$

Using the above curve C and its point P we get our main theorem.

Theorem 3.3. i) *Let P and Q be distinct Galois Weierstrass points with degree p on a curve C of genus g. Assume that $g > (p-1)^2$. Then there exist non-negative integers $i_1, \dots, i_{p-1}$ with $\sum_{q=1}^{p-1} q i_q \equiv 1 \bmod p$ and an integer s with $1 \leq s \leq p-1$ satisfying $i_s > 0$ such that*

$$S(H(P)) = \left\{ p, \sum_{q=1}^{p-1} q i_q, \dots, \sum_{q=1}^{p-1} \pi(tq) i_q, \dots, \sum_{q=1}^{p-1} \pi((p-1)q) i_q \right\},$$

$$S(H(Q)) = \left\{ p, \sum_{q=1}^{p-1} q i_q + p - 1 - s, \dots, \sum_{q=1}^{p-1} \pi(tq) i_q + p - t - \pi(ts), \dots, \right.$$
$$\left. \sum_{q=1}^{p-1} \pi((p-1)q) i_q + 1 - \pi((p-1)s) \right\}$$

and

$$\Gamma(P, Q) = \left\{ \left(\sum_{q=1}^{p-1} \pi(mq) i_q - lp, lp - \pi(ms) \right) \middle| \right.$$
$$\left. 1 \leq l \leq \left[\frac{\sum_{q=1}^{p-1} \pi(mq) i_q}{p} \right], 1 \leq m \leq p-1 \right\}.$$

ii) *Conversely, let $i_1, \dots, i_{p-1}$ be non-negative integers such that*

$$\sum_{q=1}^{p-1} q i_q \equiv 1 \bmod p.$$

Take an integer s with $i_s > 0$. Then we can construct a pair (P, Q) of Galois Weierstrass points with degree p such that $S(H(P))$, $S(H(Q))$ and $\Gamma(P, Q)$ are as in i).

Proof. i) Let C be the curve with the equation (2). We set $f^{-1}((1 : c_{st})) = \{P_{st}\}$. Since the genus of C is larger than $(p-1)^2$, we have $Q = P_{st}$ for some s and t. We transform the variable x by $X = \dfrac{1}{x - c_{st}}$. Then the equation (2) becomes

$$\frac{1}{c} z^p X^{\sum_{q=1}^{p-1} q i_q} = \left(\prod_{q=1, \neq s}^{p-1} \prod_{j=1}^{i_q} (X - c'_{qj})^q \right) \prod_{j=1, \neq t}^{i_s} (X - c'_{sj})^s$$

where $c'_{qj} = \dfrac{1}{c_{qj} - c_{st}}$ and c is some constant. Then we get

$$Z^p = X^{p-1}\left(\prod_{q=1,\neq s}^{p-1}\prod_{j=1}^{i_q}(X - c'_{qj})^q\right)\prod_{j=1,\neq t}^{i_s}(X - c'_{sj})^s,$$

where we set $Z = c^{-\frac{1}{p}}X^{\frac{u}{p}}z$ and $u = \sum_{q=1}^{p-1} qi_q + p - 1$. If $s = p-1$, then we get

$$Z^p = \left(\prod_{q=1}^{p-2}\prod_{j=1}^{i_q}(X - c'_{qj})^q\right)\left(X^{p-1}\prod_{j=1,\neq t}^{i_{p-1}}(X - c'_{p-1j})^{p-1}\right).$$

If $s \neq p-1$, then we obtain

$$Z^p = \left(\prod_{q=1,\neq s}^{p-2}\prod_{j=1}^{i_q}(X - c'_{qj})^q\right)\left(\prod_{j=1,\neq t}^{i_s}(X - c'_{sj})^s\right)\left(X^{p-1}\prod_{j=1}^{i_{p-1}}(X - c'_{p-1j})^{p-1}\right).$$

If $s = p-1$, then

$$S(H(Q)) = S(H(P)) = \{p\} \cup \left\{\sum_{q=1}^{p-1}\pi(tq)i_q \,|\, t = 1, 2, \ldots, p-1\right\}$$

If $s \neq p-1$, then by Theorem 2.1 in Kim-Komeda [5] we have

$$\begin{aligned} S(H(Q)) &= \{p\} \cup \Bigg\{\sum_{q=1,\neq s}^{p-2}\pi(tq)i_q + \pi(ts)(i_s - 1) + \pi(t(p-1))(i_{p-1}+1) \,| \\ &\qquad\qquad t = 1, 2, \ldots, p-1\Bigg\} \\ &= \{p\} \cup \left\{\sum_{q=1}^{p-1}\pi(tq)i_q + \pi(t(p-1)) - \pi(ts) \,|\, t = 1, 2, \ldots, p-1\right\} \\ &= \{p\} \cup \left\{\sum_{q=1}^{p-1}\pi(tq)i_q + p - t - \pi(ts) \,|\, t = 1, 2, \ldots, p-1\right\}. \end{aligned}$$

For any positive integer l and any $m = 1, 2, \ldots, p-1$, consider the divisor

$$\left(\frac{z^m}{(x - c_{st})^l \prod_{q=1}^{p-1}\prod_{j=1}^{i_q}(x - c_{qj})^{[\frac{mq}{p}]}}\right)$$

$$= m\left(\sum_{q=1}^{p-1}\sum_{j=1}^{i_q} qP_{qj} - \sum_{q=1}^{p-1} qi_qP\right) - l(pP_{st} - pP)$$

$$-\left(\sum_{q=1}^{p-1}\sum_{j=1}^{i_q}\left[\frac{mq}{p}\right]pP_{qj} - \sum_{q=1}^{p-1}\left[\frac{mq}{p}\right]pi_qP\right)$$

$$= \sum_{q=1,\neq s}^{p-1}\sum_{j=1}^{i_q}\left(mq - \left[\frac{mq}{p}\right]p\right)P_{qj} + \sum_{j=1,\neq t}^{i_s}\left(ms - \left[\frac{ms}{p}\right]p\right)P_{sj}$$

$$-\left(lp - ms + \left[\frac{ms}{p}\right]p\right)P_{st} - \left(m\sum_{q=1}^{p-1} qi_q - \sum_{q=1}^{p-1}\left[\frac{mq}{p}\right]pi_q - lp\right)P$$

$$= \sum_{q=1,\neq s}^{p-1}\sum_{j=1}^{i_q}\pi(mq)P_{qj} + \sum_{j=1,\neq t}^{i_s}\pi(ms)P_{sj}$$

$$-(lp - \pi(ms))P_{st} - \left(\sum_{q=1}^{p-1}\pi(mq)i_q - lp\right)P.$$

We note that $lp - \pi(ms) > 0$. Moreover, if $l \leq \left[\frac{\sum_{q=1}^{p-1}\pi(mq)i_q}{p}\right]$, then

$$\sum_{q=1}^{p-1}\pi(mq)i_q - lp > 0.$$

Hence, for $1 \leq m \leq p-1$ and $1 \leq l \leq \left[\frac{\sum_{q=1}^{p-1}\pi(mq)i_q}{p}\right]$ we get

$$\left(\sum_{q=1}^{p-1}\pi(mq)i_q - lp, lp - \pi(ms)\right) \in H(P, Q).$$

By Lemma 2 in Homma [2] we get the result.

ii) Using the integers $i_1, \ldots, i_{p-1}$ we construct the curve C with the equation (2) and its point P. Let us take P_{s1} as Q where $f^{-1}((1 : c_{s1})) = \{P_{s1}\}$. Then we get the desired result. □

We give an example of the semigroup of a pair of Galois Weierstrass points such that we can take only one s as in the above theorem.

Example 3.4. Let H be the 11-semigroup with

$$S(H) = \{11, 23, 46, 69, 92, 115, 138, 161, 184, 207, 230\}.$$

It satisfies the M-P equalities. The solution $(A_1, \ldots, A_6)$ of the system (I) associated to H is $(23, 0, 0, 0, 0, 0)$, which implies that $\mathcal{R}_H = \emptyset$. Hence H is cyclic. The solutions of the system (II) in the proof of Theorem 2.7 (1) are

$$i_1 = 23 + i_{10}, \quad i_2 = i_9, \quad i_3 = i_8, \quad i_4 = i_7,$$

$$i_5 = -i_7 - i_8 - i_9 - i_{10}, \quad i_6 = -i_7 - i_8 - i_9 - i_{10},$$

where i_7, i_8, i_9 and i_{10} are arbitrary. If $i_1, i_2, \ldots, i_{10}$ are non-negative, then we must have $i_q = 0$ for all $q = 2, 3, \ldots, 10$. Thus, $(23, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ is only one solution of (II) consisting of non-negative integers, which means that $i_s > 0$ implies $s = 1$. By Theorem 3.3 ii) we can construct Galois Weierstrass points P and Q such that

$$H(P) = H, \ S(H(Q)) = \{11, 32, 53, 74, 95, 116, 137, 158, 179, 200, 221\}$$

and

$$\Gamma(P, Q) = \{(23m - 11l, 11l - m) \mid 1 \leq l \leq 2m, 1 \leq m \leq 10\}.$$

In fact, let C be the curve defined by

$$z^{11} = \prod_{j=1}^{23}(x - c_{1j}) \quad \text{and} \quad f : C \longrightarrow \mathbb{P}^1$$

the morphism corresponding to the inclusion $k(x) \subset k(x, z)$. Set $\{P\} = f^{-1}((0\colon 1))$ and $\{Q\} = f^{-1}((1\colon c_{11}))$. We get the desired one.

For the following cyclic 11-semigroup H we may take any s with $1 \leq s \leq 10$ as in the above theorem.

Example 3.5. Let H be the 11-semigroup with

$$S(H) = \{11, 89, 90, 146, 92, 93, 149, 150, 96, 152, 153\}.$$

It satisfies the M-P equalities. The solution $(A_1, \ldots, A_6)$ of the system (I) associated to H is $(6, 0, 5, -5, 8, 8)$, which implies that $\mathcal{R}_H = \{4\}$. Since we have

$$A_4 + A_5 = -5 + 8 \geq 0 \text{ and } A_4 + A_6 = -5 + 8 \geq 0,$$

we see that H is cyclic by Theorem 2.7 (2). The solutions of the system (II) in the proof of Theorem 2.7 (1) are

$$i_1 = 6 + i_{10}, \quad i_2 = i_9, \quad i_3 = 5 + i_8, \quad i_4 = -5 + i_7,$$

$$i_5 = 8 - i_7 - i_8 - i_9 - i_{10}, \quad i_6 = 8 - i_7 - i_8 - i_9 - i_{10},$$

where i_7, i_8, i_9 and i_{10} are arbitrary. For example, $(6, 1, 5, 1, 1, 1, 6, 0, 1, 0)$ and $(7, 0, 6, 0, 1, 1, 5, 1, 0, 1)$ are solutions of (II) consisting of non-negative integers. Therefore for any s we have a solution $(i_1, \dots, i_{10})$ of (II) consisting of non-negative integers such that $i_s > 0$. In this example we set $s = 2$. Namely, let $(i_1, \dots, i_{10}) = (6, 1, 5, 1, 1, 1, 6, 0, 1, 0)$. Then by Theorem 3.3 ii) we can construct Galois Weierstrass points P and Q such that

$$H(P) = H,\ S(H(Q)) = \{11, 97, 95, 148, 91, 89, 153, 151, 94, 147, 145\}$$

and

$$\begin{gathered}\Gamma(P, Q) = \\ \{(89 - 11l, 11l - 2) \mid l = 1, \dots, 8\} \cup \{(90 - 11l, 11l - 4) \mid l = 1, \dots, 8\} \cup \\ \{(146 - 11l, 11l - 6) \mid l = 1, \dots, 13\} \cup \cdots \\ \cdots \cup \{(153 - 11l, 11l - 9) \mid l = 1, \dots, 13\}.\end{gathered}$$

In fact, let C be the curve defined by

$$z^{11} = \prod_{j=1}^{6}(x - c_{1j}) \cdot (x - c_{21})^2 \cdot \prod_{j=1}^{5}(x - c_{3j})^3 \cdot (x - c_{41})^4$$

$$\cdot (x - c_{51})^5 \cdot (x - c_{61})^6 \prod_{j=1}^{6}(x - c_{7j})^7 \cdot (x - c_{91})^9.$$

We denote by $f : C \longrightarrow \mathbb{P}^1$ the morphism corresponding to the inclusion $k(x) \subset k(x, z)$. Set $\{P\} = f^{-1}((0 : 1))$ and $\{Q\} = f^{-1}((1 : c_{21}))$. Then we get the desired one.

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