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Distribution results for lattices in $SL(2, \mathbb{Q}_p)$

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Abstract. In this paper we study the ergodic properties of the linear action of lattices Γ of $SL(2, \mathbb{Q}_p)$ on $\mathbb{Q}_p \times \mathbb{Q}_p$ and distribution results for orbits of Γ. Following Serre, one can define a "geodesic flow" for an associated tree (actually associated to $GL(2, \mathbb{Q}_p)$). The approach we use is based on an extension of this approach to "frame flows" which are a natural compact group extension of the geodesic flow.

Keywords: Equidistribution, $SL(2, \mathbb{Q}_p)$, symbolic dynamics.

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0. Introduction

It is an interesting classical problem to study the linear actions of certain discrete subgroups Γ of $SL(2, \mathbb{R})$ on $\mathbb{R}^2$. A particular useful approach is to relate their dynamical properties to those of the horocycle flow on the quotient $\Gamma\backslash\mathbb{H}^2$. The present authors considered the natural generalizations of these results to discrete groups 2×2 matrices acting on the plane of complex numbers, quaternions or Clifford numbers by studying the relationship with the horospheres for frame flows on manifolds of constant curvature [11]. In this paper we want to consider analogous problems for the non-archimedean local field $\mathbb{Q}_p$ and the linear action

$$\gamma : (x_1, x_2) \mapsto (ax_1 + bx_2, cx_1 + dx_2), \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

for Γ a lattice in $SL(2, \mathbb{Q}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Q}_p, \quad ad - bc = 1 \right\}$. We shall assume that $-I \in \Gamma$, otherwise we can replace Γ by $\Gamma \cup \{-\Gamma\}$. The following result follows easily from ergodicity of horocyclic actions [21, p. 194] on $SL(2, \mathbb{Q}_p)$ using a simple equivariance (cf. Lemma 4.2).

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Theorem A. *The linear action of* Γ *on* $\mathbb{Q}_p \times \mathbb{Q}_p$ *is ergodic with respect to the Haar measure* λ.

Our first distribution result on this action describes orbits of such linear actions.

Theorem B. *Let* f *be a continuous function with compact support on* $\mathbb{Q}_p \times \mathbb{Q}_p$. *Let* $X \neq (0,0)$ *be a point of* $\mathbb{Q}_p \times \mathbb{Q}_p$. *Then, as* $M \to \infty$,

$$\frac{1}{p^M} \sum_{\gamma \in \Gamma, ||\gamma|| \leq p^M} f(\gamma X) \to \frac{1}{\mathrm{vol}(\Gamma \backslash SL(2, \mathbb{Q}_p))} \frac{1}{||X||} \int \frac{f(Y)}{||Y||} d\lambda(Y).$$

When $P\Gamma$ is torsion-free we can also approach the action of Γ through the study of the dynamics of a natural *frame flow* associated to the action, in contrast to the usual algebraic view point in [7], [20] and [17], for example. The *frame flow* is a natural extension of the usual *geodesic flow* associated to $G = PSL(2, \mathbb{Q}_p)$ acting on the quotient space $\Gamma \backslash X$ of the associated tree X [23]. Such flows have already been considered by other authors (cf. [18], [2]). More precisely, let $\sigma : \Sigma \to \Sigma$ be a subshift of finite type representing the geodesic flow. The topological entropy of the subshift is $2 \log p$. Let S be the closed multiplicative subgroup of squares in $\mathcal{O}^\times = \{x \in \mathbb{Z}_p : |x|_p = 0\}$. Let $\Theta : \Sigma \to S$ be a Hölder continuous function. The p-adic frame flow for a lattice Γ corresponds to a simple skew product

$$\begin{aligned} \widehat{\sigma} &: \Sigma \times S \to \Sigma \times S \\ \widehat{\sigma}(x, s) &= (\sigma x, \Theta(x)s). \end{aligned} \tag{0.1}$$

Hyperbolic matrices γ have distinct eigenvalues λ_+, λ_-. If the valuations satisfy $|\lambda_+|_p < |\lambda_-|_p$ then we designate λ_+ to be the maximal eigenvalue. Our second result describes the distribution of maximal eigenvalues. The proof uses this skew product.

Theorem C. *Let* $\Gamma < SL(2, \mathbb{Q}_p)$ *be such that* $P\Gamma$ *is a torsion-free lattice in* $PSL(2, \mathbb{Q}_p)$. *Let* Γ_n *be the set of conjugacy classes of* $\gamma \in P\Gamma \subset PSL(2, \mathbb{Q}_p)$ *with* $|\mathrm{tr}\gamma|_p = -n$. *For each class* $[\gamma] \in \Gamma_n$, *denote by* $\iota([\gamma]) \in S$ *the common value of* $p^n \lambda_\gamma$, *where* λ_γ *denotes the maximal eigenvalue. Squares of eigenvalues of matrices in* Γ *are uniformly distributed in the sense that for any continuous function* ϕ *on* S, *we have:*

$$\lim_{n \to \infty} \frac{n}{p^{2n}} \sum_{[\gamma] \in \Gamma_n} \phi(\iota([\gamma])^2) = \int \phi(s) d\omega(s),$$

where ω is the Haar probability measure on S.

In particular, if $p \neq 2$ then $\iota([\gamma])^2$ has two distinct square roots, which we can associate with $\iota([\pm\gamma])$, and then we can deduce:

$$\lim_{n\to\infty} \frac{n}{p^{2n}} \sum_{[\gamma]\in\Gamma_n} \phi(\iota([\gamma])) = \int \phi(s)d\omega(s).$$

Theorem C can be viewed as a non-archimedean version of the results in [24]. We shall give a proof of this result using the frame flow and techniques from symbolic dynamics. In the particular case that $P\Gamma \subset PSL(2, \mathbb{Q}_p)$ is a congruence subgroup we can use Deligne's solution of the Ramanujan-Petersson conjecture to get the following result.

Corollary 1. *Let $P\Gamma \subset PSL(2, \mathbb{Q}_p)$ be a torsion-free congruence subgroup and let $\theta > \left(\frac{1}{p}\right)^{1/3}$. For any character χ on S,*

$$\frac{n}{p^{2n}} \sum_{[\gamma]\in\Gamma_n} \chi(\iota([\gamma])^2) = \int \chi(s)d\omega(s) + O\left(\theta^n\right). \tag{0.2}$$

Congruence subgroups $\Gamma(N)$ are torsion-free when either N or p is sufficiently large [12, p. 197] or N is equal to a square mod p.

Sections 1 and 2 are mainly preliminary. In sections 3 and 4 we prove Theorems A and B, respectively. In section 6 we describe the frame flow. In section 7 we prove the results on the distribution of orbits (Theorem C). In section 8 we discuss the error term in Corollary 1 and other generalizations.

1 p-adic numbers

We begin by establishing some notation. The p-adic numbers $\mathbb{Q}_p$ can be represented by series $x = \sum_{n=N}^{+\infty} a_n p^n$, where $a_n \in \{0, \dots, p-1\}$. These are precisely the closure of the rational numbers $\mathbb{Q}$ with respect to the norm $||x||_p = p^{-|x|_p}$ associated to the usual valuation $|\cdot|_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{+\infty\}$, i.e. $|x|_p = N$ if $a_N \neq 0$ and $a_n = 0$ for $n < N$, and $|0|_p = +\infty$. Recall that for such non-archimedean metrics we have $|x + y|_p \geq \inf\{|x|_p, |y|_p\}$ and $|xy|_p = |x|_p + |y|_p$ whenever $x, y \in \mathbb{Q}_p$. We denote by $\mathbb{Z}_p = \{x : |x|_p \geq 0\}$ the maximal compact subring consisting of those $x = \sum_{n=0}^{+\infty} a_n p^n \in \mathbb{Q}_p$ which are in the closure of $\mathbb{Z}$ with $|| \cdot ||_p$. Let us denote by $\mathcal{O}^\times = \{x \in \mathbb{Z}_p : |x|_p = 0\}$ the natural multiplicative subgroup of $\mathbb{Z}_p$.

1.1 The matrix groups

We begin by defining the groups $GL(2,\mathbb{Q}_p)$, $SL(2,\mathbb{Q}_p)$ and the most important subgroups.

Definition. We denote the set of invertible 2×2 matrices with entries in $\mathbb{Q}_p$ by

$$GL(2,\mathbb{Q}_p)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : a,b,c,d\in\mathbb{Q}_p, ad-bc\neq 0\right\},$$

and we denote by $SL(2,\mathbb{Q}_p)$ the subgroup of matrices with $ad-bc=1$. The *center* C^* of $GL(2,\mathbb{Q}_p)$ is the set of scalar matrices $C^*=\left\{\left(\begin{smallmatrix} a & 0\\ 0 & a\end{smallmatrix}\right) : a\in\mathbb{Q}_p, a\neq 0\right\}$, and the center of $SL(2,\mathbb{Q}_p)$ is $\{\pm Id\}$. We can consider the two compact subgroups of $GL(2,\mathbb{Q}_p)$ defined by

$$SL(2,\mathbb{Z}_p)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : a,b,c,d\in\mathbb{Z}_p, ad-bc=1\right\},\ \text{and}$$
$$\mathcal{K}=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : a,b,c,d\in\mathbb{Z}_p,\ ad-bc\in\mathcal{O}^\times\right\}.$$

Useful groups of diagonal matrices $\mathcal{M}\subset C\subset\mathcal{Q}\subset\mathcal{D}$ are defined as follows:

$$\mathcal{M}=\left\{\begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix} : a\in\mathcal{O}^\times\right\},$$
$$C=\left\{\begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix} : a\neq 0,\ a\in\mathbb{Q}_p\right\}$$
$$\mathcal{Q}=\left\{\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix} : |a|_p=|d|_p<\infty\right\},$$
$$\mathcal{D}=\left\{\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix} : a,d\neq 0,\ a,d\in\mathbb{Q}_p\right\}.$$

We also define the *Borel subgroup* by

$$\mathcal{B}=\left\{\begin{pmatrix} 1 & b\\ 0 & 1\end{pmatrix} : b\in\mathbb{Q}_p\right\}.$$

We can conveniently denote by $PSL(2,\mathbb{Q}_p)$, $PSL(2,\mathbb{Z}_p)$, $P\mathcal{M}$, PC the various quotients by $\{\pm Id\}$. Finally, we shall write $PGL(2,\mathbb{Q}_p)=GL(2,\mathbb{Q}_p)/C^*$, $P\mathcal{Q}=\mathcal{Q}/C^*$, and $P\mathcal{K}=\mathcal{K}/(C^*\cap\mathcal{K})$. □

Lemma 1.1. *For $p \neq 2$, we have the following natural exact sequence of groups:*

$$1 \to PSL(2,\mathbb{Q}_p) \to PGL(2,\mathbb{Q}_p) \to (\mathbb{Z}/2\mathbb{Z})^2 \to 1.$$

For $p = 2$, we have:

$$1 \to PSL(2,\mathbb{Q}_2) \to PGL(2,\mathbb{Q}_2) \to (\mathbb{Z}/2\mathbb{Z})^3 \to 1.$$

Proof. Recall that S is the group of elements of $\mathcal{O}^\times$ which can be written as a square. First assume that $p \neq 2$. Then, we have the following exact sequence of groups: [1] $1 \to S \to \mathcal{O}^\times \to \mathbb{Z}/2\mathbb{Z} \to 1$, where the factor map $\varepsilon\colon \mathcal{O}^\times \to \mathbb{Z}/2\mathbb{Z}$ satisfies $\varepsilon(x) = 0$ iff $x \in S$. The statement follows, where the factor map $\rho : PGL(2,\mathbb{Q}_p) \to (\mathbb{Z}/2\mathbb{Z})^2$ is given by

$$g \mapsto \rho(g) = \left(|\det g|_p \pmod 2,\ \varepsilon(\det g/|\det g|_p)\right).$$

For $p = 2$, we have the following exact sequence of groups: [2] $1 \to S \to \mathcal{O}^\times \to (\mathbb{Z}/2\mathbb{Z})^2 \to 1$, where the factor map $\varepsilon\colon \mathcal{O}^\times \to (\mathbb{Z}/2\mathbb{Z})^2$ satisfies $\varepsilon(x) = (0,0)$ iff $x \in S$. The factor map $\rho : PGL(2,\mathbb{Q}_p) \to (\mathbb{Z}/2\mathbb{Z})^3$ is given by the same formula. □

1.2 Trees and actions

The role of the hyperbolic upper half plane $\mathbb{H}^2$ in the usual archimedean case is taken here by a regular tree X. The construction is elegantly described by Serre [23]. We recall the principal objects.

Vertices. Given any pair of vectors $v_1, v_2 \in \mathbb{Q}_p^2$ associate a *lattice* $L = v_1\mathbb{Z}_p + v_2\mathbb{Z}_p$. We can define an equivalence relation on lattices: $L \sim L'$ if lattices L, L' are homothetically related (i.e., there exists $\alpha \in \mathbb{Q}_p$ such that $L' = \alpha L$). We take the equivalence classes $[L]$ to be the vertices of the tree X.

Edges. Given two vertices (equivalence classes) $[L_1], [L_2]$ we can associate an edge $[L_1] \leftrightarrow [L_2]$ whenever we can find a basis $\{v_1, v_2\}$ for L_1 and $\{\pi v_1, v_2\}$ for L_2, where $\pi = \frac{1}{p}$ is called the *uniformizer*.

[1] A number $x \in \mathcal{O}^\times$, $x = \sum_0^{+\infty} a_n p^n$ is a square iff a_0 is a square in the multiplicative group $\{1,\dots,p-1\}$, and then, $(-1)^{\varepsilon(x)}$ is the classical Legendre symbol $(\frac{a_0}{p})$ of a_0.

[2] A number $x \in \mathcal{O}^\times$, $x = 1 + \sum_1^{+\infty} a_n 2^n$ is a square iff $a_1 = a_2 = 0$, and we can take $\varepsilon(x) = (a_1, a_2)$.

Lemma 1.2. [23, p. 70]. *X is a homogeneous tree, with every vertex having $(p+1)$-edges.*

There is a natural action $GL(2, \mathbb{Q}_p) \times X \mapsto X$ on the tree given by $\gamma[v_1\mathbb{Z}_p + v_2\mathbb{Z}_p] = [(\gamma v_1)\mathbb{Z}_p + (\gamma v_2)\mathbb{Z}_p]$. In particular, we see that C^* acts trivially on X and we recall the following result.

Lemma 1.3 [23, Exercise 1 (b) p. 78]. *The group $GL(2, \mathbb{Q}_p)$ acts on X by tree automorphisms and the representation of $GL(2, \mathbb{Q}_p)/C^*$ is injective in $Aut(X)$.*

We also have the following straightforward result.

Lemma 1.4. *The action of $GL(2, \mathbb{Q}_p)$ on both the vertices and the edges of X is transitive.*

Fix $x_0 = e_1\mathbb{Z}_p + e_2\mathbb{Z}_p$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The stabilizer of the vertex x_0 is the set of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $L_0 \sim v_1\mathbb{Z}_p + v_2\mathbb{Z}_p$, where $v_1 = (a, c), v_2 = (b, d)$. This is precisely the group $C^*\mathcal{K}$. By contrast the action of $SL(2, \mathbb{Q}_p)$ on X is not transitive. The tree X is endowed with the usual distance $d(\cdot, \cdot)$, with neighbouring vertices having distance 1. Recall:

Lemma 1.5 [23, p. 75]. *For any $\gamma \in SL(2, \mathbb{Q}_p)$ and any vertex $x \in X$ $d(\gamma x, x)$ is even.*

Let us call a vertex x odd or even according to the parity of $d(x, x_0)$. From Lemma 1.5, the set of even (odd) vertices is invariant under $SL(2, \mathbb{Q}_p)$. The following is now straightforward:

Lemma 1.6. *The action of $SL(2, \mathbb{Q}_p)$ on both the odd and even vertices of X is transitive.*

2 The geodesic flow and subshifts of finite type

We now want to emphasize geometric viewpoint in anticipation of the introduction of symbolic dynamics later. We can now introduce analogues of familiar geometric concepts from $\mathbb{H}^n$ in the context of the tree X (cf. [18], [2]).

2.1 Geodesics

We begin with a definition.

Definitions. A *geodesic* y in X is a bi-infinite sequence of neighbouring vertices

$$\cdots \leftrightarrow [L'_{-1}] \leftrightarrow [L'_0] \leftrightarrow [L'_1] \leftrightarrow \cdots \leftrightarrow [L'_n] \leftrightarrow \dots$$

without returns. A *straight line* is a geodesic up to the place of the origin.

Let Y denote the space of geodesics in X. There is a natural induced action of $GL(2,\mathbb{Q}_p)$ on Y and on the space of straight lines. Straight lines are in one-to-one correspondence with ordered pairs of distinct points in the projective space $P^1(\mathbb{Q}_p)$ [23, p. 72]. [3] Given a geodesic y, we write $(y_{+\infty}, y_{-\infty})$ for the associated pair of endpoints in $P^1(\mathbb{Q}_p)$. The associated action of $GL(2,\mathbb{Q}_p)$ on $P^1(\mathbb{Q}_p) \times P^1(\mathbb{Q}_p)$ is the projective action given by

$$\gamma(y_{+\infty}, y_{-\infty}) = \left(\frac{ay_{+\infty}+b}{cy_{+\infty}+d}, \frac{ay_{-\infty}+b}{cy_{-\infty}+d}\right).$$

The following lemmas are now straightforward:

Lemma 2.1. *The mapping* $\varphi\colon GL(2,\mathbb{Q}_p) \mapsto (P^1(\mathbb{Q}_p) \times P^1(\mathbb{Q}_p))^*$, *where* $(P^1(\mathbb{Q}_p) \times P^1(\mathbb{Q}_p))^*$ *is the space of pairs of distinct points in* $P^1(\mathbb{Q}_p)$, *given by* $\varphi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (\frac{a}{c}, \frac{b}{d})$ *is an equivariance between left multiplication on* $GL(2,\mathbb{Q}_p)$ *and the diagonal projective action on* $(P^1(\mathbb{Q}_p) \times P^1(\mathbb{Q}_p))^*$.

Lemma 2.2. *The action of both* $GL(2,\mathbb{Q}_p)$ *and* $SL(2,\mathbb{Q}_p)$ *on the space of pair of distinct points (corresponding to straight lines) is transitive.*

Lemma 2.1 implies that the action of $GL(2,\mathbb{Q}_p)$ on the space of pair of distinct points is transitive. The stabilizer of the pair $(\infty, 0)$ is precisely the set of diagonal matrices $\mathcal{D} \subset GL(2,\mathbb{Q}_p)$. It follows that the action of $SL(2,\mathbb{Q}_p)$ is transitive on the space of distinct points in $P^1(\mathbb{Q}_p)$ and the stabilizer of the pair $(\infty, 0)$ is the set of Cartan matrices $C = SL(2,\mathbb{Q}_p) \cap \mathcal{D}$.

Notation. Let y_0 be the *reference geodesic* $y_0 = \{x(n), n \in \mathbb{Z}\}$, where $x(n)$ is the vertex associated to the lattice $L_n = v_1\mathbb{Z}_p + v_2\mathbb{Z}_p$, with $v_1 = (1,0)$, $v_2 = (0, \pi^n)$.

The stabilizer of y_0 by the action of $GL(2,\mathbb{Q}_p)$ is the group $\mathcal{Q}$ and we have:

[3] In fact, [23] actually considers unoriented straight paths, unordered pairs of distinct points in the projective space $P^1(\mathbb{Q}_p)$. The stabilizer we consider is of order 2 in the one considered there, the difference corresponding to inversions.

Corollary 2.2.1. *The group $GL(2,\mathbb{Q}_p)$ acts transitively on Y.*

The stabilizer of y_0 by the action of $SL(2,\mathbb{Q}_p)$ is the group $\mathcal{Q}\cap SL(2,\mathbb{Q}_p) = \mathcal{M}$. It follows that the action of $SL(2,\mathbb{Q}_p)$ on the space of geodesics has two orbits, according to whether the origin is odd or even. The *geodesic flow* in the non-archimedean case is the discrete transformation $T : Y \to Y$ which associates to each geodesic a new geodesic obtained by simply shifting all vertices to the left by one step (or equivalently by adding one to the indices). The geodesic flow T exchanges the two orbits of $SL(2,\mathbb{Q}_p)$ on Y, and the square $T^2 : Y \to Y$ preserves each type. Algebraically, Ty_0 is represented by the matrix $\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)$, which commutes to $\mathcal{Q}$. Thus, identifying Y with $GL(2,\mathbb{Q}_p)/\mathcal{Q} = PGL(2,\mathbb{Q}_p)/P\mathcal{Q}$, the right action of $\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)$ induces the geodesic flow on Y.

2.2 The Busemann function

By analogy with the usual upper half-plane we can introduce the following definition (cf. [3]). Given a geodesic $y = \{y(n), n\in\mathbb{Z}\}$, the *Busemann function* associates to a vertex $x\in X$ the number $b_y(x) = \lim_{n\to+\infty}(d(x,y(n)) - d(x_0,y(n)))\in\mathbb{Z}$. It is easily seen that $b_y(x)$ depends only on the endpoint $y_{+\infty}$. A geodesic $y = \{y(n) : n\in\mathbb{Z}\}$ is completely determined by the straight line it belongs to and the value of $b_y(y(0))$.

Lemma 2.3. *Let $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in GL(2,\mathbb{Q}_p)$ then we can write*

$$b_{gy_0}(gx_0) = 2\max(-|a|_p, -|d|_p) + |\det g|_p.$$

In particular, the geodesic $gy_0\in Y$ can be represented by

$$\left(\frac{a}{c},\ \frac{b}{d},\ 2\min(|a|_p,|d|_p) - |\det g|_p\right)\in (P^1(\mathbb{Q}_p)\times P^1(\mathbb{Q}_p))^*\times\mathbb{Z}.$$

Proof. It is convenient to use the KAN decomposition for $GL(2,\mathbb{Q}_p)$, with the compact group $K = \mathcal{K}$, $N = \mathcal{B}$ and A being the set of diagonal matrices $\left\{\begin{pmatrix}p^{k_1} & 0\\ 0 & p^{k_2}\end{pmatrix} : k_1,k_2\in\mathbb{Z}\right\}$. Elements $g\in K$ fix the vertex x_0, so they have no effect on the Busemann function. Similarly, elements $g = \left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)$ leave invariant b_{y_0}, so that it suffices to compute $b_{gy_0}(gx_0)$ for $g = \begin{pmatrix}p^{k_1} & 0\\ 0 & p^{k_2}\end{pmatrix}$. Then,

$$b_{y_0}(gx_0) = k_2 - k_1 = \begin{cases} -2|a|_p + |\det(g)|_p \text{ if } k_1\ge k_2\\ -2|d|_p + |\det(g)|_p \text{ if } k_1\le k_2\end{cases}$$

which agrees with the definition in the statement of the Lemma. □

In the above representation, the geodesic flow consists in subtracting 1 from the $\mathbb{Z}$ component (cf. [3], [2] and [18]).

2.3 Discrete subgroups

Throughout this note, Γ will be a discrete cocompact subgroup of $SL(2, \mathbb{Q}_p)$.

Example 1 (Quaternion groups). Given $u, v \geq 1$, the *quaternion algebra* $D = D(u, v)$ is the four dimensional algebra over the rationals $\mathbb{Q}$ with basis $1, i, j, k$ subject to $i^2 = -u$, $j^2 = -v$ and $k = ij = -ji$. For each prime p, $D(u, v)$ is said to split in $\mathbb{Q}_p$ if the algebra $G(\mathbb{Q}_p) := D \otimes \mathbb{Q}_p$ is isomorphic to the 2×2 matrices over $\mathbb{Q}_p$, where we use the convention that $\mathbb{Q}_\infty = \mathbb{R}$. It is known that $D(u, v)$ splits in $\mathbb{Q}_p$ iff $ux^2 + vy^2 = z^2$ has a non-trivial solution in $\mathbb{Q}_p$. [25, p. 11], [14, p. 79], and this happens for all but finitely many p. For example, the Hamiltonian quaternions $D(1, 1)$ is *definite*, i.e., it doesn't split over $\mathbb{R}$. It also doesn't split over $\mathbb{Q}_2$, but splits over all other $\mathbb{Q}_p$. Given a prime p satisfying $p = 1 \pmod 4$ [4] we can choose $\varepsilon \in \mathbb{Q}_p$ with $\varepsilon^2 = -1$ and set

$$\Gamma' = \left\{ \pm \begin{pmatrix} x_0 + x_1\varepsilon & x_2 + x_3\varepsilon \\ -x_2 + x_3\varepsilon & x_0 - x_1\varepsilon \end{pmatrix} : x_0, x_1, x_2, x_3 \in \mathbb{Z} \right\}.$$

This is a free group in $PGL(2, \mathbb{Q}_p)$ with the $\frac{p+1}{2}$ generators (and $\frac{p+1}{2}$ inverses)

$$S = \left\{ \pm \begin{pmatrix} x_0 + x_1\varepsilon & x_2 + x_3\varepsilon \\ -x_2 + x_3\varepsilon & x_0 - x_1\varepsilon \end{pmatrix} : \begin{matrix} x_0, x_1, x_2, x_3 \in \mathbb{Z}, \sum_{i=0}^{3} x_i^2 = p, \\ x_0 > 0 \text{ and odd }, x_1, x_2, x_3 \text{ even} \end{matrix} \right\}.$$

Moreover, Γ' is a cocompact lattice in $PGL(2, \mathbb{Q}_p)$. Then, since $\det s = p$, for $s \in S$, $\Gamma = \Gamma' \cap PSL(2, \mathbb{Q}_p)$ is a cocompact lattice in $PSL(2, \mathbb{Q}_p)$. As a subgroup of a free group, Γ is automatically free.

Example 2 (Schottky and non-arithmetic groups). There is a simple construction due to Gerritzen and van der Put [9] which gives every torsion-free discrete subgroup. Let $\Gamma \subset PSL(2, \mathbb{Q}_p)$ be a free group generated by hyperbolic elements $\gamma_1, \ldots, \gamma_l$ (i.e., none fixes a vertex). Assume that γ_i fixes a geodesic $(L_n)_{n \in \mathbb{Z}}$ and $\gamma_i L_n = L_{n+m}$ then we denote $A(\gamma_i) = \{x \in X : d(x, L_0) < d(x, L_1)\}$ and $B(\gamma_i) = \{x \in X : d(x, L_{m+1}) < d(x, L_m)\}$. The discrete group Γ is called a *Schottky group* if we can make choices so that

[4] This is a simplifying assumption to avoid working in the quadratic extension of $\mathbb{Q}_p$.

the $2l$ sets $\{A(\gamma_i), B(\gamma_i) : i = 1, \ldots, l\}$ are pairwise disjoint. Any finitely generated torsion-free subgroup of $PSL(2, \mathbb{Q}_p)$ [13, p. 409] is a Schottky group. Moreover, this construction gives rise to non-arithmetic lattices arbitrary close to any given lattice.

In general, we have the following result.

Lemma 2.4 (Ihara's Theorem) [23, p. 82]. *A discrete cocompact torsion-free subgroup Γ of $PSL(2, \mathbb{Q}_p)$ is a free group and $\Gamma\backslash X$ is a finite graph.*

If Γ is torsion-free, observe that $\Gamma \cap PSL(2, \mathbb{Z}_p)$ is compact, discrete and torsion-free, hence trivial. Moreover the intersection of Γ with any bounded group is trivial, therefore Γ cannot identify edges with a common vertex. The graph $\Gamma\backslash X$ is a homogeneous graph with $(p + 1)$ edges at each vertex, and Γ can be realized as the homotopy group of the finite graph $\Gamma\backslash X$. The following result shows that there are many more examples of torsion-free lattices.

Lemma 2.5. *Any discrete cocompact subgroup Γ of $PSL(2, \mathbb{Q}_p)$ contains a torsion-free finite index normal subgroup $\Gamma' \subset \Gamma$.*

Proof. Since $\mathbb{Q}_p$ is a characteristic zero field, Selberg's Lemma gives that Γ contains a torsion-free finite index subgroup Γ' [4, p. 87]. Moreover, the group Γ' must be finitely generated [16, p. 35]. □

The *limit set* Λ is the set of limit points of $\{\Gamma x_0\}$ in $X \cup \partial X$. It consists of the whole boundary ∂X. Observe also that the graph $\Gamma\backslash X$ is connected, since it is the quotient of the connected graph X. Since $\Gamma \subset SL(2, \mathbb{Q}_p)$, the vertices of $\Gamma\backslash X$ are Γ-orbits of either odd or even vertices.

2.4 Subshifts of finite type

Assume for the remainder of this section that Γ is a torsion-free lattice in $PSL(2, \mathbb{Q}_p)$. The group Γ acts on the left on $Y = GL(2, \mathbb{Q}_p)/\mathcal{Q}$. The geodesic flow on the quotient space $\Gamma\backslash Y$ is conjugated to a subshift of finite type:

Proposition 2.6. (cf. [2]). *There exist a subshift of finite type (Σ', σ') and a conjugacy $\Phi' : \Gamma\backslash GL(2, \mathbb{Q}_p)/\mathcal{Q} \mapsto \Sigma'$ such that $\Phi' T = \sigma' \Phi'$.*

It follows that periodic orbits of period n for Σ' are in one-to-one correspondence with periodic orbits of period n for the geodesic flow on $\Gamma\backslash GL(2, \mathbb{Q}_p)/\mathcal{Q}$, that is, with geodesics $y = gy_0 \in Y$ such that there is some $\gamma \in \Gamma$ with

$\gamma g y_0 = g a^n y_0$, where $a = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$. The elements γ associated to the same periodic orbit are conjugated in Γ. In other words, periodic orbits of period n for Σ' are in one-to-one correspondence with conjugacy classes of elements $\gamma \in \Gamma$ such that there exist $g \in GL(2,\mathbb{Q}_p)$ and $q \in \mathcal{Q}$ satisfying $g^{-1}\gamma g = a^n q$.

If the group is torsion-free, the construction is quite natural (cf. [18]). In fact, the quotient space $\Gamma\backslash Y$ can be identified with the space of geodesics of $\Gamma\backslash X$, i.e., the sequences of successive neighbouring points in $\Gamma\backslash X$ without returns. More precisely, when Γ is torsion-free the action on X does not have fixed vertices nor fixed edges, and therefore the cover from X to $\Gamma\backslash X$ is regular. This ensures that paths on $\Gamma\backslash X$ lift uniquely to X, and therefore there is a one-to-one correspondence between lifts of geodesics on X and orbits for the action of Γ on Y. The subshift of finite type of Proposition 2.6 is then built on the set of *oriented* edges of $\Gamma\backslash X$, and there are exactly p entries equal to 1 in each row and each column of the matrix of the subshift. In particular, the topological entropy is $\log p$. We have:

Proposition 2.7. *The geodesic flow T on $\Gamma\backslash Y$ is transitive and has entropy $\log p$. The measure of maximal entropy is ergodic. It corresponds to the Haar measure on $\Gamma\backslash GL(2,\mathbb{Q}_p)/\mathcal{Q}$.*

Proof. Ergodicity of the Haar measure follows from Moore's ergodicity theorem. By a result of [1], the Haar measure is the measure of maximal entropy. Since it has an ergodic measure of full support, the subshift Σ' is transitive. $\square$

We can distinguish between even and odd geodesics according to the parity of their starting point. Each one of the sets of either even or odd geodesics is in one-to-one correspondence with $\Gamma\backslash SL(2,\mathbb{Q}_p)/\mathcal{M}$. The geodesic flow T exchanges even and odd geodesics. Therefore the map T^2 preserves both sets of even and odd geodesics and, since T is ergodic, T^2 is ergodic on each of these two invariant components. We denote by Σ the set of even geodesics and σ the T^2 shift on Σ. Since the parity of the geodesic associated to a sequence depends only on the zeroth coordinate Σ is also a subshift of finite type, and we have:

Corollary 2.7.1. *The geodesic flow is transitive on the space of even geodesics of $\Gamma\backslash X$ and has entropy $2\log p$. The measure of maximal entropy is ergodic. It corresponds to the Haar measure on $\Gamma\backslash SL(2,\mathbb{Q}_p)/\mathcal{M}$.*

In other words, since the geodesic flow is given by the right multiplication by $\begin{pmatrix} \pi^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}\begin{pmatrix} \pi & 0 \\ 0 & p \end{pmatrix}$ or, equivalently, $\begin{pmatrix} \pi & 0 \\ 0 & p \end{pmatrix}$ we have the following.

Corollary 2.7.2. *The right multiplication by the matrix* $\begin{pmatrix} \pi & 0 \\ 0 & p \end{pmatrix}$ *is transitive on the space* $\Gamma\backslash SL(2,\mathbb{Q}_p)/\mathcal{M}$*. It has topological entropy* $2\log p$*. The Haar measure is the measure of maximal entropy and is ergodic.*

For any $k > 0$, there is a one-to-one correspondence between periodic orbits of length k in Σ and conjugacy classes in Γ such that the valuation of the trace is $-k$. Let π_n be the number of conjugacy classes of $\gamma \in \Gamma$ with $|\mathrm{tr}\gamma|_p = -n$. The integers n such that $\pi_n \neq 0$ have g.c.d. 1 (by a simple argument we shall present in section 5). In particular, the following well known result easily follows.

Corollary 2.7.3. *The geodesic flow is mixing on the space of even geodesics of* $\Gamma\backslash X$*. Furthermore, there is a number* $\lambda < 1$ *such that* $\pi_n = \frac{p^{2n}}{n}(1 + O(\lambda^n))$.

Remark. As we shall see in §8, more results on the error terms for π_n can be obtained using representation theory [14].

3 The proof of Theorem A

Let y denote a geodesic in Y.

Definition. The *horocycle* of y is the set of geodesics z such that

$$z_{+\infty} = y_{+\infty} \quad \text{and} \quad b_{y_{+\infty}}(z(0)) = b_{y_{+\infty}}(y(0)).$$

Equivalently, z belongs to the horocycle of y if, and only if, $z(n) = y(n)$ for all n large enough so that the horocycle of y is also the *strong stable manifold* of y under the geodesic flow.

The action of Γ preserves the horocycle relation. The following result is familiar for mixing subshifts of finite type.

Proposition 3.1. *On the space of even geodesics (odd geodesics) of* $\Gamma\backslash X$*, the horocycle relation is minimal and uniquely ergodic. The unique invariant measure is the measure of maximal entropy for the geodesic flow.*

We can rewrite this result in terms of the $SL(2,\mathbb{Q}_p)$ action.

Corollary 3.1.1. *The action of the subgroup $\mathcal{B}$ is minimal and uniquely ergodic on $\Gamma\backslash SL(2,\mathbb{Q}_p)/\mathcal{M}$.*

Let Γ be a subgroup of $SL(2,\mathbb{Q}_p)$ with $-Id \in \Gamma$. We are interested in the ergodic properties of the linear actions of Γ on $\mathbb{Q}_p \times \mathbb{Q}_p$ given by:

$$\gamma : (x_1, x_2) \mapsto (ax_1 + bx_2, cx_1 + dx_2), \text{ where } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

The following lemma is easily shown.

Lemma 3.2. *The map $\mathbb{Q}_p \times \mathbb{Q}_p \to SL(2,\mathbb{Q}_p)/\mathcal{B}$ defined by*

$$(x_1, x_2) \mapsto \begin{pmatrix} x & -x_2^{-1} \\ x_2 & 0 \end{pmatrix} \mathcal{B}, \text{ where } x_2 \neq 0, \text{ and}$$

$$(x_1, 0) \mapsto \begin{pmatrix} x_1 & 0 \\ 0 & x_1^{-1} \end{pmatrix} \mathcal{B}$$

is an equivariance between the linear action on $\mathbb{Q}_p \times \mathbb{Q}_p$ and left multiplication on cosets, e.g.,

$$\gamma \begin{pmatrix} x_1 & -x_2^{-1} \\ x_2 & 0 \end{pmatrix} \mathcal{B} = \begin{pmatrix} ax_1 + bx_2 & * \\ cx_1 + dx_2 & 0 \end{pmatrix} \mathcal{B}, \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Q}_p).$$

We saw that the $\mathcal{B}$-action on the space $\Gamma\backslash SL(2,\mathbb{Q}_p)/\mathcal{M}$ is analogous to the horocycle foliation on the space of geodesics. The $\mathcal{M}$ bundle over $\Gamma\backslash SL(2,\mathbb{Q}_p)/\mathcal{M}$ we are interested in is analogous to the frame bundle over the space of geodesics (i.e. over the unit tangent bundle) and the action is analogous to the action of the strong stable foliation of the frame flow, see §6 for details. Theorem A now follows from Lemma 3.2 and the following result.

Lemma 3.3 [21, p. 194]. *The right $\mathcal{B}$ action on $\Gamma\backslash SL(2,\mathbb{Q}_p)$, with orbits*

$$\left\{ \Gamma x \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{Q}_p \right\}$$

is ergodic (with respect to Haar measure).

4 Proof of Theorem B

4.1 Notations and results

We can define norms on $\mathbb{Q}_p$, $\mathbb{Q}_p \times \mathbb{Q}_p$ and $SL(2,\mathbb{Q}_p)$ as follows.

Proof. Definition. For $t \in \mathbb{Q}_p$, set $||t|| := p^{-|t|_p}$, for $X = (x_1, x_2) \in \mathbb{Q}_p \times \mathbb{Q}_p$, set $||X|| := \max\{||x_1||, ||x_2||\}$, and for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Q}_p)$, set $||\gamma|| := \max\{||a||, ||b||, ||c||, ||d||\}$.

We denote $|\cdot|$ the Haar measure on $\mathbb{Q}_p$ normalized such that $|\mathbb{Z}_p| = 1$. Note that $|\mathcal{O}^\times| = |\{t : ||t|| = 1\}| = |\{t : |t|_p = 0\}| = \frac{p-1}{p}$. [5] Let λ be the measure on $\mathbb{Q}_p \times \mathbb{Q}_p$ which is invariant by the action of $SL(2, \mathbb{Q}_p)$ and such that $\lambda(\mathbb{Z}_p \times \mathbb{Z}_p) = 1$. Theorem B is restated as the following result:

Theorem 4.1. *Let Γ be a discrete lattice in $SL(2, \mathbb{Q}_p)$, f be a continuous function with compact support on $\mathbb{Q}_p \times \mathbb{Q}_p$, $X \neq (0, 0)$ be a point of $\mathbb{Q}_p \times \mathbb{Q}_p$. Then, as $M \to \infty$,*

$$\frac{1}{p^M} \sum_{\gamma \in \Gamma, ||\gamma|| \leq p^M} f(\gamma X) \ \to \ \frac{1}{vol(\Gamma \backslash SL(2, \mathbb{Q}_p))} \frac{1}{||X||} \int \frac{f(Y)}{||Y||} d\lambda(Y).$$

The limit measure in the above theorems is not invariant under the action of $SL(2, \mathbb{Q}_p)$. The following observation explains why this measure is the natural limit measure for distribution problems. Let $d\gamma$ be a Haar measure on $SL(2, \mathbb{Q}_p)$.

Proposition 4.2. *Let f be a continuous function with compact support on $\mathbb{Q}_p \times \mathbb{Q}_p$, $X \neq (0, 0)$ be a point of $\mathbb{Q}_p \times \mathbb{Q}_p$. Then, as $M \to \infty$,*

$$\frac{1}{p^M} \int_{||\gamma|| \leq p^M} f(\gamma X) d\gamma \ \to \ \frac{1}{||X||} \int \frac{f(Y)}{||Y||} d\lambda(Y).$$

Proof. We first observe that we may suppose that the function f is invariant under the action of $SL(2, \mathbb{Z}_p)$. Take $X = (p^{-k}, 0)$ and choose f to be the indicator function of the sphere of norm p^{-l}, where $k, l \in \mathbb{Z}$. We have to compute $\frac{1}{p^M} \int_{||\gamma|| \leq p^M} f(\gamma X) d\gamma$, which is $\frac{1}{p^M}$ times the Haar measure of the set of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that

$$\inf\{|a|_p, |c|_p\} = l + k, \ |b|_p \geq -M, \ |d|_p \geq -M, \text{ and } ad - bc = 1.$$

We can decompose this set into the disjoint union of

$$\{|a|_p = l + k, \ |c|_p \geq l + k, \ |b|_p \geq -M \text{ and } |1 + bc|_p \geq -M + l + k\}$$

[5] On $\mathbb{Z}_p$, the measure $|.|$ is simply the fair dice Bernoulli probability measure on digits. Also, note that $\mathbb{Z}_p$ is the disjoint union of $A_k = \{t : |t|_p = k\}$ for $k \geq 0$, and $|A_k| = p^{-k}|A_0|$.

and of

$$\{|a|_p = r,\ |c|_p = l + k,\ |b|_p \geq -M \text{ and } |b + c^{-1}|_p \geq -M + r - l - k\},$$

for $r > l + k$. Taking the Haar measure of these sets and the limit, we obtain:

$$\begin{aligned}
&p^{-(l+k)} \left(\frac{p-1}{p} + \left(\frac{p-1}{p}\right)^2 \frac{1}{p-1} \right) \\
&= p^{-(l+k)} \left(\frac{p^2 - 1}{p^2} \right) \\
&= \frac{1}{p^k} \frac{1}{p^{-l}}\ \lambda(\{(x, y) : \inf\{|x|_p, |y|_p\} = l\}) \\
&= \frac{1}{||X||} \int \frac{f(Y)}{||Y||} d\lambda(Y),
\end{aligned}$$

as required. □

4.2 Proof of Theorem 4.1

The proof of Theorem 4.1 follows the ideas of the proof of Theorem 4 in [11]. In particular we start by associating to any $X = (x_1, x_2) \in \mathbb{Q}_p \times \mathbb{Q}_p$ a matrix $\Psi(X) \in SL(2, \mathbb{Q}_p)$ such that $X = \Psi(X)(1, 0)$.

Definition. For $X = (x_1, x_2) \in \mathbb{Q}_p \times \mathbb{Q}_p$, $X \neq (0, 0)$, define $\Psi(X) \in SL(2, \mathbb{Q}_p)$ by:

$$\Psi(X) = \begin{cases} \begin{pmatrix} x_1 & 0 \\ x_2 & x_1^{-1} \end{pmatrix} & \text{if} \quad |x_1|_p \leq |x_2|_p, \\ \begin{pmatrix} x_1 & -x_2^{-1} \\ x_2 & 0 \end{pmatrix} & \text{otherwise}. \end{cases}$$

Observe that for all X, the matrix $\Psi(X)$ is a product $\Psi(X) = KA$, where $K \in \mathcal{K}$ and $A = \left(\begin{smallmatrix} \pi^k & 0 \\ 0 & p^k \end{smallmatrix}\right)$, where $k = \inf\{|x_1|_p, |x_2|_p\}$, i.e. $A = \left(\begin{smallmatrix} ||X|| & 0 \\ 0 & ||X||^{-1} \end{smallmatrix}\right)$. To a compact open subset $F \subset \mathbb{Q}_p \times \mathbb{Q}_p \backslash \{(0, 0)\}$, and $m \in \mathbb{Z}$, we associate the set $\tilde{F}_m \subset SL(2, \mathbb{Q}_p)$ defined by

$$\tilde{F}_m = \{\Psi(X) \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right) : X \in F, |s|_p \geq m\}.$$

We have the following two useful lemmas:

Lemma 4.3. *Let Γ be a discrete subgroup of $SL(2, \mathbb{Q}_p)$, and F be a sphere in $\mathbb{Q}_p \times \mathbb{Q}_p$. Then, for m large enough, one can find an open cover of F by sets G^i such that the sets $\gamma \tilde{G}^i_m$, $\gamma \in \Gamma$ are pairwise disjoint.*

Proof. By homogeneity, it suffices to consider the unit sphere S. Observe that $\Psi(S) \subset \mathcal{K}$. Assume first that the group Γ is torsion-free, we know that $\gamma \notin \mathcal{K}$ for all $\gamma \neq Id$. Therefore, for $s, s' \in \mathbb{Z}_p$, $X \in S$, the matrix $\left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right) \gamma \Psi(X) \left(\begin{smallmatrix} 1 & s' \\ 0 & 1 \end{smallmatrix}\right)$ does not belong to $\mathcal{K}$ either. This shows that for all $\gamma \neq Id$, $\gamma \tilde{S}_0$ is disjoint from $\tilde{S}_0$. In general, $\Gamma \cap \mathcal{K}$ is finite, and there are only a finite number of elements γ such that $\tilde{S}_0 \cap \gamma \tilde{S}_0$ is nonempty. The lemma follows. □

Lemma 4.4. *Assume that for $X \in \mathbb{Q}_p \times \mathbb{Q}_p$, $X \neq (0, 0)$ and $\gamma \in \Gamma$, there is $s \in \mathbb{Q}_p$ such that $\Psi(\gamma X) = \gamma \Psi(X) \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$. Then, we have:*

$$||\gamma|| = \max\{||s|| \, ||X|| \, ||\gamma X||, \frac{||X||}{||\gamma X||}, \frac{||\gamma X||}{||X||}\}.$$

Proof. We have two matrices $K, K' \in \mathcal{K}$ such that:

$$K\gamma K' = \begin{pmatrix} \frac{||\gamma X||}{||X||} & s||X||||\gamma X|| \\ 0 & \frac{||X||}{||\gamma X||} \end{pmatrix}.$$

The lemma follows, since $||K\gamma K'|| = ||\gamma||$. (Compare with [11, Lemma 5].) □

Fix k and $F \subset \{(x_1, x_2) : \inf\{|x_1|_p, |x_2|_p\} = k\}$, small enough that there exists m so that the sets $\gamma \tilde{F}_m$ are pairwise disjoint. Each $\gamma \in \Gamma$ such that $\gamma X \in F$ gives rise to a ball $B_\gamma \in \mathbb{Q}_p$ with measure p^{-m} and such that $s \in B_\gamma$ if, and only if, $\gamma \Psi(X) \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right) \in \tilde{F}_m$. By our choice of F, the sets B_γ are disjoint. By Lemma 4.4, providing T is sufficiently large, we have $||\gamma|| \leq T$ if, and only if, $||s|| \leq \frac{T}{||X||p^{-k}}$ for $s \in B_\gamma$. Therefore, we can now write $\frac{1}{p^M}\text{card}\{\gamma \in \Gamma, ||\gamma|| \leq p^M, \gamma X \in F\}$ as:

$$\frac{1}{p^M} \frac{1}{p^{-m}} \left| \left\{ s \ : \ ||s|| \leq \frac{p^{M+k}}{||X||} \text{ and } \Gamma\Psi(X) \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right) \in \tilde{F}_m \right\} \right|.$$

The following distribution result for lattices is the p-adic analogue of the usual unique ergodicity results for $\mathbb{H}^2$.

Proposition 4.5 [21, p. 194]. *Let $\Gamma \subset SL(2, \mathbb{Q}_p)$ be a discrete subgroup which is a lattice:*

(1) *The normalized quotient ν of the Haar measure on $\Gamma\backslash SL(2, \mathbb{Q}_p)$ is the unique N-invariant probability measure;*

(2) *Every N-orbit is uniformly distributed, i.e., for $f \in C^0(\Gamma\backslash SL(2, \mathbb{Q}_p))$ and any $\Gamma x \in \Gamma\backslash SL(2, \mathbb{Q}_p)$ we have*

$$\lim_{T\to\infty} \frac{1}{\mu(B(T))} \int_{B(T)} f\left(\Gamma x \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) d\mu(t) = \int_{\Gamma\backslash SL(2,\mathbb{Q}_p)} f(y) d\nu(y),$$

where $B(T) = \{t \in \mathbb{Q}_p : ||t|| \leq T\}$. [21, p. 181].

Theorem 4.1 on the complement of $(0, 0)$ follows from (2) applied to the indicator function of $\Gamma\tilde{F}_m$. Observe that

$$\nu(\Gamma\tilde{F}_m) = \frac{\lambda(F)p^{-m}}{vol(\Gamma\backslash SL(2, \mathbb{Q}_p))},$$

and that $||Y|| = p^{-k}$ is constant on F. In order to prove Theorem 4.1 we still have to show that the measures $\frac{1}{T}\sum_{\gamma\in\Gamma,||\gamma||\leq T} \mathrm{Dirac}(\gamma X)$ form a tight family of measures on $\mathbb{Q}_p \times \mathbb{Q}_p$ in the neighbourhood of $(0, 0)$. By discreteness of Γ, it suffices to check that the measures $\frac{1}{T}\int_{||\gamma||\leq T} \mathrm{Dirac}(\gamma X)d\gamma$ are tight. This follows from the same computation as in the proof of the Proposition 4.2.

Remark. There are distribution results for higher rank real matrix groups due to Gorodnik, and other related results due to Maucourant.

5 Cross ratios

Although the geodesic flow is relatively easy to analyse, modeled as it is by a subshift of finite type, the natural object for us to consider is a compact abelian group extension described in the introduction. The following definition is useful in the sequel.

Definition. We define the *cross-ratio* of four distinct points $x, y, u, v \in P^1(\mathbb{Q}_p)$ by

$$(x, y, u, v) = \frac{(x-u)(y-v)}{(x-v)(y-u)} \in P^1(\mathbb{Q}_p).$$

The cross-ratio takes all values in $\mathbb{Q}_p^*$ except the value 1. It is sometimes convenient to extend the definition to $(x, x, u, v) = (x, y, u, u) = (x, x, u, u) = 1$. We call the valuation $|(x, y, u, v)|_p$ the *absolute cross ratio*. This has a particularly simple geometric interpretation. If we consider the paths $p_1 = (x, y)$ and $p_2 = (u, v)$ then $|(x, y, u, v)|_p = \{p_1, p_2\}$, where $\{p_1, p_2\}$ is simply the length of the common part of the paths, signed according to orientation. [6, p. 706].

The following simple result is also useful.

Lemma 5.1.

(1) The cross ratio is invariant under the action of $GL(2, \mathbb{Q}_p)$.

(2) For a hyperbolic matrix A the maximal eigenvalue $\lambda(A)$ belongs to $\mathbb{Q}_p$.

(3) For two hyperbolic elements A and B with respective pair of endpoints [6] (a_-, a_+) and (b_-, b_+) in $(P^1(\mathbb{Q}_p) \times P^1(\mathbb{Q}_p))^*$, we can write:

$$(a_+, b_+, a_-, b_-) = \lim_{q\to\infty} \frac{(\lambda(A^q))(\lambda(B^q))}{(\lambda(A^q B^q))}.$$

Proof. The first part follows from

$$gx - gy = \frac{ax+b}{cx+d} - \frac{ay+b}{cy+d} = \frac{(ad-bc)(x-y)}{(cx+d)(cy+d)} = \frac{\det(A)(x-y)}{(cx+d)(cy+d)}, \qquad (5.1)$$

and similar identities. Substituting into the definition of the cross ratio we have that $(x, y, u, v) = (gx, gy, gu, gv)$. For the second part, the eigenvalues $\lambda_+(A)$ and $\lambda_-(A)$ satisfy

$$\lambda_\pm(A) = \frac{\mathrm{tr}(A) \pm \sqrt{(\mathrm{tr}A)^2 - 4\det(A)}}{2}.$$

It suffices to show that $(\mathrm{tr}A)^2 - 4\det(A) \in \mathbb{Q}_p$ is a perfect square. We know that $(\mathrm{tr}A)^2$ is a square and that providing A is a hyperbolic matrix $\det(A)$ has a higher valuation, and thus we see from Hensel's Theorem that the difference is a perfect square. For the third part, the elements A^q and B^q have the same pairs of fixed points (a_-, a_+) and (b_-, b_+). Let x_q be the repelling fixed point for $A^q B^q$ then $y_q = B^q x_q$ is a fixed point for $B^q A^q$ (and, moreover, $A^q y_q = x_q$). In

[6] We use the convention that $+$ denotes an attracting fixed point, and $-$ denotes a repelling fixed point, for the associated action.

particular, $x_q \to b_-$ and $y_q \to a_-$, as $q \to +\infty$. For a given hyperbolic matrix $M = \begin{pmatrix} a(M) & b(M) \\ c(M) & d(M) \end{pmatrix}$, say, with attracting and repelling fixed points m_+ and m_-, the eigenvalues $\lambda_\pm$ satisfy:

$$\lambda_\pm(M) = (c(M)m_\pm + d(M)) \ \text{ and } \ \lambda_+(M)\lambda_-(M) = \det(M).$$

Two applications of (5.1) give

$$\begin{aligned}
(c(A^q)y_q + d(A^q)) &= \det(A^q)(a_+ - x_q)^{-1}(c(A^q)a_+ + d(A^q))^{-1}(a_+ - y_q) \\
&= \det(A^q)(a_+ - x_q)^{-1}\lambda(A^q)^{-1}(a_+ - y_q) \\
(c(B^q)x_q + d(B^q)) &= \det(B^q)(b_+ - y_q)^{-1}(c(B^q)b_+ + d(B^q))^{-1}(b_+ - x_q) \\
&= \det(B^q)(b_+ - y_q)^{-1}\lambda(B^q)^{-1}(b_+ - x_q).
\end{aligned}$$

Moreover, we have the usual relations $\det(A^q B^q) = \det(A^q)\det(B^q)$ and thus

$$\begin{aligned}
\lambda_-(A^q B^q) &= (c(A^q B^q)x_q + d(A^q B^q)) \\
&= (c(A^q)y_q + d(A^q))(c(B^q)x_q + d(B^q)) \\
&= (a_+ - y_q)(\lambda(A^q))^{-1}(a_+ - x_q)^{-1} \\
&\qquad \times (b_+ - x_q)(\lambda(B^q))^{-1}(b_+ - y_q)^{-1}\det(A^q B^q),
\end{aligned}$$

from which we deduce

$$\frac{\lambda(A^q)\lambda(B^q)}{\lambda(A^q B^q)} = \frac{(a_+ - y_q)(b_+ - x_q)}{(a_+ - x_q)(b_+ - y_q)}.$$

Finally,

$$\begin{aligned}
(a_+, b_+, a_-, b_-) &= \lim_{q\to+\infty}(a_+, b_+, y_q, x_q) \\
&= \lim_{q\to+\infty}\frac{(a_+ - y_q)(b_+ - x_q)}{(a_+ - x_q)(b_+ - y_q)} = \lim_{q\to+\infty}\frac{\lambda(A^q)\lambda(B^q)}{\lambda(A^q B^q)},
\end{aligned}$$

which completes the proof. □

To prove Corollary 2.7.3 we deduce easily from Lemma 5.1 that g.c.d. $(||\text{trace } \gamma|_p|, \gamma \in \Gamma)$ has to be 1, since otherwise g.c.d.$(||(x, y, u, v)|_p|)$ over all four distinct points $x, y, u, v \in P^1(\mathbb{Q}_p)$ would be different from 1, a contradiction. □

6 The frame flow

In this section we come to one of the key ingredients in our analysis. We shall consider a compact abelian extension of the geodesic flow, the frame flow, and study it using symbolic dynamics.

Definition. We denote by $F: PGL(2, \mathbb{Q}_p) \to PGL(2, \mathbb{Q}_p)$ multiplication on the right by the matrix $\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)$, i.e., $g \mapsto g \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We define the *frame flow* to be the map $F^2: PGL(2, \mathbb{Q}_p) \to PGL(2, \mathbb{Q}_p)$.

6.1 Coordinates for $PGL(2, \mathbb{Q}_p)$

The following identification will prove useful later.

Definition. Let Z be the set of triples (α, β, τ), where α and β are distinct points in the projective space $P^1(\mathbb{Q}_p)$ and $\tau \in \mathbb{Q}_p^*$, the set of nonzero p-adic numbers. Given any $\xi \in \mathbb{Q}_p$ we define $\Phi_\xi : PGL(2, \mathbb{Q}_p) \mapsto Z$ by

$$\Phi_\xi : g \mapsto (\alpha, \beta, \tau_\xi) := (g(\infty), g(0), -(g(\infty), g(0), \xi, g(1))) , \tag{6.1}$$

provided $\xi \neq g(0), g(\infty)$. In particular,

$$\Phi_\xi \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{a}{c}, \frac{b}{d}, \frac{a - \xi c}{b - \xi d} \right) . \tag{6.2}$$

Lemma 6.1. *The left action of g' in these coordinates is given by:*

$$g'(\alpha, \beta, \tau_\xi) \ = \ (g'(\alpha), g'(\beta), (\alpha, \beta, (g')^{-1}(\xi), \xi) \ \tau_\xi), \tag{6.3}$$

where the first two actions are the projective ones and the third is multiplication in $\mathbb{Q}_p^*$ by $(\alpha, \beta, (g')^{-1}(\xi), \xi)$.

Proof. This is by direct calculation. Suppose g and g' are generic. Observe that by definition

$$\Phi_\xi(g'g) = \left(g'(\alpha), g'(\beta), \tau_\xi(g'g)\right) .$$

To simplify the final term, we write

$$\begin{aligned} \tau_\xi(g'g) &= -(g'g(\infty), g'g(0), \xi, g'g(1)) \\ &= -(g(\infty), g(0), (g')^{-1}(\xi), g(1)) \end{aligned}$$

by Lemma 5.1.(1). By developing the last cross-ratio, we get

$$\tau_\xi(g'g) \ = \ -(g(\infty), g(0), \xi, g(1)) \ (g(\infty), g(0), (g')^{-1}(\xi), \xi),$$

which is formula (6.3). □

In addition, if g is a hyperbolic matrix, and α, β are the fixed points of g at infinity, then we get $g(\alpha, \beta, \tau) = (\alpha, \beta, \lambda^2\tau)$, where λ is the maximal eigenvalue of g, and this holds for any choice of τ_ξ, as long as ξ is not one of the fixed points of g.

6.2 Symbolic dynamics for frame flows

Let Γ be a subgroup of $SL(2, \mathbb{Q}_p)$ containing $-I$. We assume that $P\Gamma$ is torsion-free and consider the symbolic dynamics for the geodesic flow from section 2.4. Given $\underline{e}$ and $\underline{e}'$ let $n \geq 0$ be the least value such that $e_i = e_i'$ for $|i| \leq n$. We can then define a metric on Σ by $d(\underline{e}, \underline{e}') = p^{-n}$, say. The following should be compared with the measurable version in [18, Theorem 2.1].

Proposition 6.2. *Let $P\Gamma$ be a torsion-free lattice of $PGL(2, \mathbb{Q}_p)$. There there is a one-to-one conjugacy $\Psi' \colon \Sigma' \times \mathcal{O}^\times \to P\Gamma \backslash PGL(2, \mathbb{Q}_p)$ and a Lipschitz continuous function $\Theta' : \Sigma' \mapsto \mathcal{O}^\times$such that: $F\Psi'(\underline{x}, u) = \Psi'(T\underline{x}, \Theta'(\underline{x})u)$. Moreover, if $\underline{x}$ is a periodic orbit of period $2k$ in Σ', and $g \in \Gamma$ is some associated element of Γ, then*

$$|\lambda^2|_p = 2|\mathrm{tr} g|_p = -2k \ \text{ and } \ p^{2k}\lambda^2 = \Theta'(T^{2k-1}\underline{x}) \ldots \Theta'(T\underline{x})\Theta'(\underline{x}).$$

Proof. It is easy to see directly from (6.2) that, for any choice of ξ the action of the frame flow in these coordinates takes the simple form $F(\alpha, \beta, \tau_\xi) = (\alpha, \beta, \pi\tau_\xi)$. Recall that the stabilizer of the geodesic σ_0 is the subgroup $\mathcal{Q}$ of diagonal matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ such that $|a|_p = |d|_p < \infty$. The right action of $P\mathcal{Q}$ on $PGL(2, \mathbb{Q}_p)$ is given by multiplication of the third coordinate by $\frac{a}{d}$, an element of $\mathcal{O}^\times$. Moreover the $P\mathcal{Q}$-orbit is the fiber of the projection from $PGL(2, \mathbb{Q}_p)$ to Y. In other words, $(\alpha, \beta, |\tau_\xi|_p)$ defines another parameterization of Y, where the geodesic flow is subtracting 1 from the last coordinate, $PGL(2, \mathbb{Q}_p)$ is represented as $Y \times \mathcal{O}^\times$, frames going to the same geodesic are parameterized by $\tau_\xi/|\tau_\xi|_p$, and the frame flow is represented as the product of the geodesic flow and the identity in the fiber. Observe that all these properties hold even if ξ depends on α and β. From the above description of $PGL(2, \mathbb{Q}_p)$ and the corresponding coding of geodesics, we can represent $P\Gamma\backslash PGL(2, \mathbb{Q}_p)$ by $\Sigma' \times \mathcal{O}^\times$. More precisely, we can choose some fundamental domain in X for $\Gamma\backslash X$. Let $\underline{e} \in \Sigma'$ represent a geodesic and write $y(\underline{e})$ for the image of this geodesic with zeroth coordinate in the fundamental domain. For each $\underline{e}$ we choose $\xi_{\underline{e}} \in \mathbb{Q}_p \cup \{\infty\}$ different from the ends of $y(\underline{e})$. It is easy to see that we need only choose a finite number of such reference points. More precisely, we can consider two nearby geodesics $\underline{e}, \underline{e}' \in \Sigma'$ for which $e_i = e_i'$, for $-n \leq i \leq n$, with n is sufficiently large, then we can choose $\xi_{\underline{e}} = \xi_{\underline{e}'}$. Since we are assuming that $\underline{e}$ and $\underline{e}'$ agree on a very long cylinder then the end $y(\underline{e}')$ in the projective space $P^1(\mathbb{Q}_p)$ (thought as the boundary ∂X of the tree X) must be close to the end of $y(\underline{e})$. (Similarly for $g^{-1}y(T\underline{e})$ and $g^{-1}y(T\underline{e}')$, for $g \in \Gamma_0$.) If the cylinders are long enough, it is possible to find $\xi_{\underline{e}}$ outside all these sets. We thus parameterize the set of frames projecting on the geodesic $y(\underline{e})$ by the map $u_{\xi_{\underline{e}}}$:

$$u_{\xi_{\underline{e}}}(g) = \frac{\tau_{\xi_{\underline{e}}}}{|\tau_{\xi_{\underline{e}}}|_p}, \tag{6.4}$$

where $(g(\infty), g(0), |\tau_{\xi_{\underline{e}}}|_p)$ corresponds to the above parameterization of $\underline{e}$. We thus have a one-to-one representation of $\Gamma\backslash PGL(2, \mathbb{Q}_p)$ by $\Sigma' \times \mathcal{O}^\times$ which fibers over the representation of geodesics by Σ'. The frame flow preserves Γ orbits, commutes with the projection to geodesics, and factors as the geodesic flow. To compute the fiber mapping defined by the frame flow above $\underline{e}$, we successively invert $u_{\xi_{\underline{e}}}$, apply the frame flow, bring back the frame in the fundamental domain if necessary, and apply $u_{\xi_{T\underline{e}}}$ to the new frame. This is precisely multiplication by an element $\Theta'(\underline{e}) \in \mathcal{O}^\times$. Observe that, with this construction, the formula for periodic orbits is automatic. It remains to show that it is possible to choose $\underline{e} \mapsto \xi_{\underline{e}}$ in such a way that Θ' is Lipschitz continuous. Let $\underline{e} \in \Sigma'$ and let $y(\underline{e})$ be the associated geodesic described above. There is a N and two numbers $-N \leq q(\underline{e}) \leq 0$ and $0 < q'(\underline{e}) \leq N$ such that the vertex $\alpha(e_i)$ belongs to the fundamental domain if, and only if, $q \leq i < q'$. We say that $\underline{e}$ is *entering* if $q(\underline{e}) = 0$ and *leaving* if $q'(\underline{e}) = 1$. We denote by $g(\underline{e})$ the element of Γ such that $y(T^{q'(\underline{e})}\underline{e}) = g(\underline{e})y(\underline{e})$. The set Γ_0 of possible such g is finite and q, q' and g depend only on the e_i, for $|i| \leq N$. We can simplify matters by defining $\xi_{\underline{e}}$ for entering elements only. More precisely, we set $\xi_{\underline{e}} = \xi_{T^{-q(\underline{e})}\underline{e}}$, and thus we have $\Theta'(\underline{e}) = 1$ when $1 < q'$. For $q' = 1$, we need to show that the map Θ' is Lipschitz continuous. Recall that given $\underline{e}$ and $\underline{e}'$ with $e_i = e_i'$ for $|i| \leq n$, for any $n \geq N$ we can write $d(\underline{e}, \underline{e}') = p^{-n}$, say. We know that we have the same boundary identification element $g \in \Gamma_0$ for the two sequences. It is clear for the construction that the two pairs of end points $\alpha, \alpha', \beta, \beta' \in \mathbb{Q}_p$ will each be correspondingly close, i.e., $||\alpha - \alpha'|| \leq p^{-n}$ and $||\beta - \beta'|| \leq p^{-n}$. We also have that $|(\alpha,\beta,\gamma,\xi_{\underline{e}})|_p = |(\alpha',\beta',\gamma,\xi_{\underline{e}})|_p$. Finally, we observe that

$$\begin{aligned}||u_{\xi_{\underline{e}}} - u_{\xi_{\underline{e}'}}|| &= \left\|\frac{\tau_{\xi_{\underline{e}}}}{|\tau_{\xi_{\underline{e}}}|_p} - \frac{\tau_{\xi_{\underline{e}'}}}{|\tau_{\xi_{\underline{e}'}}|_p}\right\| \\ &= \left\|\frac{(g(\infty),g(0),\xi_{\underline{e}},g(1))}{|(g(\infty),g(0),\xi_{\underline{e}},g(1))|_p} - \frac{(g(\infty),g(0),\xi_{\underline{e}'},g(1))}{|(g(\infty),g(0),\xi_{\underline{e}'},g(1))|_p}\right\|.\end{aligned}$$

In particular, we see that $||u_{\xi_{\underline{e}}} - u_{\xi_{\underline{e}'}}|| = O(p^{-n})$, from the definition of the cross ratio. □

6.3 The case for $PSL(2,\mathbb{Q}_p)$

There are four $PSL(2,\mathbb{Q}_p)$ orbits (eight if $p=2$) in $P\Gamma\backslash PGL(2,\mathbb{Q}_p)$, which are described by the value of ρ. We already defined the parity of an element of Σ', and set Σ for the set of even sequences. We still have to compute $\varepsilon((\Psi')^{-1}(\underline{e},u))$. We see from (6.4) that:

$$\varepsilon((\Psi')^{-1}(\underline{e},u)) = \varepsilon(u) \;+\; \varepsilon((\alpha-\beta)(\alpha-\xi_{\underline{e}})(\beta-\xi_{\underline{e}})) \quad (\text{mod } 2), \qquad (6.5)$$

where α and β are the ends of the geodesic $y(\underline{e})$.

Proposition 6.3. *There is a one-to-one representation $\Psi\colon P\Gamma\backslash PSL(2,\mathbb{Q}_p) \mapsto \Sigma\times S$ and a Lipschitz continuous function $\Theta:\Sigma\mapsto S$ such that:*

$$\Psi F^2\Psi^{-1}(\underline{e},s) \quad = \quad (\sigma\underline{e},\Theta(\underline{e})s) \quad := \quad \hat{\sigma}(\underline{e},s),$$

where F^2 is the right multiplication by $\begin{pmatrix}\pi & 0\\ 0 & p\end{pmatrix}$ in $PSL(2,\mathbb{Q}_p)$. Furthermore, if $\underline{e}$ is a periodic orbit of period k in Σ, and $g\in\Gamma$ is any element associated to $\underline{e}$, then

$$|\lambda^2|_p = 2|\mathrm{tr} g|_p = -2k \;\; \textit{and} \;\; p^{2k}\lambda^2 = \Theta(\sigma^{k-1}\underline{e})\ldots\Theta(\sigma\underline{e})\Theta(\underline{e}).$$

Proof. We define Ψ^{-1} as the restriction of $(\Psi')^{-1}$ to $\Sigma\times S$. We have to show that we can choose the $\xi_{\underline{e}}$ in the proof of Proposition 6.2 such that all $(\Psi')^{-1}(\underline{e},s)$ are in the same $PSL(2,\mathbb{Q}_p)$ orbit. Let s vary in S. We have $\varepsilon(s)=0$ and by (6.5) we should choose $\xi_{\underline{e}}$ such that $\varepsilon\left((\alpha-\beta)(\alpha-\xi_{\underline{e}})(\beta-\xi_{\underline{e}})\right)=0$ for all $\underline{e}$. Since the cross ratio (α,β,ξ,ξ') takes all possible values, it is possible to choose such a locally constant $\xi_{\underline{e}}$, provided the ends $\alpha(y(\underline{e}))$ and $\beta(y(\underline{e}))$ vary in a small enough sets. Set $\Theta(\underline{e}) = \Theta(T\underline{e})\Theta(\underline{e})$. Then $(\Psi')^{-1}(\underline{e},s)$ and $(\Psi')^{-1}(\sigma\underline{e},\Theta(\underline{e}),s')$ are in the same $PSL(2,\mathbb{Q}_p)$ orbit for all $s,s'\in S$. The result follows from the transitivity of σ on Σ. □

Finally, we remark for completeness that Proposition 2.7 for geodesic flows easily extends to frame flows:

Proposition 6.4. *The frame flow F^2 on $P\Gamma\backslash PSL(2,\mathbb{Q}_p)$ is transitive and has entropy $2\log p$. The measure of maximal entropy is ergodic. It corresponds to the Haar measure on $\Gamma\backslash SL(2,\mathbb{Q}_p)$.*

7 Proof of Theorem C

In this section we shall make use of the symbolic dynamics described in the previous section. The following useful notion is adapted from [10].

Definition. A continuous map $\Theta\colon \Sigma \mapsto S$ is called weakly aperiodic if the relation

$$h(\sigma\underline{x}) = z\zeta(\Theta(\underline{x}))h(\underline{x}),$$

where $z \in \mathbb{C}$, ζ is a character on S, and h is a continuous function on Σ, has only the trivial solution $z = 1$, ζ trivial and h constant.

Proposition 7.1. *The map Θ defined in §6 is weakly aperiodic.*

Proof. Assume not. Then there exist z and ζ such that for any periodic orbit of period k, we have $\zeta(\Theta^{(k)}(\underline{e})) = z^{-k}$, where we write $\Theta^{(k)}(\underline{e})$ for $\Theta(\sigma^{k-1}\underline{x})\ldots\Theta(\sigma\underline{x})\Theta(\underline{x})$. Therefore we have for all $g \in \Gamma$ that

$$\zeta(p^{-|\lambda_g^2|_p}\lambda_g^2) = z^{|\lambda_g|_p},$$

i.e., $\bar{\zeta}(\lambda_g^2) = 1$, where $\bar{\zeta}(u) = \zeta(p^{-|u|_p}u)z^{-|u|_p/2}$ is a character on the group of squares in $\mathbb{Q}_p^*$. By Lemma 5.1 (3) this implies that $\bar{\zeta}((x,y,u,v)^2) = 1$ for all pairwise distinct $x,y,u,v \in \Lambda$. This implies that ζ is trivial as a character on S and $z = \pm 1$. We have $z \neq -1$ because σ is mixing. □

We also have the following useful result.

Lemma 7.2. *We can find a $\frac{1}{2}$-Hölder continuous function $U : \Sigma \to S$ such that $\Theta^+ = (U\circ\sigma)\Theta U^{-1}$ depends only on sequences in the future (i.e., can be realized as a function on the associated one sided shift space Σ^+).*

We can therefore assume (without loss of generality) that Θ is defined on Σ^+. We let $C^\alpha(\Sigma^+)$ denote the Banach space of complex valued α-continuous functions [7] with norms $||w|| = |w|_\infty + |w|_\alpha$, where $|\cdot|_\infty$ is the supremum norm and

$$|w|_\alpha = \sup \left\{ p^{-\alpha n} |w(\underline{x}) - w(\underline{y})| : x_m = y_m \text{ for } m < n \right\}.$$

Let $\chi : S \to \mathbb{C}$ be a character for the compact abelian group S. We can define a *transfer operator* $\mathcal{L} : C^\alpha(\Sigma^+) \to C^\alpha(\Sigma^+)$ by

$$\mathcal{L}_\chi w(\underline{x}) = \frac{1}{p^2} \sum_{\sigma \underline{y} = \underline{x}} \chi([\Theta(\underline{y})]^{-1}) w(\underline{y}).$$

The following result is fairly standard.

Lemma 7.3.

(1) *If* $\chi = \mathbb{I}$ *then the spectrum of* $\mathcal{L}_{\mathbb{I}}$ *has a simple maximal eigenvalue* 1, *and the rest of the spectrum is contained in the disk* $\{z \in \mathbb{C} : |z| < 1\}$

(2) *If* $\chi \neq \mathbb{I}$ *then the spectral radius of* $\mathcal{L}_\chi$ *is strictly less than* 1.

Moreover, the operator has essential spectral radius [8] *at most* p^{-1}.

Proof. Part (1) on the spectrum of $\mathcal{L}_{\mathbb{I}}$ is described in [19]. For part (2), it is easily seen that the spectral radius of $\mathcal{L}_\chi$ is at most 1 and the bound on the essential spectral radius follows from [19]. If $\mathcal{L}_\chi$ has spectral radius 1 then we can find $w \in C^\alpha(\Sigma^+)$ and $0 \leq \theta < 2\pi$ such that $\mathcal{L}_\chi w = e^{i\theta} w$. However, this implies $\chi(\Theta^{-1}(\underline{x})) w(\underline{x}) = e^{i\theta} w(\sigma \underline{x})$, for all $\underline{x} \in \Sigma^+$. By aperiodicity we deduce that w is constant and $\chi = \mathbb{I}$. □

Recall that $\Theta^{(n)}$ is the function on Σ^+ defined by

$$\Theta^{(n)}(\underline{x}) = \Theta(\sigma^{n-1}\underline{x}) \ldots \Theta(\sigma \underline{x}) \Theta(\underline{x}).$$

We can write

$$\mathcal{L}_\chi^n w(\underline{x}) = \frac{1}{p^{2n}} \sum_{\sigma^n \underline{y} = \underline{x}} \chi([\Theta^{(n)}(\underline{y})]^{-1}) w(\underline{y}).$$

By Lemma 7.3, if m denotes the measure of maximal entropy on Σ^+, we have an alternative characterization of the isolated eigenvalues.

[7] In fact, we can assume $\alpha = 1/2$ since Θ was Lipschitz and Lemma 7.2 requires us to half the exponent.

[8] If σ expands the metric by $1/\rho > 1$ then the function acting on α-Holder functions has essential spectral radius is ρ^α [19]. Here $\rho = p^{-2}$ and $\alpha = 1/2$.

Lemma 7.4. *For any character χ on S and any $\theta > \frac{1}{p}$:*

$$\mathcal{L}_\chi^n w(\underline{x}) = \int w dm \int \chi d\omega + \sum_{|\lambda_i| \geq \theta} b_{i,n,\chi}(w, \underline{x}) \lambda_i^n + O\left(\theta^n\right). \tag{7.1}$$

where λ_i are the eigenvalues of $\mathcal{L}_\chi$,

$$|b_{i,n,\chi}(w, \underline{x})| \leq \frac{(n + d_i - 1)!}{(d_i - 1)!} ||w||$$

and $d_i \geq 1$ is the multiplicity of λ_i. If λ_i is simple, we may take $|b_{i,n,\chi}(w, \underline{x})| \leq ||w||$.

Proof. We can write $\mathcal{L}_\chi^n = \sum_i P_i^n + Q_n$, where P_i are eigenprojections associated to eigenvalues λ_i (of modulus not smaller than θ) and $\limsup_{n \to +\infty} ||Q_n||^{1/n} < \theta$. By writing each P_i in Jordan canonical form we can now derive (7.1). □

The eigenvalues λ_i also have a more geometric interpretation. Let Γ_n be the set of conjugacy classes of $\gamma \in \Gamma$ with $|\mathrm{tr}\gamma|_p = -n$. For each class $[\gamma] \in \Gamma_n$, denote by $\iota([\gamma]) \in S$ the common value of $p^n \lambda_\gamma$. There is the following refinement of Corollary 2.7.3:

Proposition 7.5. *For any character χ on S and any $\gamma > \theta > \frac{1}{p}$:*

$$\frac{n}{p^{2n}} \sum_{[\gamma] \in \Gamma_n} \chi(\iota([\gamma])^2) = \int \chi d\omega + \sum_{|\lambda_i| > \theta} c_{i,n,\chi} \lambda_i^n + O\left(\gamma^n\right). \tag{7.2}$$

where λ_i are eigenvalues for $\mathcal{L}_\chi$ with multiplicities d_i and $|c_{i,n,\chi}| \leq \frac{(n+d_i-1)!}{(d_i-1)!}$. If λ_i are simple, we may take $c_{i,n,\chi}$ independent of n. In particular, eigenvalues of matrices in Γ are uniformly distributed in the sense that for any continuous function ϕ on S, we have:

$$\lim_{n \to \infty} \frac{n}{p^{2n}} \sum_{[\gamma] \in \Gamma_n} \phi(\iota([\gamma])^2) = \int \phi(s) d\omega(s),$$

where ω is the Haar probability measure on S.

The proof consists in an application of the arguments from [22] or [19]. The original argument of Ruelle gave an exponential error term, but one with a larger exponent than γ. The optimal estimate was later derived by Haydn, and it is his proof which is followed in [19]. However, subsequently a simplified proof of this result was obtained by Ruelle [22] which is the one that we will briefly describe. The main ingredient is the following lemma.

Lemma 7.6. *For any character χ on S and any $\gamma > \theta > \frac{1}{p}$:*

$$\frac{1}{p^{2n}} \sum_{\sigma^n \underline{x} = \underline{x}} \chi([\Theta^{(n)}(\underline{x})]^{-1}) = \int \chi d\omega + \sum_{|\lambda_i| > \theta} c_{i,n,\chi} \lambda_i^n + O\left(\gamma^n\right). \qquad (7.3)$$

where λ_i are eigenvalues for $\mathcal{L}_\chi$ of multiplicity d_i, and $|c_{i,n,\chi}| \leq \frac{(n+d_i-1)!}{(d_i-1)!}$. If λ_i are simple isolated eigenvalues, we may take $c_{i,n,\chi}$ independent of n.

Proof. For completeness, we briefly sketch the idea of the proof (see Ruelle [22, §2.13] and [19, pp. 157-164]). For each string $i_0, \cdots, i_{l-1}$ for Σ we choose a sequence $x_{i_0,\cdots,i_{l-1}} \in [i_0, \cdots, i_{l-1}]$ whose first symbol is i. One can then estimate

$$\begin{aligned} &\sum_{\sigma^n \underline{e} = \underline{e}} \chi([\Theta^{(n)}(\underline{e})]^{-1}) - \sum_{i_0} (\mathcal{L}_\chi^n I_{[i]})(x_i) \\ &= \sum_{l=1}^{n} \left(\sum_{\underline{i} = i_0,\ldots,i_{l-1}} (\mathcal{L}_\chi^n I_{[i_0,\ldots,i_{l-1}]})(x_{i_0,\ldots,i_{l-1}}) - (\mathcal{L}_\chi^n I_{[i_0,\ldots,i_{l-2}]})(x_{i_0,\ldots,i_{l-2}}) \right) \qquad (7.4) \\ &= O(n\theta^n), \end{aligned}$$

for any $\frac{1}{p} < \theta < 1$, When $\chi = \mathbb{I}$ we have the maximal eigenprojection for $\mathcal{L}_{\mathbb{I}}$ is $\int \cdot dm$. In particular, $\mathcal{L}_{\mathbb{I}}^n I_{[i]} \to \int I_{[i]} dm$. We deduce estimate (7.3) using (7.1) and (7.4). The proof is the same when $\chi \neq \mathbb{I}$, but now the main term is 0, that we interpret as $\int \chi d\omega$. □

To complete the proof of Proposition 7.5, recall that by Proposition 6.3, we have a one-to-one correspondence between periodic orbits of σ of period n and conjugacy classes in Γ_n. Moreover, the value of $\Theta^{(n)}$ is exactly the unitary part $\iota([\gamma])^2$ of the square of the maximal eigenvalue of the conjugacy class $[\gamma]$. It follows that we can estimate, by counting only prime periodic orbits:

$$\left|\frac{1}{n} \sum_{\sigma^n \underline{x} = \underline{x}} \chi([\Theta^{(n)}(\underline{x})]^{-1}) - \sum_{[\gamma] \in \Gamma_n} \chi(\iota([\gamma])^2)\right| = O(p^n).$$

Which completes the proof. □

Remark. In the case that $p \neq 2$, we can remove the squares in the statement of Theorem C. This follows since the eigenvalues for γ and $-\gamma$ appear as the square roots of λ_γ^2.

8 Error terms and the Ramanujan-Petersson Conjecture

In this final section we shall prove Corollary 1. We use a mixture of symbolic dynamics and representation theory, which has parallels with the Lewis-Zagier approach to cusp forms [15].

8.1 Transfer operators

We begin with some comments based on the transfer operator. By Proposition 7.5 we get:

$$\left|\frac{n}{p^{2n}}\sum_{\Gamma_n}\chi(\iota([\gamma])^2) \;-\; \int \chi\, d\omega\right| \;\leq\; C_n(\chi)|\lambda_1(\chi)|^n, \tag{8.1}$$

where $\lambda_1(\chi)$ is the eigenvalue of $\mathcal{L}_\chi$ with highest modulus (second highest when $\chi = \mathbb{I}$), $C_n(\chi)$ depends on the total multiplicities of eigenvalues with the same modulus as $\lambda_1(\chi)$, and grows polynomially in n. We want to consider

$$c(\Gamma) := \sup_\chi |\lambda_1(\chi)|.$$

If we have $c(\Gamma) < 1$, we obtain relation (0.2) for any θ, $c(\Gamma) < \theta < 1$. A related result comes from a consideration of the rate of mixing of the frame flow with respect to the measure $\mu = m \times \omega$ on $\Sigma_+ \times S$.

Lemma 8.1. *Fix $\chi \in S$, choose Lipschitz functions $G, H \in C^\alpha(\Sigma)$ and denote $G_\chi(\underline{x}, s) = G_\chi(\underline{x})\chi(s)$, $H_\chi(\underline{x}, s) = H_\chi(\underline{x})\chi(s)$ and $c_{G,H,\chi}(n) := \int G_\chi \circ (\hat{\sigma})^n H_\chi d\mu$. Then for any $\gamma > \theta > \frac{1}{p}$:*

$$c_{G,H,\chi}(n) = \int G dm \int H dm \int \chi d\omega + \sum_{|\lambda_i|>\theta} \frac{(n+d_i-1)!}{(d_i-1)!} D_i \lambda_i^n + O\left(\gamma^n\right), \tag{8.2}$$

where D_i depends on G and H. If λ_i are simple isolated eigenvalues this becomes

$$c_{G,H,\chi}(n) = \int G dm \int H dm \int \chi d\omega + \sum_{|\lambda_i|>\gamma} D_i \lambda_i^n + O\left(\gamma^n\right),$$

where D_i depends on G and H.

Proof. We can write $c_{G,H,\chi}(n) = \int G_\chi \mathcal{L}_\chi^n H_\chi d\mu$ and then the result follows from Lemma 7.4. □

8.2 Representations

Let us fix $\chi \in \hat{S}$, and consider functions G_j, $j = 1, \ldots, K(\chi)$, such that $G_{j,\chi} = G_j(x)\chi(s)$, $j = 1, \ldots, K(\chi)$ form an orthonormal basis of the sum of the eigenspaces corresponding to eigenvalues with the same modulus as $\lambda_1(\chi)$. By Lemma 8.2 we have that

$$\log|\lambda_1(\chi)| \;=\; \lim_n \frac{1}{n} \sup_{j,k} \log \left| \int G_{j,\chi} \circ (\hat{\sigma})^n \overline{G_{k,\chi}} d\mu \right|,$$

where $\hat{\sigma}$ is the symbolic frame flow. By Proposition 6.3, we get

$$\log c(\Gamma) \;\leq\; \sup_{\chi} \sup_{G,H} \lim_{n\to+\infty} \frac{1}{n} \log \left| \int_{\Gamma\backslash PSL(2,\mathbb{Q}_p)} G_\chi(\Gamma g a^n) \overline{H_\chi(\Gamma g)} dg \right|, \quad (8.3)$$

where a is the diagonal element $a = \begin{pmatrix} p & 0 \\ 0 & \pi \end{pmatrix}$ and the supremum is taken over all $\alpha-$Hölder continuous functions G, H on the associated shift Σ^+ (taken with $\int G = 0$ when $\chi = \mathbb{I}$). We can use representation theory to estimate the Right Hand Side of (8.3).

Let $G = PSL(2, \mathbb{Q}_p)$, and Γ a lattice in G, and let us denote by U_g, $g \in G$, the regular right representation of G on $L^2(\Gamma\backslash G)$. In order to effectively bound (8.3) we have to replace general Hölder functions by K-finite vectors. [9] Let φ be a typical α Hölder continuous function on Σ_+, then we can write

$$\varphi = \sum_{i=0}^{\infty} \varphi_i, \quad (8.4)$$

where $\varphi_0 = \int \varphi$ and

(a) φ_i depends only on coordinates between 0 and i on Σ^+,

(b) $\int \varphi_i = 0$, and

(c) $||\varphi_i||_\infty \leq \|\varphi\|_\alpha p^{-\alpha i}$.

We can suppose we have chosen for our identification of Σ with $\Gamma\backslash PSL(2, \mathbb{Q}_p)/\mathcal{M}$ a fundamental domain of diameter smaller than D, say, and containing the vertex e_0 (which is fixed by $K = PSL(2, \mathbb{Z}_p)$). Fix χ. Let Φ_i be associated to the function $\varphi_i(x)\chi(s)$. The images $K\Phi_i$ of Φ_i under the right action of K are associated to functions on the tree which depend only on the edges at distance to e_0 smaller than $i + D$. Moreover, the orbit of χ under K is finite. It follows that the Φ_i are K-finite, and $\dim(\langle K\Phi_i\rangle) \leq C_\chi(p^2 + 1)^{i+D}$. Recall that we can choose $\alpha = \frac{1}{2}$.

[9] i.e., functions for which the span $\langle K\Phi_i\rangle$ of the K-orbit $K\Phi$ is finite dimensional.

Lemma 8.2. *Assume there is a number $\beta < 1$ such that for the K-finite functions Φ_i, Ψ_j with norm 1 and orthogonal to constants, we have:*

$$|\langle U_{a^n}\Phi_i, \Psi_j\rangle| \quad \leq \quad \sqrt{\dim(\langle K\Phi_i\rangle)\dim(\langle K\Psi_j\rangle)}\ \beta^n. \tag{8.5}$$

Then, for $G_\chi = \sum_{i=0}^{\infty}\Phi_i$ and $H_\chi = \sum_{i=0}^{\infty}\Psi_i$, we have that, for each $\gamma < \frac{1}{3}$

$$|\langle U_{a^n}G_\chi, H_\chi\rangle| = O\left(\beta^{n\gamma}\right).$$

Proof. The proof uses decomposing the functions G_χ, H_χ from formula (8.4) into a sum of K-finite functions as above. We then apply (8.5) to get:

$$\begin{aligned}|\langle U_{a^n}G_\chi, H_\chi\rangle| &\leq \sum_{i,j\geq 0}|\langle U_{a^n}\Phi_i, \Psi_j\rangle| \\ &\leq \sum_{i,j\geq 0}\min\{C_\chi(\sqrt{p^2+1})^{i+j+2D}\beta^n, \|G\|_{\frac{1}{2}}\|H\|_{\frac{1}{2}}p^{-(i+j)/2}\}\end{aligned}$$

The estimate easily follows. $\square$

We also need the following standard result on representations.

Lemma 8.3 [8, p. 23]. *Since $\Gamma\backslash PSL(2,\mathbb{Q}_p)$ is compact, the regular right representation of G splits into a countable number of discrete unitary representations.*

A K-finite function Φ in $L^2(\Gamma\backslash PSL(2,\mathbb{Q}_p))$ decomposes into an orthogonal sum of Φ_k in the irreducible representations. It is clear that the Φ_k are K-finite and that

$$\dim(\langle K\Phi\rangle) = \sum_k \dim(\langle K\Phi_k\rangle).$$

To prove (8.5), it suffices to know that there is a number $\beta < 1$ such that for any irreducible representation ρ in $L^2(\Gamma\backslash G)$, any K-finite unit vectors u, v orthogonal to the constant functions, we have:

$$\langle\rho(a^n)u, v\rangle \quad \leq \quad \sqrt{\dim(\langle Ku\rangle)\dim(\langle Kv\rangle)}\ \beta^n. \tag{8.6}$$

In order to minimize β, we shall concentrate on congruence subgroups.

8.3 Congruence subgroups

For definiteness, consider the quaternion algebra $D = D(1, 1)$. Recall from §2.3 that if we write $G(\mathbb{Q}_p) = D \otimes \mathbb{Q}_p$ then

(1) $G(\mathbb{R})$ is compact,

(2) $G(\mathbb{Q}_2)$ is compact, and

(3) $G(\mathbb{Q}_p)$ is not compact for $p \neq 2, \infty$.

For $p \equiv 1\,(4)$, $PG(\mathbb{Q}_p)$ is isomorphic to $PGL(2,\mathbb{Q}_p)$, and $\Gamma = \Gamma(1) = PG(\mathbb{Z}[1/p])$ is a cocompact lattice in $PGL(2,\mathbb{Q}_p)$.

Definition. For $p \equiv 1\,(4)$ and $N \geq 1$, $(N|p) = 1$, the *congruence* subgroup $\Gamma(N)$ of $PGL(2,\mathbb{Q}_p)$ associated to D and N is the group

$$\Gamma(N) \;=\; \mathrm{Ker}(G(\mathbb{Z}[1/p]) \mapsto G(\mathbb{Z}[1/p]/N\mathbb{Z}[1/p])).$$

Clearly, a congruence subgroup is a cocompact lattice in $PGL(2,\mathbb{Q}_p)$. By Lemma 1.1, its intersection with $PSL(2,\mathbb{Q}_p)$ is a cocompact lattice in $PSL(2,\mathbb{Q}_p)$. Our objective is to show the following.

Proposition 8.4. *Let Γ be a congruence subgroup of $PGL(2,\mathbb{Q}_p)$. Then, for any irreducible representation ρ in $L^2(\Gamma\backslash PGL(2,\mathbb{Q}_p))$, any K-finite unit vectors v_1, v_2 orthogonal to the constant functions, we have:*

$$\langle \rho(a^n)u, v\rangle \quad \leq \quad \sqrt{\dim(\langle Ku\rangle)\dim(\langle Kv\rangle)}\,\left(\frac{1}{p}\right)^n. \tag{8.7}$$

Proposition 8.4 itself follows from deep results of Jacquet-Langlands [12, Theorem 18, p. 186] and Deligne [12, Theorem 6, p.137], [14, pp. 79-82]. More precisely, for $q \neq p$, let $K_q \subset G(\mathbb{Q}_q)$ be a compact open set and let $\Gamma = G(\mathbb{A}) \cap \prod_{q\neq p} K_q$, where $\mathbb{A}$ denotes the adeles of $\mathbb{Q}$. There is a natural bijection between $G(\mathbb{Q})\backslash G(\mathbb{A})/\left(G(\mathbb{R}) \times \prod_{q\neq p} K_q\right)$ and $\Gamma\backslash G(\mathbb{Q}_p)$, which induces a $G(\mathbb{Q}_p)$-equivariant isometry on the corresponding L^2 spaces. In particular, if ρ_p is an irreducible representation appearing in $L^2(\Gamma\backslash\mathbb{Q}_p)$ then there exists an irreducible representation ρ in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ such that the pth component of ρ is ρ_p. By the Jacquet-Langlands theorem, for all such ρ_p there exists an irreducible representation of $\widetilde{\rho} \in L^2(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$ such that $\widetilde{\rho}_p = \rho_p$

and $\widetilde{\rho}_\infty$ is a discrete series for $SL(2,\mathbb{R})$. It then follows from Deligne's Theorem that $\widetilde{\rho}_p$ is tempered. In particular, for all $\epsilon > 0$ and K-finite vectors u, v we have that $\langle \rho_p \begin{pmatrix} p^n & 0 \\ 0 & \pi^n \end{pmatrix} u, v\rangle = O\left(\sqrt{\dim(\langle Ku\rangle)\dim(\langle Kv\rangle)}\, p^{-n}\right)$ [5].

We can now complete the proof of Corollary 1.

By sections 8.1 and 8.2 we see that the spectral gap can be estimated by the speed of mixing and thus by the decay of coefficients in some representations (Lemma 8.3). This decay is given by (8.7). It follows that no eigenvalue for any operator $\mathcal{L}_\chi$ occurs in the region $|z| > \left(\frac{1}{p}\right)^{1/3}$ and Corollary 1 follows.

Remarks.

(i) There are a number of closely related invariants associated to graphs such as $\Gamma\backslash G/K$. Perhaps the most familiar is the *Ihara zeta function* defined by

$$Z(s) = \prod_\gamma \left(1 - p^{-sl(\gamma)}\right)^{-1}$$

where the product is over prime closed curves γ in $\Gamma\backslash G/K$ of length $l(\gamma)$. Ihara showed that $Z(s)$ is rational in $u = q^{-s}$ and can be expressed in terms of the transition matrix A by $Z(s) = (1-u^2)^{-1}\det(I - Au + qu^2)^{-1}$. In particular, the poles of $Z(s)$ are determined by the eigenvalues of the transition matrix A and for congruence subgroups we have the analogue of the Riemann hypothesis on the location of the zeros. The above analysis shows that the corresponding L-function

$$L(s,\chi) = \prod_{[\gamma]} \left(1 - \chi(\iota([\gamma]))p^{-sl(\gamma)}\right)^{-1}$$

has a meromorphic extension to a half-plane $Re(s) > c$, where $c < 1$.

(ii) It would be interesting to extend Theorem 3 and Corollary 1 to the case of lattices Γ containing torsion. However, this would require a significant revision of the method of coding geodesics since $\Gamma\backslash Y$ is not obviously a subshift of finite type. There is a discussion of this problem in [2].

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