BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 36(3), 285-307

On Vertices, focal curvatures and differential geometry of space curves

Ricardo Uribe-Vargas

Abstract. The *focal curve* of an immersed smooth curve $\gamma : \theta \mapsto \gamma(\theta)$, in Euclidean space $\mathbb{R}^{m+1}$, consists of the centres of its osculating hyperspheres. This curve may be parametrised in terms of the Frenet frame of γ $(\mathbf{t}, \mathbf{n}_1, \ldots, \mathbf{n}_m)$, as $C_\gamma(\theta) = (\gamma + c_1\mathbf{n}_1 + c_2\mathbf{n}_2 + \cdots + c_m\mathbf{n}_m)(\theta)$, where the coefficients $c_1, \ldots, c_{m-1}$ are smooth functions that we call the *focal curvatures* of γ. We discovered a remarkable formula relating the Euclidean curvatures κ_i, $i = 1, \ldots, m$, of γ with its focal curvatures. We show that the focal curvatures satisfy a system of Frenet equations (not vectorial, but scalar!). We use the properties of the focal curvatures in order to give, for $\ell = 1, \ldots, m$, necessary and sufficient conditions for the radius of the osculating ℓ-dimensional sphere to be critical. We also give necessary and sufficient conditions for a point of γ to be a vertex. Finally, we show explicitly the relations of the Frenet frame and the Euclidean curvatures of γ with the Frenet frame and the Euclidean curvatures of its focal curve C_γ.

Keywords: vertex, space curve, focal curvatures, singularity, caustic.

Mathematical subject classification: 51L15, 53A04, 53D12.

Introduction

The differential geometry of space curves is a classical subject which usually relates geometrical intuition with analysis and topology. Last years, the ideas and techniques of singularity theory of wave fronts and caustics ([1], [2]), revealed to be a powerful tool to discover new theorems on the differential geometry of curves and surfaces (c.f. [3]-[6], [13], [17], [22]-[31]).

The *focal surface* or *caustic* of a curve γ in Euclidean 3-space is the envelope of the normal planes of γ. The study of the focal surface of a curve can provide useful geometric information about that curve and vice versa. Darboux found how to determine the *evolutes* of a curve γ, that is, the curves whose tangents

Received 6 April 2005.

are normals of γ. Moreover, he showed (proved) that *the focal surface of γ is foliated by the evolutes, and all of them lie on the focal surface*, see [10].

The focal surface of γ is singular along a curve C_γ (it has a cuspidal edge along C_γ) which is called the *focal curve* of γ (in [9], it is called the *evolute of second type* of γ). *The osculating planes of C_γ are the normal planes of γ, and the points of C_γ are the centres of the osculating spheres of γ*, see [9].

In this paper, we study the geometry of the focal surface, focusing on the properties of the focal curve C_γ. Using these properties, we formulate and prove new results for curves in Euclidean n-space for arbitrary $n \geq 2$.

Let $\gamma : \mathbb{R} \to \mathbb{R}^{m+1}$ be a smooth curve (a source of light). The *caustic* of γ (defined as the envelope of the normal lines of γ) is a singular and stratified hypersurface. The *focal curve* of γ, C_γ, is defined as the singular stratum of dimension 1 of the caustic and *it consists of the centres of the osculating hyperspheres of γ*. Since the centre of any hypersphere tangent to γ at a point lies on the normal plane to γ at that point, the focal curve of γ may be parametrised using the Frenet frame $(\mathbf{t}, \mathbf{n}_1, \ldots, \mathbf{n}_m)$ of γ as follows: $C_\gamma(\theta) = (\gamma + c_1\mathbf{n}_1 + c_2\mathbf{n}_2 + \cdots + c_m\mathbf{n}_m)(\theta)$, where the coefficients $c_1, \ldots, c_{m-1}$ are smooth functions that we call the *focal curvatures* of γ.

The Euclidean curvatures of γ, $\kappa_1, \kappa_2, \ldots, \kappa_m$, form a system of m functions which determine the curve γ up to translation and rotation. Let us denote with a prime the derivation with respect to the arc-length parameter. We prove that the following formula holds (Theorem 2):

$$\kappa_i = \frac{c_1c_1' + c_2c_2' + \cdots + c_{i-1}c_{i-1}'}{c_{i-1}c_i}, \quad \text{for } i \geq 2,$$

showing that the focal curvatures also determine the curve up to translation and rotation (if all zeros of c_k, $2 \leq k \leq m-1$, are simple).

In Theorem 1, we show that *the focal curvatures of γ satisfy a system of Frenet equations* (not vectorial, but scalar equations and with the same Frenet matrix of γ !).

For $k = 1, \ldots, m-1$, we give necessary and sufficient conditions, in terms of the focal curvatures, for the radius of the k-dimensional osculating sphere of a generic curve in $\mathbb{R}^{m+1}$ to be critical (Theorem 4).

We prove that: *A point of γ is a vertex* (that is, a point at which the order of contact of γ with its osculating hypersphere is higher than the usual one) *if and only if $c_m' + c_{m-1}\kappa_m = 0$ at that point* (Theorem 3). So, in terms of the focal curvatures, the equation characterising the curves lying on a hypersphere in $\mathbb{R}^{m+1}$ is very simple: $c_m' + c_{m-1}\kappa_m \equiv 0$.

In Theorem 5, we show explicitly that the Frenet frame of the focal curve C_γ consist (up to signs) of the same vectors of the Frenet frame of γ but the order of the vectors is reversed. Moreover, the Euclidean curvatures $K_1, \dots, K_m$ of the focal curve C_γ are related to those of γ by

$$\frac{K_1}{|\kappa_m|} = \frac{K_2}{\kappa_{m-1}} = \cdots = \frac{|K_m|}{\kappa_1} = \frac{1}{|c_m' + c_{m-1}\kappa_m|}.$$

These relations, together with the stratification of the caustic described in §2, provide a partial solution to the *inverse problem*: given the caustic, reconstruct the source of light.

In §0, we define the order of contact of a curve with a submanifold of $\mathbb{R}^n$ and we recall some basic notions and results on the differential geometry of space curves. In §1, we state the results of the paper. In §2, we use the techniques of singularity theory (in symplectic geometry) to study the geometry and the natural stratification of the *focal set* of a curve γ in Euclidean n-space (the codimension 1 strata being the focal curve of γ). In §3, we prove our results.

§0. Preliminary Definitions and Remarks

In order to give the definition of osculating k-spheres of a curve (at a point of it) we need to introduce the following definition:

Definition. Let M be a d-dimensional submanifold of $\mathbb{R}^n$, considered as a complete intersection: $M = \{x \in \mathbb{R}^n : g_1(x) = \cdots = g_{n-d}(x) = 0\}$. We say that a (regularly parametrised) smooth curve $\gamma : \theta \mapsto \gamma(\theta) \in \mathbb{R}^n$ has *k-point contact* with M or that its *order of contact* is k, at a point $\gamma(\theta_0)$, if at $\theta = \theta_0$ each function $g_1 \circ \gamma, \dots, g_{n-d} \circ \gamma$ has a zero of multiplicity at least k and at least one of them has a zero of multiplicity k.

Remark. To make this definition more invariant, one could denote the image of γ by Γ and then write that the *order of contact* at a point is the minimum of the multiplicities of zero among the functions of the form $g_{|\Gamma} : \Gamma \to \mathbb{R}$ at that point, where g belongs to the generating ideal of M and we assume that 0 is a regular value of g.

In this paper, M will be an affine subspace or a sphere of dimension d.

Remark. Do not confuse our order of contact with the order of tangency: two perpendicular lines in the plane have order of contact 1 at the point of intersection, but the order of tangency is 0.

Example 1. A smooth curve in Euclidean (or affine) space $\mathbb{R}^n$ has 2-point contact with its tangent line (at the point of tangency) for the generic points of the curve. The plane curve $y = x^3$ has 3-point contact with the line $y = 0$, at the origin: the equation $x^3 = 0$ has a root of multiplicity 3.

Conventions. Write $n = m + 1$. In the sequel $\mathbb{R}^{m+1}$ denotes a Euclidean space, θ denotes any regular parameter of the curve and s denotes the arc length parameter. A parametrised curve $\gamma = \gamma(\theta)$ in $\mathbb{R}^{m+1}$ is said to be *good* if its derivatives of order $1, \ldots, m$, are linearly independent at any point. *A generic curve is good*. We will consider only good curves.

The *osculating k-plane* of a curve at a point is the affine subspace spanned by the first k derivatives of the curve at that point. *A curve has at least $(k+1)$-point contact with its osculating k-plane at the point of osculation*. For $k = m$ we will simply write *osculating hyperplane*.

Given a point of a generic smoothly immersed curve in $\mathbb{R}^{m+1}$, the sequence consisting of that point and of the osculating k-planes, $k = 1, \ldots, m$, form a complete flag, which is called the *osculating flag* of the curve at that point.

By convention, the k-dimensional affine subspaces of the Euclidean space $\mathbb{R}^{m+1}$ will be also considered as k-dimensional spheres of infinite radius.

Definition. For $k = 1, \ldots, m$, a *k-osculating sphere* at a point of a curve in Euclidean space $\mathbb{R}^{m+1}$ is a k-dimensional sphere having at least $(k + 2)$-point contact with the curve at that point. For $k = m$ we will simply write *osculating hypersphere*.

Example 2. A generic plane curve and its osculating circle have 3-point contact at an ordinary point of the curve.

Remark. For $1 \leq \ell < m$, the osculating ℓ-sphere at a point of a curve in $\mathbb{R}^{m+1}$ is the intersection of the osculating hypersphere with the osculating $(\ell+1)$-plane at that point.

Curvature, Frenet frame and higher order curvatures. For a curve γ in $\mathbb{R}^3$ parametrised by arc-length (from a fixed point of it) the tangent vector $\mathbf{t}(s) = \gamma'(s)$ is unitary and it is orthogonal to $\mathbf{t}'(s) = \gamma(s)''$. If $\gamma(s)'' \neq 0$ these vectors span the (unique) osculating plane of γ at s. Write $\mathbf{t}'(s) = \kappa_1(s)\mathbf{n}_1(s)$, where $\mathbf{n}_1(s)$ is the unit vector orthogonal to $\mathbf{t}(s)$ such that the coefficient $\kappa_1(s)$, called

the *curvature of* γ *at* s, is positive. The radius of the osculating circle of γ at s is given by $R_1(s) = 1/\kappa_1(s)$ and it is called the *radius of curvature of* γ *at* s.

Assume that $\mathbb{R}^3$ is oriented and take the unit vector $\mathbf{n}_2(s)$ such that the basis $\mathbf{t}(s)$, $\mathbf{n}_1(s)$, $\mathbf{n}_2(s)$, called *Frenet frame*, is positive (right-handed), that is $\mathbf{n}_2 = \mathbf{t} \times \mathbf{n}_1$. One easily proves that there is a number $\kappa_2 = \kappa_2(s)$, called the *torsion* or *second curvature* of γ at s, such that $\mathbf{n}_2' = -\kappa_2\mathbf{n}_1$. It is the speed of rotation of the vector $\mathbf{n}_2$. For any good curve we have the following formulas:

$$\mathbf{t}' = \kappa_1\mathbf{n}_1, \quad \mathbf{n}_1' = -\kappa_1\mathbf{t} + \kappa_2\mathbf{n}_2, \quad \mathbf{n}_2' = -\kappa_2\mathbf{n}_1,$$

which are called *Frenet equations* of the curve γ.

Consider a good curve γ in the oriented space $\mathbb{R}^{m+1}$, that is, the vectors $\gamma'(s), \dots, \gamma^{(m)}(s)$ are linearly independent for any s. Apply Gram-Schmidt process to these vectors to obtain the orthonormal system $\mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{m-1}(s)$. Let $\mathbf{n}_m(s)$ be the (unique) vector such that the basis $\mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_m(s)$, called *Frenet frame* of γ at s, is orthonormal and positive. The derivatives of the Frenet frame vectors are given by the so called *system of Frenet equations* of γ:

$$\begin{pmatrix} \mathbf{t}' \\ \mathbf{n}_1' \\ \mathbf{n}_2' \\ \mathbf{n}_3' \\ \vdots \\ \mathbf{n}_{m-2}' \\ \mathbf{n}_{m-1}' \\ \mathbf{n}_m' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & \cdots & 0 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & \cdots & 0 & 0 & 0 \\ 0 & -\kappa_2 & 0 & & & & \vdots \\ 0 & 0 & -\kappa_3 & & & & \vdots \\ \vdots & 0 & & & & & \\ & & & & 0 & \kappa_{m-1} & 0 \\ \vdots & & & & -\kappa_{m-1} & 0 & \kappa_m \\ 0 & 0 & & \cdots & 0 & -\kappa_m & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \\ \vdots \\ \mathbf{n}_{m-2} \\ \mathbf{n}_{m-1} \\ \mathbf{n}_m \end{pmatrix}.$$

The functions $\kappa_1 = \kappa_1(s), \dots, \kappa_m = \kappa_m(s)$ are called *Euclidean curvatures* of the curve and are defined only for good curves. Note that the ℓ-th Euclidean curvature κ_ℓ gives the speed of rotation of the osculating ℓ-plane around the osculating $(\ell-1)$-plane, with respect to the variation of the arc-length parameter (one can find other geometric interpretations of the Euclidean curvatures). The curvatures $\kappa_1, \dots, \kappa_{m-1}$ of any good curve are strictly positive, while κ_m can take any real value.

A point of a smooth curve in $\mathbb{R}^{m+1}$ for which the derivative of the curve of order $m+1$ belongs to the osculating hyperplane is said to be a *flattening*. *At a flattening the last Euclidean curvature* κ_m *vanishes and the curve has at least* $(m+2)$*-point contact with its osculating hyperplane at that point.*

Remark about flattenings. At a flattening of a generic curve the osculating hypersphere is unique and it coincides with the osculating hyperplane. In this case, the centre of the osculating hypersphere is not defined and we will say that "it is at infinity". If at a point the order of contact of γ with its osculating sphere of codimension 2, S^{m-1}, is greater than the usual one, then the point is a non generic flattening. In this case, all hyperspheres containing S^{m-1} are osculating, i.e. the centre of the osculating hypersphere is not uniquely defined.

Example. These conditions (not satisfied for any point of a generic curve) are however satisfied by the flattenings of a generic spherical curve (that is, a generic curve among the curves lying on a hypersphere).

For these reasons we will assume that our curves are good and have no flattening, unless we consider (explicitly) spherical curves.

§1. Statement of Results

Definition. The curve $C_\gamma : \theta \mapsto C_\gamma(\theta) \in \mathbb{R}^{m+1}$ consisting of the centres of the osculating hyperspheres of a good curve (without its flattenings) $\gamma : \theta \mapsto \gamma(\theta) \in \mathbb{R}^{m+1}$ is called the *parametrised focal curve* of γ.

Remark. In geometrical optics, a curve γ in Euclidean 3-space can be considered as a source of light. The envelope of all light rays normal to γ is the *focal surface* or *caustic* of γ. The light intensity is much more concentrated on the caustic than in all other points of the space. Moreover, the caustic itself is more illuminated along its cuspidal edge, which is the focal curve of γ.

Consider a good curve $\gamma : \mathbb{R} \to \mathbb{R}^{m+1}$. Write $\kappa_1, \kappa_2, \ldots, \kappa_m$ for its Euclidean curvatures and $\mathbf{t}, \mathbf{n}_1, \ldots, \mathbf{n}_m$ for its Frenet frame. The hyperplane normal to γ at a point consists of the set of centres of all hyperspheres tangent to γ at that point. Hence the centre of the osculating hypersphere at that point lies in such normal hyperplane. Therefore (denoting $C_\gamma(\theta)$ by C_γ, $\gamma(\theta)$ by γ and so on,...) we can write

$$C_\gamma = \gamma + c_1\mathbf{n}_1 + c_2\mathbf{n}_2 + \cdots + c_m\mathbf{n}_m,$$

where the coefficients $c_1, \ldots, c_{m-1}$ are smooth functions of the parameter of the curve γ.

Definition. The coefficient c_i, $i = 1, \ldots, m$, is called the ith *focal curvature* of γ.

Remark. *The first focal curvature c_1 never vanishes: $c_1 = 1/\kappa_1$.*

The Frenet equations of a curve in $(m+1)$-Euclidean space is a system of $m+1$ vectorial equations involving the unit vectors of the Frenet frame and their derivatives. The following theorem shows that the focal curvatures of that curve satisfy a system of *scalar Frenet equations* which "is obtained from the usual Frenet equations by replacing the ith normal vector of the Frenet frame by the ith focal curvature".

Theorem 1. *The focal curvatures of a curve lying on a hypersphere $\gamma : \mathbb{R} \to \mathbb{S}^n \subset \mathbb{R}^{m+1}$, parametrised by arc length s, satisfy the following "scalar Frenet equations":*

$$\begin{pmatrix} 1 \\ c_1' \\ c_2' \\ c_3' \\ \vdots \\ c_{m-2}' \\ c_{m-1}' \\ c_m' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & \cdots & 0 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & \cdots & 0 & 0 & 0 \\ 0 & -\kappa_2 & 0 & & & & \vdots \\ 0 & 0 & -\kappa_3 & & & & \vdots \\ \vdots & 0 & & & & & \\ & & & & 0 & \kappa_{m-1} & 0 \\ \vdots & & & & -\kappa_{m-1} & 0 & \kappa_m \\ 0 & 0 & & \cdots & 0 & -\kappa_m & 0 \end{pmatrix} \begin{pmatrix} 0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{m-2} \\ c_{m-1} \\ c_m \end{pmatrix}.$$

Remark. If the curve is not spherical then the correcting term $-\frac{(R_m^2)'}{2c_m}$ must be added to the last component of the left hand side vector to obtain $c_m' - \frac{(R_m^2)'}{2c_m}$, for $c_m \neq 0$.

Theorem 2. *The Euclidean curvatures of a good curve γ (with $\kappa_m \neq 0$) in $\mathbb{R}^{m+1}$, parametrised by arc length, are given in terms of the focal curvatures of γ by the formula:*

$$\kappa_i = \frac{c_1 c_1' + c_2 c_2' + \cdots + c_{i-1} c_{i-1}'}{c_{i-1} c_i}, \quad \textit{for } i \geq 2.$$

Remark. For a generic curve, the focal curvatures c_i or c_{i-1} can vanish at isolated points. At these points the function $c_1 c_1' + c_2 c_2' + \cdots + c_{i-1} c_{i-1}'$ also vanishes, and the corresponding value of the Euclidean curvature κ_i may be obtained by l'Hôpital rule.

Definition. A *vertex* of a curve in $\mathbb{R}^n$ is a point at which the curve has at least $(n+2)$-point contact with its osculating hypersphere.

Example 3. *The vertices of a curve in Euclidean plane $\mathbb{R}^2$ are the points at which the curvature is critical*. For instance, a non-circular ellipse has 4 vertices: They are the points at which the ellipse intersects its principal axes.

The interest on the vertices of curves came, for instance, from geometrical optics (c.f. Huygens) and from the *geometry in the large*. Namely the classical 4-vertex theorem states that *a smooth closed convex plane curve has at least* 4 *different vertices*, [16]. Besides several important works generalising this theorem (c.f. [14, 15, 7, 21, 18, 20]), the recent progress in symplectic geometry and singularity theory have revived the interest on the study of vertices together with the different variants of its definition (c.f. [22, 11, 13, 17], [24]-[31]). Here we are mainly concerned with local properties of vertices.

The next theorem (implicitly contained in [19]) provides necessary and sufficient conditions for a point to be a vertex.

Theorem 3. *A non-flattening point of a good curve parametrised by arc length in $\mathbb{R}^{m+1}$, $m > 1$, is a vertex if and only if*

$$c_m' + c_{m-1}\kappa_m = 0 \text{ at that point.}$$

Corollary 1. *A good curve parametrised by arc length in Euclidean space $\mathbb{R}^{m+1}$, $m > 1$, lies on a hypersphere if and only if*

$$c_m' + c_{m-1}\kappa_m \equiv 0.$$

Example 4. For curves in Euclidean 3-space, Corollary 1 provides the following classical result on spherical curves (see for instance [8]):

A smoothly immersed curve of $\mathbb{R}^3$, with curvature κ and torsion τ both nowhere zero, lies on a sphere if and only if

$$c_2' + c_1\tau \equiv 0, \text{ i.e. if and only if } \left(\frac{R_1'}{\tau}\right)' + R_1\tau \equiv 0,$$

where derivation is taken with respect to the arc length of the curve and $R_1 = 1/\kappa$, is the radius of curvature.

There is a small mistake in the beautiful Hilbert–Cohn Vossen's book, [12]:

A curve of $\mathbb{R}^3$ lies on a sphere if and only if

$$R_1^2 + (R_1')^2 \frac{1}{\tau^2} = \text{const.} \tag{W}$$

Of course, a curve lying on a sphere satisfies condition (W), which means that the radius of the osculating sphere is constant. However, the dimension of the space of non-spherical curves satisfying condition (W) is infinite: *If a curve with nowhere vanishing torsion has constant curvature $\kappa \neq 0$ then the radius of its osculating sphere is constant and equal to $R = 1/\kappa$.* This follows from condition (W). One example is the circular helix $t \mapsto (\cos t, \sin t, t))$. The above statement becomes true if one suppose the genericity condition $R_1' \neq 0$.

The radius of the osculating hypersphere of a curve in $\mathbb{R}^{m+1}$ is critical at each vertex of that curve; the converse statement is not always true for $m > 1$ (see [23], [31]): There are examples of curves having points for which the radius of the osculating hypersphere is critical, but which are not vertices. The geometric meaning of such points becomes clear from Proposition 0, below.

The following two theorems give necessary and sufficient conditions for the radius of the osculating sphere of dimension $\ell \leq m$ to be critical.

Theorem 4. *For $1 \leq \ell < m$, the radius of the osculating ℓ-sphere of a generic curve in $\mathbb{R}^{m+1}$ is critical if and only if either*

$$c_\ell = 0 \text{ or } c_{\ell+1} = 0.$$

Moreover, c_1 never vanishes.

Remark. *At a point of a curve γ, the first ℓ focal curvatures $c_1, \ldots, c_\ell$ are the coordinates (with respect to the Frenet frame) of the centre of the ℓ-dimensional osculating sphere of γ at that point.* Therefore the curve γ_ℓ described by the centre of the ℓ-dimensional osculating sphere is parametrised by:

$$\gamma_\ell = \gamma + c_1\mathbf{n}_1 + c_2\mathbf{n}_2 + \cdots + c_\ell\mathbf{n}_\ell.$$

Of course, $\gamma_m = C_\gamma$. Theorem 4 implies for instance that *the curves γ_1 and γ_2 intersect at least twice, and the curve γ_ℓ intersects either $\gamma_{\ell-1}$ or $\gamma_{\ell+1}$, at least at two points, $1 < \ell < m$.*

Corollary 2. *If the ℓth focal curvature c_ℓ vanishes at a point, then both the radii of the osculating spheres of dimensions $\ell - 1$ and ℓ are critical at that point.*

Remark. The mth focal curvature c_m at a point of a smooth curve in $\mathbb{R}^{m+1}$ is the signed distance between the osculating hyperplane and the centre of the osculating hypersphere at that point.

Definition. A point of a curve is said to be a *pseudo-vertex* of that curve if the centre of the osculating hypersphere at that point lies in the osculating hyperplane at that point (that is, if $c_m = 0$).

Corollary 3. *A generic closed curve in $\mathbb{R}^3$ has at least two vertices or two pseudo-vertices.*

Corollary 4. *At a pseudo-vertex of a smooth curve in $\mathbb{R}^{m+1}$, $m > 1$, both the radius of the osculating hypersphere and the radius of the osculating $(m-1)$-sphere are critical.*

Proposition 0. *The radius of the osculating hypersphere at a point of a good curve in $\mathbb{R}^{m+1}$, $m > 1$, is critical if and only if such point is either a vertex or a pseudo-vertex.*

A point of a generic smooth curve at which the last Euclidean curvature vanish, $\kappa_m = 0$, is a *flattening* of the curve (see our Remark about flattenings above). The following statement is a consequence of Proposition 0.

Corollary 5. *Write V, F and P for the number of vertices, flattenings and pseudo-vertices of a generic closed curve smoothly immersed in $\mathbb{R}^{m+1}$. The following inequalities hold:*

$$V + P \geq F \quad \text{and} \quad V + P \geq 2.$$

We reformulate Proposition 0 (and we will prove it, in §3) in terms of the focal curvatures c_m and c_{m-1}:

Proposition $\tilde{0}$. *The radius of the osculating hypersphere of a good curve in $\mathbb{R}^{m+1}$, $m > 1$, parametrised by arc length, is critical at a point if and only if either $c_m = 0$ or $c_m' + c_{m-1}\kappa_m = 0$ at that point.*

After I have sent this paper to V.D. Sedykh, he communicated to me that he had discovered independently Proposition 0 and Corollary 5, but he had not published them and he urged me to publish all results of this paper.

Remark. By definition, the first $m-1$ Euclidean curvatures of a generic curve $\gamma : \mathbb{R} \to \mathbb{R}^{m+1}$ are positive everywhere, while the last one, κ_m, can take any real value. The sign of the last Euclidean curvature at a non-flattening point of a curve is defined only when the orientation on the ambient space $\mathbb{R}^{m+1}$ is fixed : κ_m is positive (negative) at the points of the curve where the derivatives of order $1, \dots, m+1$ form a positive (negative, resp.) basis of $\mathbb{R}^{m+1}$.

Remark. Consider a curve $\gamma : \mathbb{R} \to \mathbb{R}^{m+1}$ in the oriented Euclidean space $\mathbb{R}^{m+1}$. *If the number $m > 0$ is of the form $4k$ or $4k+1$, with $k \in \mathbb{N}$, then sign of the last Euclidean curvature of γ at a non-flattening point depends on the orientation of the curve*. That is, the last Euclidean curvature of a curve at a non-flattening point is a function whose sign depends not only on the point of the curve but also on the orientation of the curve given by the parametrisation.

Proof. Let $\gamma : \mathbb{R} \to \mathbb{R}^{m+1}$ be a generic curve in $\mathbb{R}^{m+1}$, such that $\gamma(0)$ is not a flattening. Write $\tau(t) = -t$ and consider the parametrisation in the opposite direction $\tilde{\gamma} = \gamma \circ \tau : t \mapsto \gamma(-t)$. The derivative of order r of $\tilde{\gamma}$ at $t = 0$ is $\tilde{\gamma}^{(r)}(0) = \gamma^{(r)}(0) \cdot (-1)^r$. So the derivatives of odd order of γ and $\tilde{\gamma}$ at $t = 0$ have opposite directions while the derivatives of even order of γ and $\tilde{\gamma}$ at $t = 0$ coincide. Therefore the basis obtained from the derivatives of order $1, \dots, m+1$ of $\tilde{\gamma}$ at $t = 0$ and the basis obtained from the derivatives of order $1, \dots, m+1$ of γ at $t = 0$ give different orientations of $\mathbb{R}^{m+1}$ if and only if the cardinality of the set $\{r \in \mathbb{N} : \ r \text{ is odd and } r \leq m+1\}$ is odd, i.e. if and only if the number $m > 0$ is of the form $4k$ or $4k+1$, with $k \in \mathbb{N}$.

Theorem 5. *Let $\gamma : s \mapsto \gamma(s) \in \mathbb{R}^{m+1}$ be a good curve without flattenings. Write $\kappa_1, \dots, \kappa_m$ for its Euclidean curvatures and $\{\mathbf{t}, \mathbf{n}_1, \dots, \mathbf{n}_m\}$ for its Frenet frame. For each non-vertex $\gamma(s)$ of γ, write $\varepsilon(s)$ for the sign of $(c_m' + c_{m-1}\kappa_m)(s)$ and $\delta_k(s)$ for the sign of $(-1)^k \varepsilon(s)\kappa_m(s)$, $k = 1, \dots, m$. For any non-vertex of γ the following holds:*

a) *The Frenet frame $\{\mathbf{T}, \mathbf{N}_1, \dots, \mathbf{N}_m\}$ of C_γ at $C_\gamma(s)$ is well-defined and its vectors are given by $\mathbf{T} = \varepsilon \mathbf{n}_m$, $\mathbf{N}_k = \delta_k \mathbf{n}_{m-k}$, for $k = 1, \dots, m-1$, and $\mathbf{N}_m = \pm\mathbf{t}$, the sign in $\pm\mathbf{t}$ is chosen in order to obtain a positive basis.*

b) *The Euclidean curvatures $K_1, \dots, K_m$ of the parametrised focal curve of γ, $C_\gamma : s \mapsto C_\gamma(s)$, are related to those of γ by :*

$$\frac{K_1}{|\kappa_m|} = \frac{K_2}{\kappa_{m-1}} = \cdots = \frac{|K_m|}{\kappa_1} = \frac{1}{|c_m' + c_{m-1}\kappa_m|},$$

the sign of K_m is equal to δ_m times the sign chosen in $\pm\mathbf{t}$.

That is, the Frenet matrix of C_γ at $C_\gamma(s)$ is

$$\frac{1}{|c_m' + c_{m-1}\kappa_m|}\begin{pmatrix} 0 & |\kappa_m| & 0 & \cdots & 0 & 0 & 0 \\ -|\kappa_m| & 0 & \kappa_{m-1} & \cdots & 0 & 0 & 0 \\ 0 & -\kappa_{m-1} & 0 & & & & \vdots \\ 0 & 0 & -\kappa_{m-2} & & & & \vdots \\ \vdots & 0 & & & & & \\ & & & & 0 & \kappa_2 & 0 \\ \vdots & & & & -\kappa_2 & 0 & \mp\delta_m\kappa_1 \\ 0 & 0 & & \cdots & 0 & \pm\delta_m\kappa_1 & 0 \end{pmatrix}.$$

Application to self-congruent curves. A curve of $\mathbb{R}^{m+1}$ is said to be *self-congruent* if for any two points a and b of it, there is a preserving orientation orthogonal transformation of $\mathbb{R}^{m+1}$ sending the curve to itself and sending a to b. One can prove that *the class of self-congruent curves coincides with the class of curves whose Euclidean curvatures are constant*.

The focal curvatures of these curves are therefore constant and the scalar Frenet equations imply that

$$c_{2\ell} = 0 \quad \text{and} \quad c_{2\ell+1} = \prod_{j=0}^{\ell}\left(\frac{\kappa_{2j}}{\kappa_{2j+1}}\right),$$

where the convention $\kappa_0 = 1$ is used, and the subindices 2ℓ and $2\ell+1$ are taken over all values of ℓ for which $2 \le 2\ell \le m$ and $1 \le 2\ell+1 \le m$, respectively.

Proposition. *For any $\ell \in \mathbb{N}$ such that $0 < 2\ell \le m$, the following holds: At any point of a self-congruent curve of $\mathbb{R}^{m+1}$ the centre of the osculating 2ℓ-sphere lies in the osculating 2ℓ-plane.*

Proof. This follows from the above equalities $c_{2\ell} = 0$.

§2. Study of the Focal Set (caustic) of a Curve

The *focal set* or *caustic* of a submanifold of positive codimension in Euclidean space $\mathbb{R}^{m+1}$ (for instance, of a curve in $\mathbb{R}^3$) is defined as the envelope of the family of normal lines to the submanifold.

Remark. Similarly to geometrical optics in Euclidean 3-space, a submanifold of positive codimension in Euclidean space $\mathbb{R}^{m+1}$ may be considered as a source of light (or as an initial wave front). The normal lines to this source submanifold are called *normal light rays* and its focal set (on which the light intensity is much more concentrated than in the other points of the space) is called the *caustic* of that submanifold.

We will study the focal set of a generic curve $\gamma : \mathbb{R} \to \mathbb{R}^{m+1}$.

The hyperplane normal to γ at a point is the union of all lines normal to γ at that point. The envelope of all hyperplanes normal to γ is thus a component of the focal set. We call it the *main component* (the other component is the curve γ itself, but we will not consider it).

The normal hyperplanes of a curve at two neighbouring points intersect along an affine subspace of codimension 2 which approaches a limiting position as the points move into coincidence. The affine subspace that assumes this limiting position is called the 2-*codimensional focal subspace* of the curve at the point under consideration.

When the point moves along the curve the 2-codimensional focal subspace generates a hypersurface which, by construction, is the envelope of the hyperplanes normal to γ, i.e. it is the main component of the focal set.

So the main component of the focal set of a curve is the union (in a one-parameter family) of affine subspaces of codimension 2 (see Claim 3 in subsection 2.2).

Example 5. At a point of a curve in $\mathbb{R}^3$, the 2-codimensional focal subspace is the line through the centre of the osculating circle, which is parallel to the binormal vector. In classical differential geometry of curves in Euclidean 3-space, it is called the *polar line* (see [10]).

2.1 The caustic of a family of functions

We will use techniques of singularity theory in order to have a more detailed study of the focal set.

Definition. The *caustic* of a family of functions depending smoothly on parameters consists of the parameter values for which the corresponding function has a non-Morse critical point.

Example 6. Given a generic curve $\gamma : \mathbb{R} \to \mathbb{R}^{m+1}$, let $F : \mathbb{R}^{m+1} \times \mathbb{R} \to \mathbb{R}$ be the $(m+1)$-parameter family of real functions given by

$$F(q, \theta) = \frac{1}{2} \parallel q - \gamma(\theta) \parallel^2 .$$

The caustic of the family F is given by the set

$$\{q \in \mathbb{R}^{m+1} : \exists \theta \in \mathbb{R} : F'_q(\theta) = 0 \text{ and } F''_q(\theta) = 0\}.$$

Proposition A. *The caustic of the family $F(q, \theta) = \frac{1}{2} \parallel q - \gamma(\theta) \parallel^2$ coincides with the focal set of the curve $\gamma : \mathbb{R} \to \mathbb{R}^{m+1}$.*

Proof. The caustic of F is defined by the pair of equations $F'_q(\theta) = 0$, $F''_q(\theta) = 0$. For each fixed value of θ, the set of points $q \in \mathbb{R}^{m+1}$ satisfying the first equation form the hyperplane normal to γ at $\gamma(\theta)$:

$$F'_q(\theta) = -\langle q - \gamma(\theta), \gamma'(\theta)\rangle = 0.$$

The set of points $q \in \mathbb{R}^{m+1}$ satisfying both equations for a fixed θ are thus the stationary points of the normal hyperplane at $\gamma(\theta)$ under an infinitesimal variation of it. They form an affine subspace of codimension 2 in $\mathbb{R}^{m+1}$:

$$F''_q(\theta) = -\langle q - \gamma(\theta), \gamma''(\theta)\rangle + \langle \gamma'(\theta), \gamma'(\theta)\rangle = 0.$$

Of course this subspace coincides with the 2-codimensional focal plane of the curve at $\gamma(\theta)$, considered above.

2.2 The natural stratification of the focal set

The focal set of a curve $\gamma : \mathbb{R} \to \mathbb{R}^{m+1}$ is stratified in a natural way. The following claims describe the geometry of such stratification for curves without flattenings.

Denote by $A^k_\gamma(\theta)$, $k = 1, \ldots, m+2$, the set consisting of the centres of all hyperspheres having at least $(k+1)$-point contact with γ at $\gamma(\theta)$.

Claim 1. *The set $A^k_\gamma(\theta)$, $k = 1, \ldots, m+1$ is an affine subspace of codimension k in $\mathbb{R}^{m+1}$.*

Claim 2. *The set $A^1_\gamma(\theta)$ (consisting of the centres of all hyperspheres having at least 2-point contact with γ at $\gamma(\theta)$) is the hyperplane normal to γ at the point $\gamma(\theta)$.*

Definition. The affine subspace $A^k_\gamma(\theta)$ is called *k-codimensional focal plane* of γ at $\gamma(\theta)$.

Corollary (of claims 1 and 2). *The sequence of focal subspaces* $A^1_\gamma(\theta) \supset A^2_\gamma(\theta) \supset \cdots \supset A^{m+1}_\gamma(\theta)$ *defines a complete flag on the hyperplane normal to* γ *at* $\gamma(\theta)$.

Remark. The complete flag $A^1_\gamma(\theta) \supset A^2_\gamma(\theta) \supset \cdots \supset A^{m+1}_\gamma(\theta)$ defines a natural stratification on the hyperplane normal to γ at $\gamma(\theta)$. This stratification induces a natural stratification on the focal set of γ. The stratum of dimension 1 being the focal curve of γ. The 0-dimensional stratum consists of isolated points at which the focal curve is singular (it has a cusp, see Proposition 1 in §3). These singular points of the focal curve of γ correspond to the vertices of γ (for these points the set $A^{m+2}_\gamma(\theta)$ is not empty).

Claim 3. *The focal set of a smooth curve consists of the centres of all hyperspheres having at least* 3*-point contact with that curve at a point of it (i.e. it is the union of all the* 2*-codimensional focal planes of the curve).*

Proposition B. *The complete flag* $A^1_\gamma(\theta) \supset A^2_\gamma(\theta) \supset \cdots \supset A^{m+1}_\gamma(\theta)$ *is the osculating flag of the focal curve of* γ *at the point* $C_\gamma(\theta)$. *In particular, the hyperplane normal to* γ *at* $\gamma(\theta)$ *coincides with the osculating hyperplane of the focal curve of* γ *at the point* $C_\gamma(\theta)$.

Lemma 0. *A point* $q \in \mathbb{R}^{m+1}$ *is the centre of a hypersphere having k-point contact with* γ *at the point* $\gamma(\theta_0)$ *if and only if the function* $F_q(\theta) = \frac{1}{2} \parallel q - \gamma(\theta) \parallel^2$ *has a critical point of multiplicity* $k-1$ *at* θ_0*:*

$$F'_q(\theta_0) = F''_q(\theta_0) = \ldots = F^{(k-1)}_q(\theta_0) = 0 \text{ and } F^k_q(\theta_0) \neq 0.$$

Proof. The sphere of radius r with centre at q is defined by the equation

$$g_r(x) = \frac{1}{2}(\parallel q - x \parallel^2 - r^2) = 0.$$

So a point q is the centre of a hypersphere having k-point contact with γ at the point $\gamma(\theta_0)$ if and only if the function $g_r \circ \gamma$ has a zero of multiplicity k at $\theta = \theta_0$, for some r, i.e. if and only if the function $F_q(\theta) = \frac{1}{2} \parallel q - \gamma(\theta) \parallel^2$ has a critical point of multiplicity $k-1$ at θ_0. $\square$

Proof of Claims 2 and 3. To prove Claims 2 and 3, use Lemma 0 and repeat the proof of Proposition A. Another proof of Claim 3 follows from Example 6, Lemma 0 and Proposition A. □

Proof of Claim 1. Consider the following system of $(m+2)$ equations

$$\begin{array}{rcl} F_q'(\theta) & = & 0 \\ F_q''(\theta) & = & 0 \\ & \vdots & \\ F_q^{(m+2)}(\theta) & = & 0. \end{array}$$

For each fixed value of θ, it can be easily seen that the first k equations — written explicitly— define an affine subspace of codimension k in $\mathbb{R}^{m+1}$ (the cases $k = 1, 2$, are in the proof of Proposition A). So the set $A_\gamma^k(\theta)$ of centres of all hyperspheres having at least $(k+1)$-point contact with γ at $\gamma(\theta)$ is an affine subspace of $\mathbb{R}^{m+1}$. □

Remark. The (generating) family $F(q,\theta) = \frac{1}{2} \parallel q - \gamma(\theta) \parallel^2$ together with Sturm theory can be used to calculate the number of vertices of the curve γ, see [31].

Remark (for Singularity Theory Specialists). In the setting of the theory of Lagrangian singularities, Lagrangian maps and the caustics of Lagrangian maps, the focal set of the curve γ is the caustic of the *Normal map* associated to γ, which is a Lagrangian map defined by the generating family $F(q,\theta)$ (for the notions of caustic, Lagrangian map, Lagrangian singularity and generating family, we refer the reader to [1] and [2]). Thus *the vertices of a curve in* $\mathbb{R}^{m+1}$ *correspond to a Lagrangian singularity* A_{m+2} *of the normal map, that is, the focal set has a "swallowtail" singularity at the centres of the osculating hyperspheres corresponding to the vertices of the curve.*

§3. The Proofs of the Results

As we mentioned in the introduction, the ideas and techniques of the theory of Lagrangian and Legendrian singularities (singularities of caustics and wave fronts) were an important tool for the discovery of the results of this paper and also for their initial proofs. Some of these results would be difficult to discover only using Frenet frame theory. However, once the results were discovered and

proved, the author has made an effort in order to present the proofs as short as possible and as elementary as possible. The author hopes the proofs will be understandable for anyone.

To prove our results we will prove before some lemmas related to the focal curve. Below, θ denotes any regular parameter of the curve and s denotes the arc length parameter.

Lemma 1. *Let $\gamma : \theta \mapsto (\varphi_1(\theta), \ldots, \varphi_{m+1}(\theta))$ be a good curve in $\mathbb{R}^{m+1}$. The velocity vector $q'(\theta)$ of the focal curve of γ at θ is proportional to the mth-normal vector $\mathbf{n}_m(\theta)$ of γ.*

Proof. Consider the (generating) family of functions $F : \mathbb{R} \times \mathbb{R}^{m+1} \to \mathbb{R}$ defined by

$$F_q(\theta) = \frac{1}{2} \parallel q - \gamma(\theta) \parallel^2 .$$

Write $g = \frac{\gamma^2}{2}$. As in §2, use the fact that $-F = \gamma \cdot q - \frac{\gamma^2}{2} - \frac{q^2}{2}$ to recall that the following system of $m+1$ equations defines the *focal curve* $q(\theta)$ of γ:

$$\begin{array}{rcl} \gamma' \cdot q(\theta) - g' & = & 0, \\ \gamma'' \cdot q(\theta) - g'' & = & 0, \\ & \vdots & \\ \gamma^{(m+1)} \cdot q(\theta) - g^{(m+1)} & = & 0. \end{array} \qquad (*)$$

Derive each equation with respect to θ to obtain a second system of equations:

$$\begin{array}{rcl} \gamma' \cdot q'(\theta) + \gamma'' \cdot q(\theta) - g'' & = & 0, \\ \gamma'' \cdot q'(\theta) + \gamma''' \cdot q(\theta) - g''' & = & 0, \\ & \vdots & \\ \gamma^{(m)} \cdot q'(\theta) + g^{(m+1)} \cdot q(\theta) - g^{(m+1)} & = & 0, \\ \gamma^{(m+1)} \cdot q'(\theta) + g^{(m+2)} \cdot q(\theta) - g^{(m+2)} & = & 0. \end{array} \qquad (**)$$

Combine the ith equation of system $(**)$ with the $(i+1)^{th}$ equation of system $(*)$, for $i = 1, \ldots, m$, to obtain

$$\begin{array}{rcl} \gamma' \cdot q'(\theta) & = & 0, \\ \gamma'' \cdot q'(\theta) & = & 0, \\ & \vdots & \\ \gamma^{(m)} \cdot q'(\theta) & = & 0. \end{array} \qquad (***)$$

This means that the velocity vector $q'(\theta)$ is orthogonal to the osculating hyperplane of γ, i.e. $q'(\theta)$ is proportional to the mth-normal vector $\mathbf{n}_m$. $\square$

Proposition 1. *A non-flattening point of a good curve in $\mathbb{R}^{m+1}$ is a vertex if and only if the velocity vector of the focal curve is zero.*

Proof. If the point $\gamma(\theta)$ is a vertex of γ, then besides the system of equations $(*)$ obtained in the proof of Lemma 1, it also satisfies the equation:

$$\gamma^{(m+2)} \cdot q(\theta) - g^{(m+2)} = 0,$$

which combined with the last equation of system $(**)$ gives the equation

$$\gamma^{(m+1)} \cdot q'(\theta) = 0.$$

The preceding equation together with the system $(***)$ imply that for a non-flat vertex $\gamma(\theta)$ of the curve γ the velocity vector $q'(\theta)$ of the focal curve is zero.

Conversely, if a point $\gamma(\theta_0)$ is not a vertex then the corresponding point of the focal curve satisfies the relation

$$\gamma^{(m+2)}(\theta_0) \cdot q(\theta_0) - g^{(m+2)}(\theta_0) \neq 0,$$

which together with the last equation of $(**)$, for $\theta = \theta_0$, imply that $q'(\theta_0) \neq 0$. □

Lemma 1 and Proposition 1 were also stated in [19], where the condition to the point to be a non-flattening is unfortunately absent. Without this condition Proposition 1 does not hold.

Lemma 2. *Let $\gamma : \mathbb{R} \to \mathbb{R}^{m+1}$ be a good curve with $\kappa_m \neq 0$. The derivative of its parametrised focal curve C_γ with respect the arc length s of γ is*

$$C'_\gamma = (c'_m + c_{m-1}\kappa_m)\mathbf{n}_m.$$

Proof of Theorem 1, Proposition 0 and Lemma 2. Consider the parametrised focal curve of γ:

$$C_\gamma(s) = (\gamma + c_1\mathbf{n}_1 + c_2\mathbf{n}_2 + \cdots + c_m\mathbf{n}_m)(s).$$

Denote $C_\gamma(\theta)$, $\gamma(\theta)$ and so on by C_γ, γ, etc. Derive C_γ with respect to the arc length of γ and use Frenet equations of γ to obtain:

$$\begin{aligned} C'_\gamma &= \mathbf{t} + c_1(-\kappa_1\mathbf{t} + \kappa_2\mathbf{n}_2) + c'_1\mathbf{n}_1 + \cdots + c'_{m-1}\mathbf{n}_{m-1} + c_m(-\kappa_m\mathbf{n}_{m-1}) + c'_m\mathbf{n}_m \\ &= (1 - c_1\kappa_1)\mathbf{t} + (c'_1 - \kappa_2 c_2)\mathbf{n}_1 + (c'_2 + c_1\kappa_2 - c_3\kappa_3)\mathbf{n}_2 + \cdots \\ &\quad + (c'_i + c_{i-1}\kappa_i - c_{i+1}\kappa_{i+1})\mathbf{n}_i + \cdots + (c'_m + c_{m-1}\kappa_m)\mathbf{n}_m. \end{aligned}$$

By Lemma 1, the first $m-1$ components of C'_γ vanish. Consequently

$$C'_\gamma = (c'_m + c_{m-1}\kappa_m)\mathbf{n}_m \tag{1}$$

and the following equalities hold:

$$\begin{array}{rcl} 1 & = & \kappa_1 c_1, \\ c'_1 & = & \kappa_2 c_2, \\ c'_2 & = & -\kappa_2 c_1 + \kappa_3 c_3, \\ \vdots & \vdots & \vdots \\ c'_{m-1} & = & -c_{m-2}\kappa_{m-1} + c_m \kappa_m. \end{array} \tag{2}$$

Equation (1) proves Lemma 2. Use the fact that the radius R_m of the osculating hypersphere satisfies $R_m^2 = \| C_\gamma - \gamma \|^2$ to obtain

$$\begin{array}{rcl} (R_m^2)' & = & \langle C_\gamma - \gamma, C_\gamma - \gamma \rangle' \\ & = & 2\langle C'_\gamma - \gamma', C_\gamma - \gamma \rangle \\ & = & 2\langle (c'_m + c_{m-1}\kappa_m)\mathbf{n}_m - \mathbf{t}, c_1\mathbf{n}_1 + \cdots + c_m\mathbf{n}_m \rangle \\ & = & 2c_m(c'_m + c_{m-1}\kappa_m); \end{array}$$

$$\text{i.e. } (R_m^2)' = 2c_m(c'_m + c_{m-1}\kappa_m). \tag{3}$$

Thus for $c_m \neq 0$, $c'_m - \frac{(R_m^2)'}{2c_m} = -c_{m-1}\kappa_m$. This equation together with the set of equations (2) (using our conventions $c_0 = 0$ and $c'_0 = 1$) prove Theorem 1. Equation (3) and Theorem 3 prove Proposition 0. □

Proof of Theorem 3 and of its Corollary. By Lemma 2, we have that

$$C'_\gamma = (c'_m + c_{m-1}\kappa_m)\mathbf{n}_m.$$

Proposition 1 implies thus that a point of the curve γ is a vertex if and only if $c'_m + c_{m-1}\kappa_m = 0$. □

Proof of Theorem 2. The proof will be done by induction. Use the scalar Frenet equations of Theorem 1 to obtain that

$$\kappa_1 = \frac{1}{c_1}, \quad \kappa_2 = \frac{c'_1}{c_2} = \frac{c_1 c'_1}{c_1 c_2} \text{ and } \kappa_3 = \frac{c'_2 + c_1\kappa_2}{c_3} = \frac{c'_2 + c_1\frac{c'_1}{c_2}}{c_3} = \frac{c_2 c'_2 + c_1 c'_1}{c_2 c_3}.$$

Suppose that

$$\kappa_i = \frac{c_{i-1}c'_{i-1} + \cdots + c_2c'_2 + c_1c'_1}{c_{i-1}c_i}. \tag{4}$$

The scalar Frenet equations of Theorem 1 imply that $c_{i+1}\kappa_{i+1} = c'_i + c_{i-1}\kappa_i$. Substitute equation (4) to obtain

$$c_{i+1}\kappa_{i+1} = c'_i + \frac{c_{i-1}c'_{i-1} + \cdots + c_2c'_2 + c_1c'_1}{c_i} = \frac{c_ic'_i + \cdots + c_2c'_2 + c_1c'_1}{c_i}. \quad \square$$

Proof of Theorem 4. We have $R_\ell^2 = c_1^2 + \cdots + c_\ell^2$. Thus $R_\ell R'_\ell = c_1c'_1 + \cdots + c_\ell c'_\ell$. Combine last equation with the formula of Theorem 2 to obtain

$$R_\ell R'_\ell = c_\ell c_{\ell+1}\kappa_{\ell+1}, \quad \text{for } 1 \le \ell < m.$$

For a generic curve in $\mathbb{R}^{m+1}$ the first $m-1$ Euclidean curvatures are nowhere vanishing and the mth Euclidean curvature may vanish at isolated points, which do not coincide with the points at which R_{m-1} is critical. Thus for a generic curve in $\mathbb{R}^{m+1}$, $m > 1$, $R'_\ell = 0$ if and only if either $c_\ell = 0$ or $c_{\ell+1} = 0$ for $1 \le \ell < m$. Moreover, for a smoothly immersed curve the function $c_1 = R_1 = 1/\kappa_1$ never vanishes. This proves Theorem 4. $\square$

Proof of Theorem 5. Write $\sigma(s)$ for the value of the arc length parameter of C_γ at $C_\gamma(s)$. We assume that the orientations of the parametrised focal curve C_γ given by the arc length parameter s of γ and by the arc length parameter σ of C_γ coincide. Lemma 2 and Theorem 3 imply that, at a non-vertex of γ, the unit tangent vector of the parametrised focal curve C_γ is

$$\mathbf{T} = \frac{(c'_m + c_{m-1}\kappa_m)}{|c'_m + c_{m-1}\kappa_m|}\mathbf{n}_m = \varepsilon\mathbf{n}_m. \tag{5}$$

Moreover, for any non vertex

$$\frac{ds}{d\sigma} = \frac{1}{|c'_m + c_{m-1}\kappa_m|}.$$

In order to obtain that

$$\mathbf{N}_1 = \delta_1\mathbf{n}_{m-1} \tag{6}$$

$$\text{and } K_1 = \frac{|\kappa_m|}{|c'_m + c_{m-1}\kappa_m|},$$

derive equation (5) with respect to σ and apply the Frenet equations of γ taking into account that the first $m-1$ Euclidean curvatures of a generic curve are always positive. In the same way, use equation (6) to obtain

$$\mathbf{N}_2 = \delta_2 \mathbf{n}_{m-2} \text{ and } K_2 = \frac{\kappa_{m-1}}{|c'_m + c_{m-1}\kappa_m|}.$$

To finish the proof, apply induction process. □

References

[1] Arnold V.I., Varchenko A.N. and Gussein–Zade S.M., *Singularities of Differentiable Maps*, Vol. 1, Birkhäser (1986).

[2] Arnold V.I., *Singularities of caustics and wave fronts*, Kluwer, Maths. and its Appls., Soviet series, **62** (1991).

[3] Arnold V.I., *Sur les propriétés des projections Lagrangiennes en géométrie symplectique des caustiques*, Rev. mat. Univ. Complut. Madrid, **8**(1): (1995), 109–119.

[4] Arnold V.I., *On the Number of Flattening Points of Space Curves*, Amer. Math. Soc. Trans. Ser. **171** (1995), 11–22.

[5] Arnold V.I., *The Geometry of Spherical Curves and the Algebra of Quaternions*, Russian Math. Surveys **50**(1): (1995), 1–68.

[6] Arnold V.I., *Towards the Legendrian Sturm Theory of Space Curves*, Funct. Anal. and Appl. **32**(2): (1998), 75–80.

[7] Barner M., *Über die Mindestanzahl stationärer Schmiegeebenen bei geschlossenen Streng-Konvexen Raumkurven*, Abh. Math. Sem. Univ. Hamburg, **20** (1956), 196–215.

[8] Blaschke W., *Vorlesungen über Differentialgeometrie* I, 3th ed. Spriger-Verlag, Berlin, (1930).

[9] Blaschke W. and Leichweiß K., *Elementare Differentialgeometrie*, 5th reworked Edition, Spriger-Verlag, Berlin, (1973).

[10] Darboux G., *Leçons sur la théorie des surfaces*, Vol. 1, Chap. 1, 2nd revised and enlarged Edition, Gauthier-Villars, Paris, (1914).

[11] Ghys E., *Cercles osculateurs et géométrie lorentzienne*, Talk at the Jounée inaugurale de CMI, Marseille, February (1995).

[12] Hilbert D. and Cohn Vossen S., *Geometry and the Imagination*, Chelsea Pub. Comp., N. Y. (1952).

[13] Kazarian M., *Nonlinear Version of Arnol'd's Theorem on Flattening Points*, C.R. Acad. Sci. Paris, t. **323**, Série I, no.1, (1996), 63–68.

[14] Kneser A., *Bemerkungen über die Anzahl der Extreme der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nicht–euklidischen Geometrie*, Festschrift Heinrich Weber zu Seinem Siebzigsten Geburstag, Teubner, Leipzig, (1912), 170–180.

[15] Mohrmann H., *Die minimalzahl der stationären Ebenen eines räumlichen Ovals*, Sitz. Ber. klg. Bayerichen Akad. Wiss. Math. Phys., Kl., (1917), 1–3.

[16] Mukhopadhyaya S., *New Methods in the Geometry of a Plane Arc I*, Bull. Calcutta Math. Soc. **1** (1909), 31–37.

[17] Ovsienko V. and Tabachnikov S., *Sturm theory, Ghys Theorem on zeroes of the Schwarzian derivative and flattenings of Legendrian curves*, Selecta Math. New Series **2**(2): (1996), 297–307.

[18] Romero-Fuster M.C., *Convexly-generic curves in* $\mathbb{R}^3$, Geometriae Dedicata **28** (1988), 7–29.

[19] Romero-Fuster M.C. and Sanabria-Codesal E., *Generalized evolutes, vertices and conformal invariants of curves in* $\mathbb{R}^{n+1}$, Indag. Math. N.S. **10**(2): (1999), 297–305.

[20] Sedykh V.D., *The Theorem About Four Vertices of a Convex Space Curve*, Functional Anal. Appl. **26**(1): (1992), 28–32.

[21] Segre B., *Alcune proprietà differenziali delle curve chiuse sghembe*, Rend. Mat. **6**(1): (1968), 237–297.

[22] Shcherbak O.P., *Projectively Dual Space Curves and Legendre Singularities*, Proc. Tbilisi Univ., **232–233**, 280–336, (1982); English transl. in Selecta Math. Soviet., **5**(4): (1986), 391–421.

[23] Uribe-Vargas R., *On the Higher Dimensional Four-Vertex Theorem*, C.R. Acad. Sci. Paris, t. **321**, Série I, (1995), 1353–1358.

[24] Uribe-Vargas R., *On the* $(2k+2)$*-Vertex and* $(2k+2)$*-Flattening Theorems in Higher Dimensional Lobatchevskian Space*, C.R. Acad. Sci. Paris, t. **325**, Série I, (1997), 505–510.

[25] Uribe-Vargas R., *Four-Vertex Theorems in Higher Dimensional Spaces for a Larger Class of Curves than the Convex Ones*, C.R. Acad. Sci. Paris, t. **330**, Série I, (2000), 1085–1090.

[26] Uribe-Vargas R., *Singularités symplectiques et de contact en géométrie différentielle des courbes et des surfaces*, PhD. Thesis, Université Paris 7, (2001). (In English).

[27] Uribe-Vargas R., *Rigid Body Motions and Arnold's Theory of Fronts on* $\mathbb{S}^2 \subset \mathbb{R}^3$, J. of Geometry and Physics, 45/1-2, (2003), 91–104.

[28] Uribe-Vargas R., *On Polar Duality, Lagrange and Legendre Singularities and Stereographic Projection to Quadrics*, Proceedings of London Mathematical Society, **87**/3 (2003), 701–724.

[29] Uribe-Vargas R., *On 4-Flattening Theorems and the Curves of Carathéodory, Barner and Segre*, Journal of Geometry, **77** (2003), 184a–192.

[30] Uribe-Vargas R., *On Singularities, "Perestroikas" and Differential Geometry of Space Curves, L'Enseigement Mathématique*, t. **50** (2004), 69–101.

[31] Uribe-Vargas R., *4-Vertex Theorems, Sturm Theory and Lagrangian Singularities, Mathematical Physics, Geometry and Analysis*, **7** (2004), 223–237.

Ricardo Uribe-Vargas
Collège de France
3 rue d'Ulm, 75005 Paris
FRANCE

E-mail: uribe@math.jussieu.fr
www.math.jussieu.fr/∼uribe/