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Positive periodic solutions for a nonlinear difference system via a continution theorem

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Abstract. Based on a continuation theorem of Mawhin, the existence of a positive periodic solution for a nonlinear difference system is studied.

Keywords: Nonlinear difference system, positive periodic solution, continution theorem.

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1 Introduction

In [1], we explained that scalar difference equations of the form

$$y_{n+1} = y_n \exp\{f(n, y_n, y_{n-1}, ..., y_{n-k})\}, \ n \in Z = \{0, \pm 1, \pm 2, ...\}, \tag{1}$$

where $f = f(t, u_0, u_1, ..., u_k)$ is a real continuous function defined on R^{k+2} such that

$$f(t+\omega, u_0, ..., u_k) = f(t, u_0, ..., u_k), \ (t, u_0, ..., u_k) \in R^{k+2},$$

and ω is a positive integer, are of interest since they include well known equations such as

$$y_{n+1} = y_n \exp\left\{\frac{\mu(1-y_n)}{K}\right\}, \ K > 0,$$

and they are intimately related to delay differential equations with piecewise constant independent arguments [2]:

$$y'(t) = y(t)\ f([t], y([t]), y([t-1]), y([t-2]), ..., y([t-k])), \ t \in R.$$

We also show that continuation theorems can be used to show existence of periodic solutions of these equations.

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Note that in the above equations, only one time dependent variable y_t is involved. In real problems, multiple time dependent variables may interact, and therefore it is natural to study systems of difference equations.

In this paper, we consider one such system of the form

$$y_i^{(n+1)} = y_i^{(n)} \exp\left(r_i^{(n)} - \sum_{j=1}^{k} a_{ij}^{(n)} y_j^{(n)} - \sum_{j=1}^{k} b_{ij}^{(n)} y_j^{\left(n-\tau_{ij}^{(n)}\right)} \right),$$
$$i \in \{1, ..., k\}, n \in Z, \tag{2}$$

where

$$r_i = \left\{ r_i^{(n)} \right\}_{n\in Z}, \quad a_{ij} = \left\{ a_{ij}^{(n)} \right\}_{n\in Z}, \quad b_{ij} = \left\{ b_{ij}^{(n)} \right\}_{n\in Z} \quad \text{and} \quad \tau_{ij} = \left\{ \tau_{ij}^{(n)} \right\}_{n\in Z},$$

are real ω-periodic sequences such that

$$\begin{aligned} r_i^{(n)} &= r_i^{(n+\omega)}, \ n \in Z \\ a_{ij}^{(n)} &= a_{ij}^{(n+\omega)}, \ n \in Z \\ b_{ij}^{(n)} &= b_{ij}^{(n+\omega)}, \ n \in Z \\ \tau_{ij}^{(n)} &= \tau_{ij}^{(n+\omega)}, \ n \in Z \end{aligned}$$

for $i, j \in \{1, ..., k\}$. We assume further that

$$a_{ij}^{(n)}, b_{ij}^{(n)} \geqslant 0, \ i, j \in \{1, ..., k\}; n \in Z,$$

$$\sum_{0 \le n \le \omega-1} r_i^{(n)} > 0, \ i \in \{1, ..., k\},$$

and

$$\sum_{0 \le n \le \omega-1} \left(a_{ii}^{(n)} + b_{ii}^{(n)} \right) \neq 0, \ i \in \{1, ..., k\}.$$

The number ω is a positive integer as before.

A solution of (2) is a real vector sequence of the form $y = \left\{ y^{(n)} \right\}_{n\in Z}$ where $y^{(n)} = \left(y_1^{(n)}, y_2^{(n)}, ..., y_k^{(n)} \right)^{\dagger}$ which renders (2) into an identity after substitution. As in [1], we are concerned with the existence of positive solutions which are ω-periodic, that is, solutions that satisfy $y^{(n+\omega)} = y^{(n)}$ for $n \in Z$ and $y_i^{(n)} > 0$ for $n \in Z$ and $i \in \{1, ..., k\}$.

Our system (2) can be used to describe multispecies ecological competition systems or multi-nation competition models. The analogous problem for differential systems has been treated by Smith [4], Cushing [5], Zanolin [6], Fan and

Wang [7] and others. In particular, in [7], the authors study differential systems of the form

$$y_i'(t) = y_i(t)\left(r_i(t) - \sum_{j=1}^{k} a_{ij}(t)y_j(t) - \sum_{j=1}^{k} b_{ij}(t)y_j(t-\tau_{ij})\right),\ i = 1, 2, ..., k.$$

As for our system, we can also show that it is related to differential systems with piecewise constant independent arguments of the form

$$y_i'(t) = y_i(t)\left(r_i([t]) - \sum_{j=1}^{k} a_{ij}([t])\, y_j(n) - \sum_{j=1}^{k} b_{ij}([t])\, y_j\left([t] - \tau_{ij}([t])\right)\right),$$
$$i \in \{1, ..., k\}, t \in R, \tag{3}$$

where $[x]$ is the greatest-integer function, $r_i(t)$, $a_{ij}(t)$ and $b_{ij}(t)$ are real continuous ω-periodic functions defined on R. Indeed, once the existence of a positive ω-periodic solution of (2) can be demonstrated, we may then make immediate statements about the existence of positive ω-periodic solutions of (3). The proof of our assertion is not much different from that of Theorem 1 in [1], and hence is not included here.

As in [1], we will invoke a continuation theorem of Mawhin for obtaining periodic solutions of (2). For the sake of easy reference, we briefly describe this result here. Let X and Y be two Banach spaces and $L : \mathrm{Dom}L \subset X \to Y$ is a linear mapping and $N : X \to Y$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \mathrm{Ker}L = \mathrm{codim}\ \mathrm{Im}L < +\infty$, and $\mathrm{Im}L$ is closed in Y. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I - Q)$. It follows that $L_{|\mathrm{Dom}L\cap\mathrm{Ker}P} : (I - P)X \to \mathrm{Im}L$ has an inverse which will be denoted by K_P. If Ω is an open and bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $\overline{K_P(I - Q)N(\bar{\Omega})}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$ there exist an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Theorem A (Mawhin's continuation theorem [1]). *Let L be a Fredholm mapping of index zero, and let N be L-compact on $\bar{\Omega}$. Suppose*

(i) *for each* $\lambda \in (0, 1)$, $x \in \partial\Omega$, $Lx \neq \lambda Nx$; *and*

(ii) *for each* $x \in \partial\Omega \cap \mathrm{Ker}L$, $QNx \neq 0$ *and* $\deg(JQN, \Omega \cap \mathrm{Ker}L, 0) \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \mathrm{dom}L$.

We recall the useful nonstandard "summation" operation [1] for any real sequence $\{u^{(n)}\}_{n\in Z}$:

$$\bigoplus_{n=\gamma}^{\beta} u^{(n)} = \begin{cases} \sum_{n=\gamma}^{\beta} u^{(n)}, & \gamma \le \beta \\ 0, & \beta = \gamma - 1 \\ -\sum_{n=\beta+1}^{\gamma-1} u^{(n)}, & \beta < \gamma - 1 \end{cases}.$$

As usual, the forward difference is defined by $\Delta u^{(k)} = u^{(k+1)} - u^{(k)}$.

We will also employ the following notations for the 'time' averages:

$$\begin{aligned} \overline{r}_i &= \frac{1}{\omega} \sum_{0\le n\le \omega-1} r_i^{(n)}, \\ \overline{R}_i &= \frac{1}{\omega} \sum_{0\le n\le \omega-1} \left| r_i^{(n)} \right|, \\ \overline{a}_{ij} &= \frac{1}{\omega} \sum_{0\le n\le \omega-1} a_{ij}^{(n)}, \\ \overline{b}_{ij} &= \frac{1}{\omega} \sum_{0\le n\le \omega-1} b_{ij}^{(n)}. \end{aligned}$$

2 Existence Criteria

The main result of our paper is the following.

Theorem 1. *Suppose the following set of conditions hold:*

(i) *for each $i \in \{1, ..., k\}$, $\overline{r}_i > 0$,*

(ii) *for $i, j \in \{1, ..., k\}$, the inverse of the matrix $\left(\overline{a}_{ij} + \overline{b}_{ij}\right)_{k\times k}$ exists and all its components are positive, and*

(iii) *for each $i \in \{1, ..., k\}$,*

$$\overline{r}_i > \sum_{1\le j\le k, j\ne i} \left(\overline{a}_{ij} + \overline{b}_{ij}\right) \frac{\overline{r}_j}{\overline{a}_{jj} + \overline{b}_{jj}} \exp\left(\frac{1}{2}\left(\overline{R}_j + \overline{r}_j\right)\omega\right).$$

Then (2) has a positive ω-periodic solution.

In order to provide a proof, we proceed in a manner similar to that of Theorem 1 in [1]. However, there are sufficient difference to warrant some details in the following discussions. We first note that if

$$x = \left\{ \left(x_1^{(n)}, x_2^{(n)}, ..., x_k^{(n)} \right)^{\dagger} \right\}_{n \in Z}$$

is a ω-periodic solution of the following system

$$x_i^{(n)} = x_i^{(0)} + \bigoplus_{s=0}^{n-1} \left(r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp\left(x_j^{(s)} \right) - \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{\left(s - \tau_{ij}^{(s)}\right)} \right),$$

$$i \in \{1, ..., k\}, n \in Z, \tag{4}$$

then we can easily check that

$$y = \left\{ \left(y_1^{(n)}, y_2^{(n)}, ..., y_k^{(n)} \right)^{\dagger} \right\}_{n \in Z} = \left\{ \left(e^{x_1^{(n)}}, e^{x_2^{(n)}}, ..., e^{x_k^{(n)}} \right)^{\dagger} \right\}_{n \in Z}$$

is a positive ω-periodic solution of (1).

We will therefore seek an ω-periodic solution of (4). Let X_ω be the Banach space of all real vector ω-periodic sequences of the form $x = \{x^{(n)}\}_{n \in Z}$ where $x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, ..., x_k^{(n)} \right)^{\dagger}$ and endowed with the usual linear structure as well as the norm

$$\|x\|_1 = \left(\sum_{1 \le i \le k} \left(\max_{0 \le n \le \omega - 1} \left| x_i^{(n)} \right| \right)^2 \right)^{\frac{1}{2}}.$$

Let Y_ω be the Banach space of all real sequences of the form

$$y = \{y^{(n)}\}_{n \in Z} = \left\{ n\alpha + h^{(n)} \right\}_{n \in Z}$$

such that $y^{(0)} = 0$, where $\alpha = (\alpha_1, ..., \alpha_k)^{\dagger} \in R^k$ and $\{h^{(n)}\}_{n \in Z} \in X_\omega$, and endowed with the usual linear structure as well as the norm $\|y\|_2 = |\alpha| + \|h\|_1$, here $|\alpha| = \left(\sum_{1 \le i \le k} \alpha_i^2 \right)^{\frac{1}{2}}$. Let the zero element of X_ω and Y_ω be denoted by θ_1 and θ_2 respectively.

Define the mappings $L : X_\omega \to Y_\omega$ and $N : X_\omega \to Y_\omega$ respectively by

$$(Lx)^{(n)} = x^{(n)} - x^{(0)}, \; n \in Z. \tag{5}$$

and

$$(Nx)_i^{(n)} = \bigoplus_{s=0}^{n-1} \left(r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{\left(s-\tau_{ij}^{(s)}\right)} \right),$$
$$i \in \{1, ..., k\}, n \in Z. \tag{6}$$

Let

$$\begin{aligned} \bar{h}_i^{(n)} = & \bigoplus_{s=0}^{n-1} \left(r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp \left(x_j^{\left(s-\tau_{ij}^{(s)}\right)} \right) \right) \\ & - \frac{n}{\omega} \bigoplus_{s=0}^{\omega-1} \left(r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp \left(x_j^{\left(s-\tau_{ij}^{(s)}\right)} \right) \right) \end{aligned} \tag{7}$$

for $i = 1, ..., k$ and $n \in Z$.

Since $\bar{h} = \{\bar{h}^{(n)}\}_{n\in Z} \in X_\omega$ and $\bar{h}^{(0)} = \theta_1$, N is a well-defined operator from X_ω to Y_ω. On the other hand, direct calculation shows that $\mathrm{Ker}L = \{x \in X_\omega \mid x^{(n)} = x^{(0)}, n \in Z, x^{(0)} \in R^k\}$ and $\mathrm{Im}L = X_\omega \cap Y_\omega$. Let us define $P : X_\omega \to X_\omega$ and $Q : Y_\omega \to Y_\omega$ respectively by

$$(Px)^{(n)} = x^{(0)}, \ \ n \in Z, \tag{8}$$

for $x = \{x^{(n)}\}_{n\in Z} \in$ and

$$(Qy)^{(n)} = n\alpha \tag{9}$$

for $y = \{n\alpha + h^{(n)}\}_{n\in Z} \in Y_\omega$. The operators P and Q are projections and $X_\omega = \mathrm{Ker}P \oplus \mathrm{Ker}L$, $Y_\omega = \mathrm{Im}L \oplus \mathrm{Im}Q$. It is easy to see that $\dim \mathrm{Ker}L = k = \dim \mathrm{Im}Q = \mathrm{codimIm}L$, and

$$\mathrm{Im}L = \{y \in X_\omega \mid y(0) = 0\} \subset Y_\omega.$$

It follows that $\mathrm{Im}L$ is closed in Y_ω. Thus the following Lemma is true.

Lemma 1. *The mapping L defined by (5) is a Fredholm mapping of index zero.*

Lemma 2. *Let L and N defined by (5) and (6) respectively. Suppose Ω is an open and bounded subset of X_ω. Then N is L-compact on $\overline{\Omega}$.*

Proof. From (6), (7) and (9), we see that for any $x = \{x^{(n)}\}_{n\in Z} \in X_\omega$,

$$(QNx)_i^{(n)} = \frac{n}{\omega}\bigoplus_{s=0}^{\omega-1}\left(r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp\left(x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right)\right),$$

$$i \in \{1, 2, .., k\}, n \in Z. \tag{10}$$

We denote the inverse of the mapping $L\mid_{\mathrm{Dom}L\cap\mathrm{Ker}P}: (I-P)\,X \to \mathrm{Im}L$ by K_P. Direct calculation leads to

$$(K_P(I-Q)Nx)_i^{(n)} = \bigoplus_{s=0}^{n-1}\left(r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right)$$

$$-\frac{n}{\omega}\bigoplus_{s=0}^{\omega-1}\left(r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right). \tag{11}$$

It is easy to see that QN and $K_P(I-Q)N$ are continuous on X_ω and takes bounded sets into bounded sets respectively. Since the Banach space X_ω is finite dimensional, N is L-compact on $\overline{\Omega}$. The proof is complete. □

Let l_ω, where $\omega \geqslant 2$ is positive number, be the space of all real ω-periodic sequences of the form $u = \left\{u^{(n)}\right\}_{n\in Z}$.

Lemma 3. *If $u = \left\{u^{(n)}\right\}_{n\in Z} \in l_\omega$, then*

$$\max_{0\le s,i\le\omega-1}\left|u^{(s)} - u^{(i)}\right| \le \frac{1}{2}\sum_{k=0}^{\omega-1}\left|\Delta u^{(k)}\right|, \tag{12}$$

where the constant factor $1/2$ *is the best possible.*

Proof. Let $u = \left\{u^{(n)}\right\}_{n\in Z} \in l_\omega$ and $s, i \in \{0, 1, ..., \omega-1\}$. Without loss of any generality, let $s \in \{i+1, ..., i+\omega-1\}$, we have

$$u^{(s)} = u^{(i)} + \sum_{k=i}^{s-1}\Delta u^{(k)} \tag{13}$$

and

$$u^{(i)} = u^{(i+\omega)} = u^{(s)} + \sum_{k=s}^{i+\omega-1}\Delta u^{(k)}. \tag{14}$$

From (13) and (14), we see that for any $s \in \{i, i+1, \dots, i+\omega-1\}$,

$$2u^{(s)} = 2u^{(i)} + \sum_{k=i}^{s-1} \Delta u^{(k)} - \sum_{k=s}^{i+\omega-1} \Delta u^{(k)}, \tag{15}$$

that is

$$u^{(s)} = u^{(i)} + \frac{1}{2}\left\{\sum_{k=i}^{s-1} \Delta u^{(k)} - \sum_{k=s}^{i+\omega-1} \Delta u^{(k)}\right\}. \tag{16}$$

Thus for any $s \in \{i, i+1, \dots, i+\omega-1\}$,

$$\left|u^{(s)} - u^{(i)}\right| \le \frac{1}{2}\sum_{k=i}^{i+\omega-1} \left|\Delta u^{(k)}\right| = \frac{1}{2}\sum_{k=0}^{\omega-1}\left|\Delta u^{(k)}\right|, \tag{17}$$

so that

$$\max_{0\le s,i\le\omega-1}\left|u^{(s)} - u^{(i)}\right| \le \frac{1}{2}\sum_{k=0}^{\omega-1}\left|\Delta u^{(k)}\right|. \tag{18}$$

Now we assert that if β is a constant and $\beta < 1/2$, then there are $u = \left\{u^{(n)}\right\}_{n\in Z} \in l_\omega$ and such that

$$\max_{0\le s,i\le\omega-1}\left|u^{(s)} - u^{(i)}\right| > \beta\sum_{k=0}^{\omega-1}\left|\Delta u^{(k)}\right|. \tag{19}$$

Indeed, if we let $u^{(n)} = j$ for $n = k\omega + j$, $k \in Z$ and $j = 0, 1, ..., \omega-1$, then $\max_{0\le s,i\le\omega-1}\left|u^{(s)} - u^{(i)}\right| = \omega - 1$ and

$$\Delta u^{(n)} = \begin{cases} 1, & n = 0, 1, .., \omega-2 \\ -(\omega-1), & n = \omega-1 \end{cases}, \tag{20}$$

and

$$\beta\sum_{k=0}^{\omega-1}\left|\Delta u^{(k)}\right| = 2\beta(\omega-1) < \max_{0\le s,i\le\omega-1}\left|u^{(s)} - u^{(i)}\right|$$

as required. This shows that the constant $1/2$ in (12) is the best possible. The proof is complete. $\square$

Now, we consider the following system

$$x_i^{(n)} - x_i^{(0)} = \lambda \bigoplus_{s=0}^{n-1}\left(r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp\left(x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right)\right),$$
$$i \in \{1, ..., k\}, n \in Z, \tag{21}$$

where $\lambda \in (0, 1)$.

Lemma 4. *Suppose the condition (iii) in Theorem 1 holds. Then there exist positive constants $H_1, ..., H_k$ such that for any solution $x = \{x^{(n)}\}_{n\in Z} = \left\{\left(x_1^{(n)}, ..., x_k^{(n)}\right)^\dagger\right\}_{n\in Z} \in X_\omega$ of (21), we have the following inequalities*

$$\max_{0\le n\le\omega-1}\left|x_i^{(n)}\right| \le H_i,\ \ i\in\{1, ..., k\}. \tag{22}$$

Proof. Let $x = \{x^{(n)}\}_{n\in Z}$ be a ω-periodic solution of (21). Then

$$\bigoplus_{s=0}^{\omega-1}\left(r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)}\exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)}\exp x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right) = 0,\ \ i\in\{1, ..., k\}. \tag{23}$$

It leads to

$$\bigoplus_{s=0}^{\omega-1}\left(\sum_{j=1}^{k} a_{ij}^{(s)}\exp x_j^{(s)} + \sum_{j=1}^{k} b_{ij}^{(s)}\exp x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right) = \omega\overline{r_i}. \tag{24}$$

From (21), we have

$$\Delta x_i^{(n)} = \lambda\left(r_i^{(n)} - \sum_{j=1}^{k} a_{ij}^{(n)}\exp x_j^{(n)} - \sum_{j=1}^{k} b_{ij}^{(n)}\exp\left(x_j^{\left(n-\tau_{ij}^{(n)}\right)}\right)\right),$$
$$i\in\{1, ..., k\}, n\in Z. \tag{25}$$

By (24) and (25), we see that

$$\begin{aligned}\bigoplus_{s=0}^{\omega-1}\left|\Delta x_i^{(s)}\right| &\le \bigoplus_{s=0}^{\omega-1}\left(\left|r_i^{(s)}\right| + \sum_{j=1}^{k} a_{ij}^{(s)}\exp x_j^{(s)} + \sum_{j=1}^{k} b_{ij}^{(s)}\exp x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right)\\ &= \bigoplus_{s=0}^{\omega-1}\left|r_i^{(s)}\right| + \bigoplus_{s=0}^{\omega-1}\left(\sum_{j=1}^{k} a_{ij}^{(s)}\exp\left(x_j^{(s)}\right) + \sum_{j=1}^{k} b_{ij}^{(s)}\exp x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right)\\ &= \left(\overline{R}_i + \overline{r}_i\right)\omega.\end{aligned} \tag{26}$$

Let $x_i^{(\mu_i)} = \max_{0\le n\le\omega-1} x_i^{(n)}$ and $x_i^{(\nu_i)} = \min_{0\le n\le\omega-1} x_i^{(n)}$, where $0 \le \mu_i, \nu_i \le \omega - 1$. From (24), we have

$$\begin{aligned}
\omega\bar{r}_i &\geqslant \bigoplus_{s=0}^{\omega-1}\left(\sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(\nu_j)} + \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{(\nu_j)}\right)\\
&= \sum_{j=1}^{k}\left(\bar{a}_{ij} + \bar{b}_{ij}\right)\omega \exp x_j^{(\nu_j)}\\
&\geqslant \left(\bar{a}_{ii} + \bar{b}\right)\omega \exp x_i^{(\nu_i)},
\end{aligned} \tag{27}$$

that is,

$$x_i^{(\nu_i)} \le \ln\left\{\frac{\bar{r}_i}{\bar{a}_{ii} + \bar{b}_{ii}}\right\}. \tag{28}$$

In view of Lemma 3, (26) and (28), we see that for any $n = 0, 1, ..., \omega - 1$,

$$x_i^{(n)} \le x_i^{(\nu_i)} + \frac{1}{2}\sum_{k=0}^{\omega-1}\left|\Delta x_i^{(k)}\right| \le B_i, \tag{29}$$

where

$$B_i = \ln\left\{\frac{\bar{r}_i}{\bar{a}_{ii} + \bar{b}_{ii}}\right\} + \frac{1}{2}\left(\bar{R}_i + \bar{r}_i\right)\omega. \tag{30}$$

Furthermore, from (24), we have

$$\begin{aligned}
\omega\bar{r}_i &\le \bigoplus_{s=0}^{\omega-1}\left(\sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(\mu_j)} + \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{(\mu_j)}\right)\\
&= \sum_{j=1}^{k}\left(\bar{a}_{ij} + \bar{b}_{ij}\right)\omega \exp x_j^{(\mu_j)}.
\end{aligned} \tag{31}$$

By (29) and (31), we see that

$$\begin{aligned}
&\left(\bar{a}_{ii} + \bar{b}_{ii}\right)\exp x_i^{(\mu_i)}\\
&\geqslant \bar{r}_i - \sum_{1\le j\le k, j\ne i}\left(\bar{a}_{ij} + \bar{b}_{ij}\right)\exp x_j^{(\mu_j)}\\
&\geqslant \bar{r}_i - \sum_{1\le j\le k, j\ne i}\left(\bar{a}_{ij} + \bar{b}_{ij}\right)\frac{\bar{r}_j}{\bar{a}_{jj} + \bar{b}_{jj}}\exp\left(\frac{1}{2}\left(\bar{R}_j + \bar{r}_j\right)\omega\right),
\end{aligned} \tag{32}$$

that is

$$x_i^{(\mu_i)} \geqslant C_i, \tag{33}$$

where

$$C_i = \ln \left\{ \frac{\overline{r}_i - \sum_{1\le j\le k, j\ne i} \left(\overline{a}_{ij} + \overline{b}_{ij}\right) \frac{\overline{r}_j}{\overline{a}_{jj}+\overline{b}_{jj}} \exp\left(\frac{1}{2}\left(\overline{R}_j + \overline{r}_j\right)\omega\right)}{\overline{a}_{ii} + \overline{b}_{ii}} \right\}. \tag{34}$$

In view of Lemma 3, (26) and (33), we see that for any $n = 0, 1, ..., \omega - 1$,

$$x_i^{(n)} \geqslant x_i^{(\mu_i)} - \frac{1}{2}\sum_{k=0}^{\omega-1}\left|\Delta x_i^{(k)}\right| \geqslant C_i - \frac{1}{2}\left(\overline{R_i} + \overline{r_i}\right)\omega. \tag{35}$$

From (29) and (35), we have

$$\max_{0\le n\le\omega-1}\left|x_i^{(n)}\right| \le H_{i,} \tag{36}$$

where $H_i = \max\left\{|B_i|, \left|C_i - \frac{1}{2}\left(\overline{R_i} + \overline{r_i}\right)\omega\right|\right\} + 1$. The proof is complete. $\square$

Proof of Theorem 1. Let L, N, P and Q be defined by (5), (6), (8) and (9) respectively. By conditions (i) and (ii), we know that the linear system of the form

$$\overline{r}_i - \sum_{j=1,}^{k}\left(\overline{a}_{ij} + \overline{b}_{ij}\right) v_j = 0, \ i \in \{1, ..., k\}, \tag{37}$$

has the unique solution $v^* = \left(v_1^*, v_2^*, ..., v_k^*\right)^\dagger$ and $v_i^* > 0$ for $i \in \{1, ..., k\}$. Pick M such that

$$\left(\sum_{i=1}^{k}\left(\ln v_i^*\right)^2\right)^{\frac{1}{2}} < M. \tag{38}$$

From Lemma 4, we know there exist positive constants $H_1, ..., H_k$ such that for any solution $x = \{x^{(n)}\}_{n\in Z} = \left\{\left(x_1^{(n)}, ..., x_k^{(n)}\right)^\dagger\right\}_{n\in Z} \in X_\omega$ of (21), we have the following inequalities

$$\max_{0\le n\le\omega-1}\left|x_i^{(n)}\right| \le H_i, \ i = 1, ..., k. \tag{39}$$

Let $H = \left(\sum_{i=1}^{k} H_i^2\right)^{\frac{1}{2}} + M$. Then $\|x\|_1 < H$. Set

$$\Omega = \{x \in X_\omega | \ \|x\|_1 < H\}.$$

It is easy to see that Ω is an open and bounded subset of X_ω and for each $\lambda \in (0, 1)$ and $x \in \partial\Omega$, $Lx \neq \lambda Nx$. Furthermore, in view of Lemma 1 and Lemma 2, L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$. Noting that $H > \left(\sum_{i=1}^{k} H_i^2\right)^{\frac{1}{2}}$, by Lemma 4, for each $\lambda \in (0, 1)$ and $x \in \partial\Omega$, $Lx \neq \lambda Nx$. Next note that a vector sequence $x = \{x^{(n)}\}_{n\in Z} \in \partial\Omega \cap \mathrm{Ker}L$ must be a constant vector and $\|x\|_1 = H > M$. Hence

$$\|QNx\|_2 = \left\{\sum_{i=1}^{k}\left(\left(\bar{r}_i - \sum_{j=1}^{k}\left(\bar{a}_{ij} + \bar{b}_{ij}\right)\exp x_j\right)\right)^2\right\}^{\frac{1}{2}} \neq 0.$$

So

$$QNx \neq \theta_2.$$

The isomorphism $J\colon\ \mathrm{Im}\,Q \to \mathrm{Ker}L$ is defined by $JQy = \alpha$ for $y = \{n\alpha + h^{(n)}\}_{n\in Z} \in Y_\omega$. Then

$$\begin{aligned}(JQNx)_i^{(n)} &= \frac{1}{\omega}\bigoplus_{s=0}^{\omega-1}\left(r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)}\exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)}\exp x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right)\\ &= \bar{r}_i - \sum_{j=1}^{k}\left(\bar{a}_{ij} + \bar{b}_{ij}\right)\exp x_j,\end{aligned} \tag{40}$$

for $n \in Z$ and $i \in \{1, ..., k\}$. Since (37) has the unique solution $v^* = \left(v_1^*, v_2^*, ..., v_k^*\right)^\dagger$ with positive components and such that (38) is satisfied, we see that the system

$$\bar{r}_i - \sum_{j=1}^{k}\left(\bar{a}_{ij} + \bar{b}_{ij}\right)\exp\left(x_j\right) = 0,\ i \in \{1, ..., k\} \tag{41}$$

has a unique solution $\bar{x} = \left(x_1^*, x_2^*, ..., x_k^*\right)^\dagger$ in $\Omega \cap \mathrm{Ker}\,L$, so that from the condition (ii) we have

$$\deg\left(JQNx, \Omega \cap \mathrm{Ker}\,L, \theta_1\right) = \text{ sign } \det \Upsilon_{JQN}\,(\bar{x}) \neq 0.$$

where $\Upsilon_{JQN}\,(\bar{x})$ is the Jacobi matrix of JQN at $\bar{x}$. By Theorem A, we see that equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \mathrm{Dom}L$. In other words, (4) has a ω-periodic solution $x = \{x^{(n)}\}_{n\in Z}$, and hence $\left\{\left(e^{x_1^{(n)}}, ..., e^{x_k^{(n)}}\right)\right\}_{n\in Z}$ is a positive ω-periodic solution of (2). The proof is complete. □

We remark that by the relationship that exists between (2) and (3), under the same assumption of Theorem 1, system (3) has a positive ω-periodic solution.

We now illustrate our main result by considering the following system

$$\begin{cases} y_1^{(n+1)} = y_1^{(n)} \exp\left(r_1^{(n)} - a_{11}^{(n)} y_1^{(n)} - b_{12}^{(n)} y_2^{\left(n-\tau_{12}^{(n)}\right)} \right) & , \\ y_2^{(n+1)} = y_2^{(n)} \exp\left(r_2^{(n)} - a_{22}^{(n)} y_2^{(n)} - b_{21}^{(n)} y_1^{\left(n-\tau_{21}^{(n)}\right)} \right) & , \end{cases}$$

where r_i, b_{ij}, a_{ii} and τ_{ij} for $i,\ j \in \{1, 2\}$ are 2-periodic sequences and

$$r_1^{(0)} = 0,\ r_1^{(1)} = 1,\ r_2^{(0)} = 1,\ r_2^{(1)} = 0,\ a_{11}^{(0)} = 1/3,\ a_{11}^{(1)} = 2/3,\ a_{22}^{(0)} = 2/3,$$

$$a_{22}^{(1)} = 1/3,\ b_{12}^{(0)} = 1/6e,\ b_{12}^{(1)} = 1/4e,\ b_{21}^{(0)} = 1/5e,\ b_{21}^{(1)} = 1/7e.$$

It is easily verified from Theorem 1 that it has a positive 2-periodic solution.

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