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Some beta distributions

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Abstract. Four new generalizations of the standard beta distribution are introduced. Various properties are derived for each distribution, including its hazard rate function and moments.

Keywords: beta distributions, Hazard rate function, moments.

Mathematical subject classification: 33C90, 62E99.

1 Introduction

Beta distributions are very versatile and a variety of uncertainties can be usefully modeled by them. Many of the finite range distributions encountered in practice can be easily transformed into the standard distribution. In reliability and life testing experiments, many times the data are modeled by finite range distributions, see for example Barlow and Proschan (1975).

A random variable X is said to have the standard beta distribution with parameters a and b if its probability density function (pdf) is:

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \tag{1}$$

for $0 < x < 1$, $a > 0$ and $b > 0$, where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$$

denotes the beta function. Many generalizations of (1) involving algebraic, exponential and hypergeometric functions have been proposed in the literature; see Chapter 25 in Johnson et al. (1995) and Gupta and Nadarajah (2004) for detailed accounts. In this paper, we introduce four new generalizations of (1). We derive

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various properties for each distribution, including its pdf, cdf (cumulative distribution function), moments and the hazard rate function. These results are given in Sections 2, 3, 4 and 5. The calculations of this paper use the Anger function, the Fresnel cosine integral, the Struve function, the incomplete gamma function, the Lerch function, the incomplete beta function, and the hypergeometric functions defined by

$$\mathbf{J}_a(x) = \frac{1}{\pi} \int_0^{\pi} \cos(at - x \sin t) dt,$$

$$C(x) = \frac{2}{\sqrt{2\pi}} \int_0^{x} \cos\left(t^2\right) dt,$$

$$\mathbf{H}_a(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+a+1}}{\Gamma(k+3/2)\,\Gamma(a+k+3/2)},$$

$$\gamma(a, x) = \int_0^{x} t^{a-1} \exp(-t) dt,$$

$$\Phi(x, a, b) = \sum_{k=0}^{\infty} (b+k)^{-a} x^k,$$

$$B_x(a, b) = \int_0^{x} t^{a-1} (1-t)^{b-1} dt,$$

$${}_1F_1(\alpha; \beta; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} \frac{x^k}{k!},$$

$${}_1F_2(\alpha; \beta, \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k (\gamma)_k} \frac{x^k}{k!},$$

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!},$$

and

$$
{}_3F_2\left(\alpha, \beta, \gamma; \lambda, \mu; x\right) \quad = \quad \sum_{k=0}^{\infty} \frac{(\alpha)_k\,(\beta)_k\,(\gamma)_k}{(\lambda)_k\,(\mu)_k} \frac{x^k}{k!},
$$

where $(c)_k = c(c+1)\cdots(c+k-1)$ denotes the ascending factorial. A useful relationship between the incomplete beta function and the ${}_2F_1$ function (also known as the Gauss hypergeometric function) is that

$$
B_x(a, b) \quad = \quad \frac{x^a}{a}\,{}_2F_1\left(a, 1-b; a+1; x\right). \tag{2}
$$

The properties of these special functions can be found in Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000).

2 Compound beta

This generalization is based on the characterization that if X and Y are independent gamma random variables then the ratio $X/(X+Y)$ has the pdf (1). Here, we take X and Y to have the compound gamma distribution with the pdfs

$$
f(x) \quad = \quad \frac{x^{a-1}(1+x)^{-(a+b)}}{B(a, b)} \tag{3}
$$

and

$$
f(y) \quad = \quad \frac{y^{\alpha-1}(1+y)^{-(\alpha+\beta)}}{B(\alpha, \beta)} \tag{4}
$$

for $x > 0$ and $y > 0$, respectively, and then consider the distribution of $W = X/(X+Y)$. We refer to this as the *compound beta* distribution. A practical example of the use of W can be described as follows: suppose there are two financial institutions A and B and that one is interested in quantifying their relative performance. Assume that some financial indices for the two institutions A and B are compound gamma distributed according to (3) and (4), respectively. This assumption is very reasonable because the compound gamma distribution (which is also known as the beta distribution of the second kind) has been heavily applied in the areas of finance and economics. Under the assumption, it is clear that the variable $W = X/(X+Y)$ will reflect the relative performance of institution A with respect to institution B. Hence, measures of relative performance can be based on the distribution of W.

The variable of interest is $W = X/(X+Y)$. The cdf corresponding to (4) is

$$\begin{aligned} F(y) &= \frac{\int_0^y t^{\alpha-1}(1+t)^{-(\alpha+\beta)}dt}{B(\alpha,\beta)} \\ &= \frac{\int_{1/(1+y)}^1 w^{\beta-1}(1-w)^{\alpha-1}dw}{B(\alpha,\beta)} \\ &= 1-\frac{B_{1/(1+y)}(\beta,\alpha)}{B(\alpha,\beta)}. \end{aligned}$$

Thus, the cdf of W can be written as

$$\begin{aligned} F(w) &= \Pr\left(\frac{X}{X+Y} \le w\right) \\ &= \Pr\left(Y \ge \frac{1-w}{w}X\right) \\ &= 1-\Pr\left(Y \le \frac{1-w}{w}X\right) \\ &= 1-\int_0^\infty \left\{\frac{B_{w/\{w+(1-w)x\}}(\beta,\alpha)}{B(\alpha,\beta)}\right\} \frac{x^{a-1}(1+x)^{-(a+b)}}{B(a,b)}dx \\ &= \frac{I(w)}{B(a,b)B(\alpha,\beta)}, \end{aligned} \tag{5}$$

where

$$I(w) = \int_0^\infty x^{a-1}(1+x)^{-(a+b)} B_{w/\{w+(1-w)x\}}(\beta,\alpha)dx. \tag{6}$$

Using the relationship (2), (6) can be rewritten as

$$\begin{aligned} I(w) &= \frac{1}{\beta}\int_0^\infty x^{a-1}(1+x)^{-(a+b)}\left(1+\frac{1-w}{w}x\right)^{\beta} \\ &\quad {}_2F_1\left(\beta, 1-\alpha; \beta+1; \frac{w}{w+(1-w)x}\right)dx. \end{aligned} \tag{7}$$

Setting $y = w/\{w+(1-w)x\}$, (7) can be further rewritten as

$$I(w) = \frac{1}{\beta}w^a(1-w)^b(1-2w)^{-(a+b)}J(w), \tag{8}$$

where

$$J(w) = \int_0^1 y^{b-\beta-1}(1-y)^{a-1}\left(\frac{w}{1-2w}+y\right)^{-(a+b)} {}_2F_1(\beta, 1-\alpha; \beta+1; y)\, dy. \tag{9}$$

Using the series representation

$$(1+z)^\alpha = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j+1)} \frac{z^j}{j!},$$

(9) can be expanded as

$$J(w) = \sum_{k=0}^{\infty} \frac{(-1)^k (a+b)_k}{k!} \left(\frac{w}{1-2w}\right)^{-(a+b+k)} J_k, \tag{10}$$

where

$$J_k = \int_0^1 y^{k+b-\beta-1}(1-y)^{a-1}\, {}_2F_1(\beta, 1-\alpha; \beta+1; y)\, dy. \tag{11}$$

By application of equation (2.21.1.5) in Prudnikov et al. (1986, volume 3), (11) can be evaluated as

$$J_k = B(k+b-\beta, a)\ {}_3F_2(\beta, 1-\alpha, b-\beta; \beta+1, b+a-\beta; 1). \tag{12}$$

Combining (5), (8), (10) and (12), one obtains the cdf of W as

$$F(w) = C\left(\frac{1-w}{w}\right)^b \sum_{k=0}^{\infty} \frac{(-1)^k (a+b)_k}{k!} \left(\frac{1-2w}{w}\right)^k B(k+b-\beta, a), \tag{13}$$

where the constant C is given by

$$C = \frac{{}_3F_2(\beta, 1-\alpha, b-\beta; \beta+1, b+a-\beta; 1)}{\beta B(a,b) B(\alpha,\beta)}. \tag{14}$$

Differentiating (13) with respect to w, one obtains the corresponding pdf as

$$f(w) = \frac{C(1-w)^{b-1}}{w^{b+1}(1-2w)} \sum_{k=0}^{\infty} \frac{(-1)^k (a+b)_k B(k+b-\beta, a)}{k!} \times \left(\frac{1-2w}{w}\right)^k \{(2b+k)w-b-k\}. \tag{15}$$

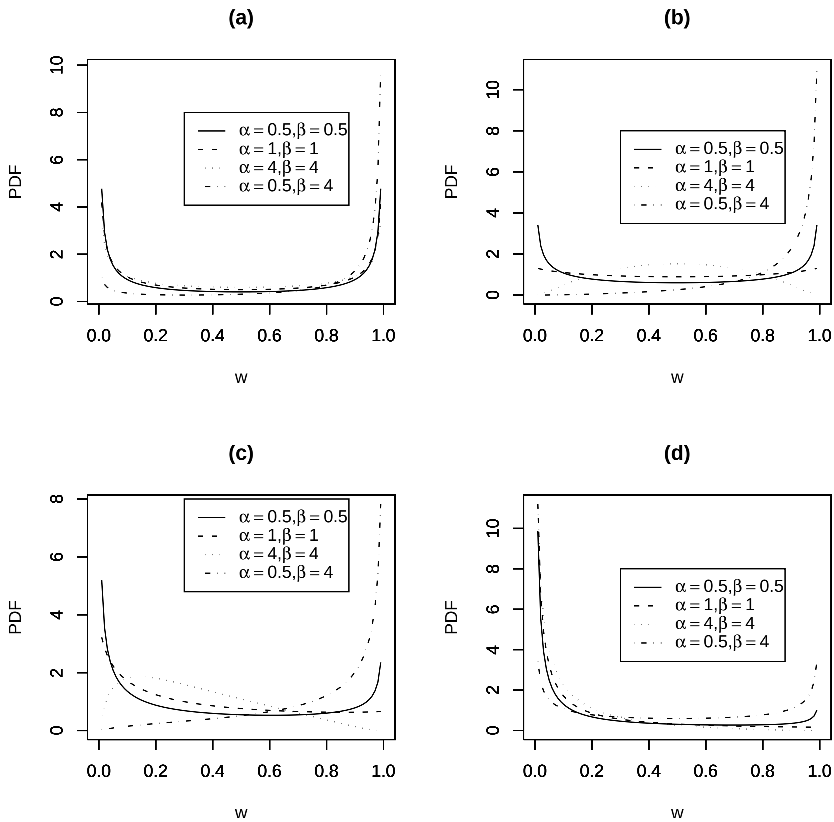

Figure 1: The pdf (15) for (a): $(a, b) = (0.5, 0.5)$; (b): $(a, b) = (4, 4)$; (c): $(a, b) = (2, 4)$; and, (d): $(a, b) = (0.5, 4)$.

Figure 1 illustrates the shape of (15) for selected values of the parameters (a, b, α, β).

Several particular cases of (13) can be obtained by using special properties of the Gauss hypergeometric function (see Sections 9.10 to 9.13 of Gradshteyn and Ryzhik (2000)). We consider two cases: if $b = 2\beta + 1$ then (13) reduces to

$$\begin{aligned} F(w) &= \frac{\Gamma(\beta)\Gamma(a+\alpha)}{a\Gamma(a+\alpha+\beta)B(a,b)B(\alpha,\beta)} \\ &\quad {}_3F_2\left(a, a+b, a+\alpha; a+1, a+\alpha+\beta; \frac{1-2w}{1-w}\right), \end{aligned} \tag{16}$$

and, if $b = 1 - a + \beta - \alpha$ then

$$F(w) = C \left(\frac{1-w}{w} \right)^b \sum_{k=0}^{\infty} \frac{(-1)^k (a+b)_k \Gamma(1+k-a-\alpha)}{k! \Gamma(2a+b+2\alpha-k-2) \Gamma(1+k-\alpha)} \left(\frac{1-2w}{w} \right)^k, \tag{17}$$

where the constant C is given by

$$C = \frac{\Gamma(a)\Gamma(\beta)\Gamma(a+\alpha)\, {}_2F_1(\beta, b-\beta; \beta+1; 1)}{B(a,b)B(\alpha,\beta)}.$$

The hazard rate function defined by $h(x) = f(x)/\{1-F(x)\}$ is an important quantity characterizing life phenomena. It can be loosely interpreted as follows: if there is a large number of items (say, $n(x)$) in operation at time x, then $n(x) \times h(x)$ is approximately equal to the number of failures per unit time (or $h(x)$ is approximately equal to the number of failures per unit time per unit at risk). It is immediate from (13) and (15) that the hazard rate function is given by

$$h(x) = \frac{\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k (a+b)_k B(k+b-\beta, a)}{k!} \left(\frac{1-2w}{w} \right)^k \{(2b+k)w - b - k\}}{\displaystyle 1 - w(1-w)(1-2w) \sum_{k=0}^{\infty} \frac{(-1)^k (a+b)_k}{k!} \left(\frac{1-2w}{w} \right)^k B(k+b-\beta, a)}. \tag{18}$$

Some possible shapes of (18) for $a = b = 1/2$ are shown in Figure 2. Ideally one would like a "bathtub" shape for $h(x)$ because most systems in real–life capture the three distinct hazard regimes: the region of infant mortality (where $h(x)$ decreases with x), the random failure region (where $h(x)$ does not change rapidly with x) and the wear-out region (where $h(x)$ increases with x due to deterioration processes). It is pleasing to see that the shapes exhibited in Figure 2 are exactly of this type.

A series representation for the nth moment of W can be obtained as follows. Note from (13) that one can write

$$F(w) = C \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{(-1)^{k+\ell} 2^\ell (a+b)_k}{k!} B(k+b-\beta, a)\, w^{\ell-b-k} (1-w)^b,$$

where C is given by (14). Thus, using the definition that

$$E\left(W^n\right) = n \int_0^1 w^{n-1} \{1 - F(w)\}\, dw,$$

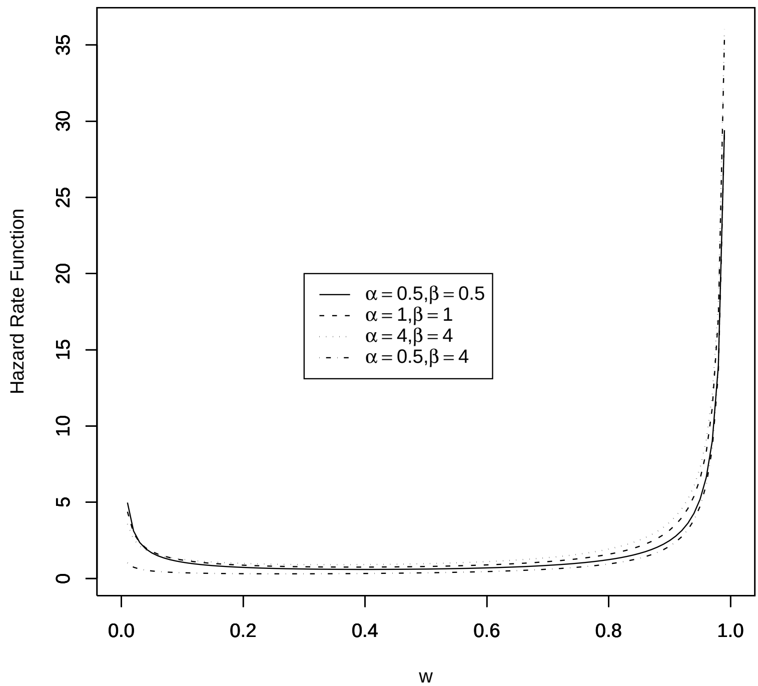

Figure 2: The hazard rate function (18) for $a = b = 0.5$.

one can obtain

$$E\left(W^n\right) = 1 - nC \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{(-1)^{k+\ell} 2^{\ell} (a+b)_k}{k!} B\left(k+b-\beta, a\right) B(n+\ell-b-k, b+1). \tag{19}$$

One can also obtain simpler expressions than (19) by considering the special cases (16) and (17).

3 Power beta

The generalization of this section is also based on the basic characterization that if X and Y are independent gamma distributed random variables with common scale parameter then the ratio $X/(X+Y)$ has the pdf (1). More specifically, if X and Y have the pdfs

$$f(x) = \frac{x^{a-1} \exp\left(-x/\beta\right)}{\beta^a \Gamma(a)} \tag{20}$$

and

$$f(y) = \frac{y^{b-1}\exp(-y/\beta)}{\beta^b\Gamma(b)} \tag{21}$$

for $x > 0$ and $y > 0$, respectively, then the ratio $X/(X+Y)$ will have the pdf (1). We consider the distribution of $W = X^c/(X^c+Y^c)$, $c > 0$, which we refer to as *power beta*. A practical example of the use of W can be described as follows: suppose that a physical plant has two components A and B and that one is interested in quantifying their relative performance. Assume that the inter-failure times for the two components A and B are gamma distributed according to (20) and (21), respectively. Assume further that the parameter c reflects the physical process which leads to the failures, i.e. the number of ways in which the failures can occur. Under these assumptions, it is clear that the variable $W = X^c/(X^c+Y^c)$ will reflect the relative performance of component A with respect to component B. Hence, measures of relative performance can be based on the distribution of W.

The variable of interest is $W = X^c/(X^c+Y^c)$, $c > 0$. The cdf corresponding to (21) is

$$F(y) = \frac{\gamma(b, x)}{\Gamma(b)}.$$

Thus, the cdf of W can be written as

$$\begin{aligned} F(w) &= \Pr\left(\frac{X^c}{X^c+Y^c} \le w\right) \\ &= \Pr\left(Y \ge \left(\frac{1-w}{w}\right)^{1/c} X\right) \\ &= 1 - \Pr\left(Y \le \left(\frac{1-w}{w}\right)^{1/c} X\right) \\ &= 1 - \int_0^\infty \frac{\gamma\left(b, \left(\frac{1-w}{w}\right)^{1/c}\frac{x}{\beta}\right)}{\Gamma(b)} \frac{x^{a-1}\exp\left(-\frac{x}{\beta}\right)}{\beta^a\Gamma(a)} dx \\ &= 1 - \frac{I(w)}{\beta^a\Gamma(a)\Gamma(b)}, \end{aligned} \tag{22}$$

where

$$I(w) = \int_0^\infty x^{a-1} \exp\left(-\frac{x}{\beta}\right) \gamma\left(b, \left(\frac{1-w}{w}\right)^{1/c} \frac{x}{\beta}\right) dx. \tag{23}$$

By direct application of equation (6.455.2) in Gradshteyn and Ryzhik (2000), (23) can be evaluated as

$$\begin{aligned} I(w) &= \frac{\beta^a \Gamma(a+b)}{b} \left(\frac{1-w}{w}\right)^{b/c} \\ &\quad {}_2F_1\left(b, a+b; b+1; -\left(\frac{1-w}{w}\right)^{1/c}\right). \end{aligned}$$

Substituting this into (22), one obtains the cdf of W as

$$\begin{aligned} F(w) &= 1 - \frac{1}{bB(a,b)} \left(\frac{1-w}{w}\right)^{b/c} \\ &\quad {}_2F_1\left(b, a+b; b+1; -\left(\frac{1-w}{w}\right)^{1/c}\right). \end{aligned} \tag{24}$$

Differentiating (24) with respect to w, one obtains the corresponding pdf as

$$\begin{aligned} f(w) &= \frac{w^{-\left(1+\frac{b}{c}\right)}(1-w)^{\frac{b}{c}-1}}{bcB(a,b)} \left\{ b\, {}_2F_1\left(b, a+b; b+1; -\left(\frac{1-w}{w}\right)^{1/c}\right) \right. \\ &\quad \left. - \left(\frac{1-w}{w}\right)^{1/c} {}_2F_1\left(b+1, a+b+1; b+2; -\left(\frac{1-w}{w}\right)^{1/c}\right) \right\}. \end{aligned} \tag{25}$$

Figure 3 illustrates the shape of (25) for selected values of the parameters (a, b, c). The effect of the parameter c can be clearly seen – higher values of c have the effect of increasing the concentration of mass around the end points 0 and 1.

Several particular cases of (24) can be obtained by using special properties of the Gauss hypergeometric function (see Sections 9.10 to 9.13 of Gradshteyn and Ryzhik (2000)). If $c = 1$ then (24) reduces to

$$F(w) = \frac{B_w(a,b)}{B(a,b)}, \tag{26}$$

which is the cdf of (1). If $a = 1$ then

$$F(w) = 1 - \left(\frac{1-w}{w}\right)^{b/c} \left\{1 + \left(\frac{1-w}{w}\right)^{1/c}\right\}^{-b}. \tag{27}$$

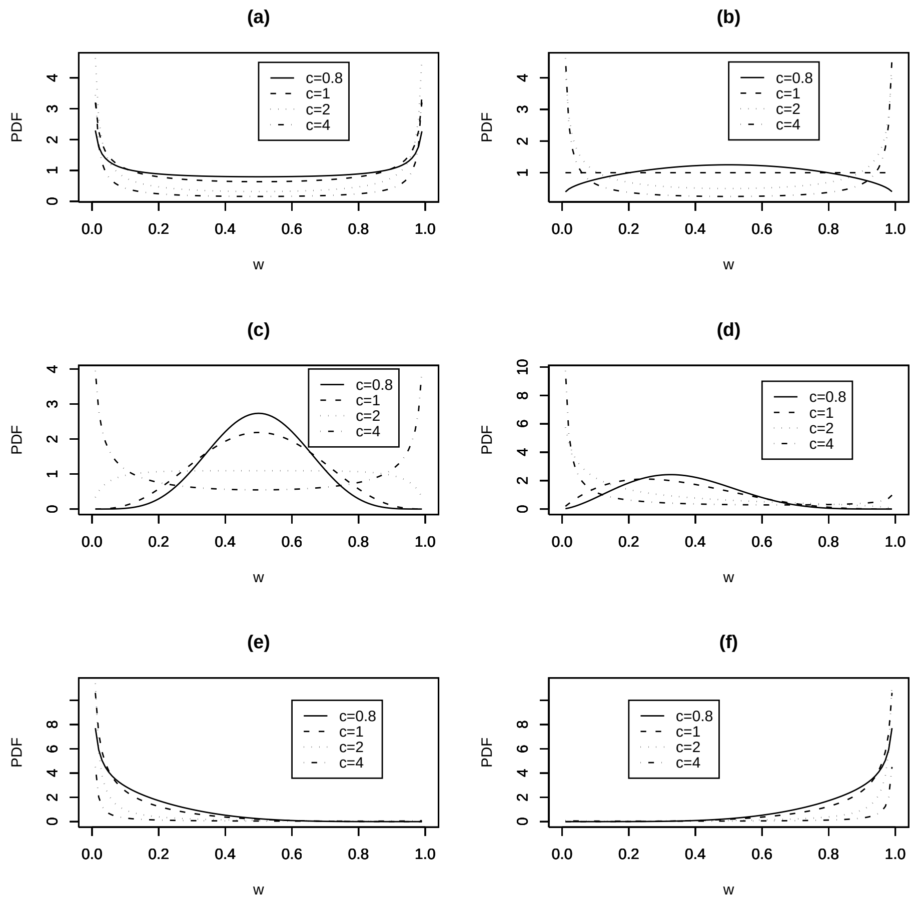

Figure 3: The pdf (25) for (a): $(a, b) = (0.5, 0.5)$; (b): $(a, b) = (1, 1)$; (c): $(a, b) = (4, 4)$; (d): $(a, b) = (2, 4)$; (e): $(a, b) = (0.5, 4)$; and, (f): $(a, b) = (4, 0.5)$.

If b is an integer then

$$\begin{aligned} F(w) &= 1 - \frac{(-1)^b B(1-a-b, b)}{B(a, b)} \left[1 + \left\{ 1 + \left(\frac{1-w}{w} \right)^{1/c} \right\}^b \right. \\ &\qquad \left. \times \sum_{k=1}^{b} \frac{(-1)^k \Gamma(b+k-1)}{\Gamma(b)\Gamma(k)} \left(\frac{1-w}{w} \right)^{\frac{k-1}{c}} \right]. \end{aligned} \tag{28}$$

If $a+b=1$ then

$$F(w) = 1 - \frac{(1-w)^{b/c}}{\Gamma(a)\Gamma(1-a)w^{b/c}} \Phi\left(-\left(\frac{1-w}{w}\right)^{1/c}, 1, b\right). \tag{29}$$

Finally, if $a=b=1/2$ then

$$F(w) = 1 - \frac{2}{\pi i}\tanh^{-1}\left(i\left(\frac{1-w}{w}\right)^{1/(2c)}\right), \tag{30}$$

where $i=\sqrt{-1}$ is the imaginary unit.

It is immediate from (24) and (25) that the hazard rate function is given by

$$\begin{aligned} h(x) &= \frac{b}{cw(1-w)} \\ &- \frac{(1-w)^{1/c-1}\ {}_2F_1\left(b+1, a+b+1; b+2; -\left(\frac{1-w}{w}\right)^{1/c}\right)}{cw^{1/c+1}\ {}_2F_1\left(b, a+b; b+1; -\left(\frac{1-w}{w}\right)^{1/c}\right)}. \end{aligned} \tag{31}$$

Some possible shapes of (31) for $a=b=1/2$ and $c=1,2,4,10$ are shown in Figure 4. As with (18), it is pleasing to see that "bathtub" shapes are exhibited.

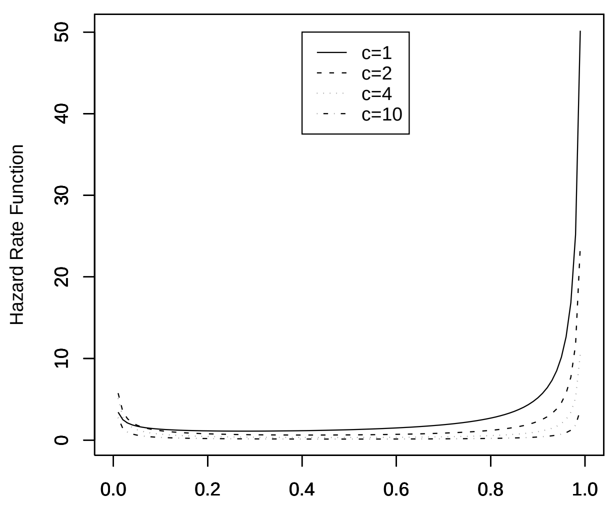

Figure 4: The hazard rate function (31) for $a=b=0.5$.

Using (24) and the definition that

$$E\left(W^k\right) = k \int_0^1 w^{k-1} \left\{1 - F(w)\right\} dw,$$

one can write the kth moment of W as

$$\begin{aligned} E\left(W^k\right) = \frac{k}{bB(a,b)} \int_0^1 w^{k-1} \left(\frac{1-w}{w}\right)^{b/c} \\ {}_2F_1\left(b, a+b; b+1; -\left(\frac{1-w}{w}\right)^{1/c}\right) dw. \end{aligned} \tag{32}$$

Setting $x = ((1-w)/w)^{1/c}$, (32) can be reduced to

$$E\left(W^k\right) = \frac{ck}{bB(a,b)} \int_0^\infty \frac{x^{b+c-1}}{(1+x^c)^{k+1}}\ {}_2F_1(b, a+b; b+1; -x)\, dx. \tag{33}$$

The integral on the right of (33) is difficult to calculate for general k. However, if $k \leq -1$ then on using the binomial expansion one can rewrite (33) as

$$E\left(W^k\right) = \frac{ck}{bB(a,b)} \sum_{\ell=0}^{-(k+1)} \binom{-(k+1)}{\ell} I(\ell), \tag{34}$$

where

$$I(\ell) = \int_0^\infty x^{c\ell+b+c-1}\ {}_2F_1(b, a+b; b+1; -x)\, dx.$$

By equation (7.511) in Gradshteyn and Ryzhik (2000),

$$I(\ell) = \frac{b\Gamma(c\ell+b+c)\Gamma(-c\ell-c)\Gamma(a-c\ell-c)}{\Gamma(a+b)\Gamma(1-c\ell-c)}. \tag{35}$$

Combining (34) and (35), one obtains the kth moment for $k \leq -1$ as

$$E\left(W^k\right) =$$

$$\frac{ck}{\Gamma(a)\Gamma(b)} \sum_{\ell=0}^{-(k+1)} \binom{-(k+1)}{\ell} \frac{\Gamma(c\ell+b+c)\Gamma(-c\ell-c)\Gamma(a-c\ell-c)}{\Gamma(1-c\ell-c)}.$$

One can also obtain simpler expressions than (33) by considering the special cases (26)–(30). For instance, if $a = 1$ then one has

$$E\left(W^k\right) = ck \int_0^\infty \frac{x^{b+c-1}(1+x)^{-b}}{(1+x^c)^{k+1}} dx.$$

4 Hypergeometric beta

In this section, we introduce a generalization of (1) involving the Gauss hypergeometric function. We define its pdf by

$$f(x) = \frac{bB(a,b)}{B(a,b+\gamma)} x^{a+b-1} \; {}_2F_1(1-\gamma, a; a+b; x) \tag{36}$$

for $0 < x < 1$, $a > 0$, $b > 0$ and $\gamma > 0$. The corresponding cdf is:

$$F(x) = \frac{bB(a,b)}{B(a,b+\gamma)} \int_0^x y^{a+b-1} \, {}_2F_1(1-\gamma, a; a+b; y) \, dy. \tag{37}$$

By an application of equation (2.21.1.4) in Prudnikov et al. (1986, volume 3), (37) can be reduced to

$$F(x) = \frac{bB(a,b)}{(a+b)B(a,b+\gamma)} x^{a+b} \, {}_2F_1(1-\gamma, a; a+b+1; x). \tag{38}$$

The nth moment associated with (36) is:

$$E\left(X^n\right) = \frac{bB(a,b)}{B(a,b+\gamma)} \int_0^1 x^{n+a+b-1} \; {}_2F_1(1-\gamma, a; a+b; x) \, dx. \tag{39}$$

Now, by an application of equation (2.21.1.5) in Prudnikov et al. (1986, volume 3), (39) can be reduced to

$$\begin{aligned} E\left(X^n\right) &= \frac{bB(a,b)}{(n+a+b)B(a,b+\gamma)} \\ &\quad {}_3F_2(1-\gamma, a, n+a+b; a+b, n+a+b+1; 1). \end{aligned}$$

Note that in the particular case $b = 0$, (36) reduces to the standard beta pdf (1) with parameters a and γ. Figure 5 illustrates the shape of (36) for selected values of (a, b, γ). The effect of the parameter b can be clearly seen.

It is immediate from (36) and (38) that the hazard rate function is given by

$$\begin{aligned} &h(x) = \\ &\frac{b(a+b)B(a,b)x^{a+b-1} \, {}_2F_1(1-\gamma, a; a+b; x)}{(a+b)B(a,b+\gamma) - bB(a,b)x^{a+b} \, {}_2F_1(1-\gamma, a; a+b+1; x)}. \end{aligned} \tag{40}$$

Some possible shapes of (40) for $a = \gamma = 1/2$ and $b = 0.1, 1, 2, 5$ are shown in Figure 6. As with (18) and (31), it is pleasing to see that "bathtub" shapes are exhibited.

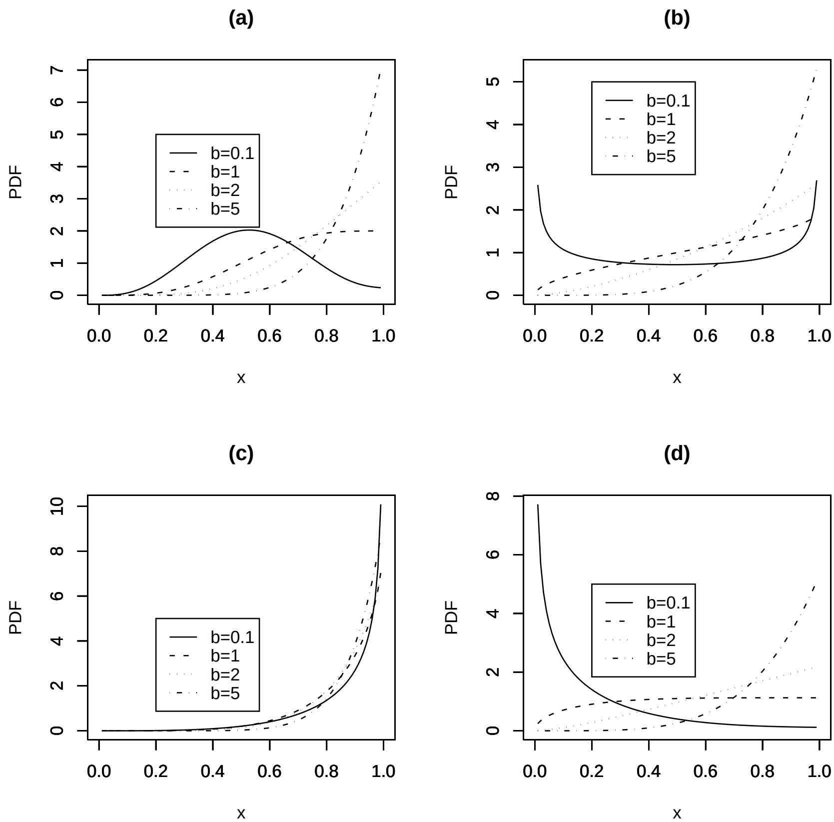

Figure 5: The pdf (36) for (a): $(a, \gamma) = (4, 4)$; (b): $(a, \gamma) = (0.5, 0.5)$; (c): $(a, \gamma) = (4, 0.5)$; and, (d): $(a, \gamma) = (0.5, 4)$.

Several particular cases of (36) can be obtained by using special properties of the Gauss hypergeometric function (see Section 7.3 of Prudnikov et al. (1986) and Sections 9.10 to 9.13 of Gradshteyn and Ryzhik (2000)). Some of these cases are:

1. If $a + b + \gamma = 1$ then (36) reduces to

$$f(x) = \frac{b\Gamma(b)x^{a+b-1}(1-x)^{-a}}{\Gamma(1-a)\Gamma(a+b)}.$$

2. If $a + b + \gamma = 2$ then

$$f(x) = \frac{b(a+b-1)B(a,b)}{B(a,2-a)} B_x(a+b-1, 1-a).$$

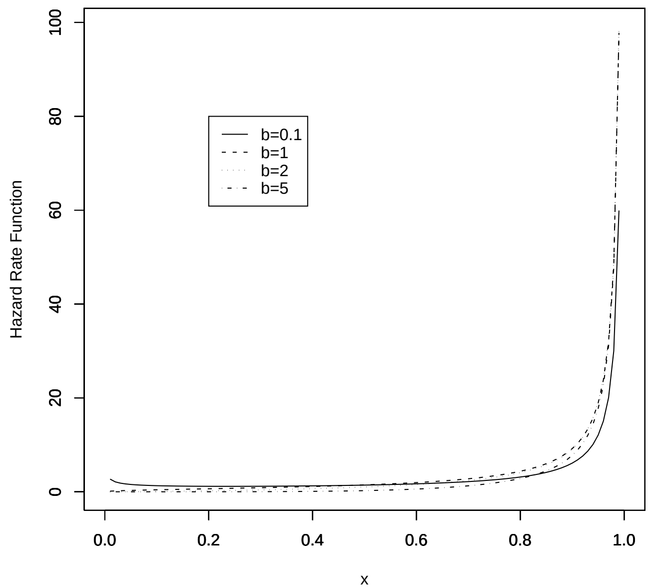

Figure 6: The hazard rate function (40) for $a = \gamma = 0.5$.

If in addition

(i) $a + b - 1$ is an integer then

$$f(x) = \frac{b(a+b-1)B(a,b)B(a+b-1,1-a)}{B(a,2-a)} \times \left\{ 1 - \sum_{\ell=1}^{a+b-1} \frac{\Gamma(\ell - a)}{\Gamma(1-a)\Gamma(\ell)} x^{\ell-1}(1-x)^{1-a} \right\}.$$

(ii) $a = 1/2$ and $b = 1$ then

$$f(x) = \frac{4}{\pi} \arctan \sqrt{\frac{x}{1-x}}.$$

(iii) $a = 1/2$ and $b = k$ then

$$f(x) = \frac{k(2k-1)B\left(1/2, k\right) B\left(1/2, k - 1/2\right)}{\pi} \times \left\{ \frac{2}{\pi} \arctan \sqrt{\frac{x}{1-x}} - \sqrt{x(1-x)} \sum_{\ell=1}^{k-1} d_\ell \right\}.$$

3. If $\gamma = 0$ then

$$f(x) = b(a+b-1)(1-x)^{b-1} B_x(a+b-1, 1-b).$$

If in addition

(i) $a+b-1$ is an integer then

$$\begin{aligned} f(x) &= b(a+b-1) B(a+b-1, 1-b)(1-x)^{b-1} \\ &\times \left\{ 1 - \sum_{\ell=1}^{a+b-1} \frac{\Gamma(\ell-b)}{\Gamma(1-b)\Gamma(\ell)} x^{\ell-1}(1-x)^{1-b} \right\}. \end{aligned}$$

(ii) $a=1$ and $b=1/2$ then

$$f(x) = \frac{1}{2\sqrt{1-x}} \arctan \sqrt{\frac{x}{1-x}}.$$

(iii) $a=k$ and $b=1/2$ then

$$f(x) = \frac{(2k-1)B(1/2, k-1/2)}{4\sqrt{1-x}} \left\{ \frac{2}{\pi} \arctan \sqrt{\frac{x}{1-x}} - \sqrt{x(1-x)} \sum_{\ell=1}^{k-1} d_\ell \right\}.$$

4. If $\gamma = 1$ then

$$f(x) = (a+b)x^{a+b-1},$$

a power function pdf.

5. If $a = 0$ then

$$f(x) = bx^{b-1},$$

another power function pdf.

6. If $b = 1$ then

$$f(x) = \frac{B_x(a,\gamma)}{B(a,\gamma+1)}.$$

If in addition

(i) a is an integer then

$$f(x) = \left(\frac{a}{\gamma}+1\right)\left\{1-\sum_{\ell=1}^{a}\frac{\Gamma(\gamma+\ell-1)}{\Gamma(\gamma)\Gamma(\ell)}x^{\ell-1}(1-x)^{\gamma}\right\}.$$

(ii) γ is an integer then

$$f(x) = \left(\frac{a}{\gamma}+1\right)\left\{\sum_{\ell=1}^{\gamma}\frac{\Gamma(a+\ell-1)}{\Gamma(a)\Gamma(\ell)}x^{a}(1-x)^{\ell-1}\right\}.$$

(iii) $a=\gamma=1/2$ then

$$f(x) = \frac{4}{\pi}\arctan\sqrt{\frac{x}{1-x}}.$$

(iv) $a=k-1/2$ and $\gamma=j-1/2$ then

$$f(x) = \left(\frac{a}{\gamma}+1\right)\left\{\frac{2}{\pi}\arctan\sqrt{\frac{x}{1-x}}-\sqrt{x(1-x)}\sum_{\ell=1}^{k-1}d_{\ell}+\sum_{\ell=1}^{j-1}c_{\ell}\right\}.$$

It should be noted above that the constants c_ℓ and d_ℓ are given by

$$c_{\ell} = \frac{\Gamma(k+\ell-1)x^{k-1/2}(1-x)^{\ell-1/2}}{\Gamma\left(k-1/2\right)\Gamma\left(\ell+1/2\right)}$$

and

$$d_{\ell} = \frac{\Gamma(\ell)x^{\ell-1}}{\Gamma\left(\ell+1/2\right)\Gamma\left(1/2\right)},$$

respectively.

5 Trigonometric beta

In this section, we introduce generalizations of (1) involving trigonometric functions. We refer to them as the beta trigonometric beta (TB) distributions. We propose four TB distributions in all: two of these involve the cosine function and the other two are complementary distributions involving the sine function.

The first generalization is given by the pdf

$$f(x) = Cx^{\nu-1}(1-x)^{\mu-1}\cos(ax) \tag{41}$$

for $0 < x < 1$, $\nu > 0$, $\mu > 0$ and $0 \leq a < \pi/2$, where the constant C is given by

$$\frac{1}{C} = \frac{1}{2} B(\nu, \mu) \left\{ {}_1F_1(\nu; \nu + \mu; ia) + {}_1F_1(\nu; \nu + \mu; -ia) \right\}$$

and $i = \sqrt{-1}$ is the imaginary unit. The standard beta pdf (1) arises as the particular case of (41) for $a = 0$. Some other particular cases of (41) are:

$$f(x) = C_1 x^{\mu-1} (1-x)^{\mu-1} \cos(ax)$$

for $0 < x < 1$, $\mu > 0$ and $0 \leq a < \pi/2$;

$$f(x) = C_2 (1-x)^{\nu} \cos(ax)$$

for $0 < x < 1$, $\nu > -1$ and $0 \leq a < \pi/2$; and,

$$f(x) = C_3 x^{-1/2} \cos(ax)$$

for $0 < x < 1$ and $0 \leq a < \pi/2$, where the constants C_1, C_2 and C_3 are given by

$$\frac{1}{C_1} = \sqrt{\pi} a^{1/2-\mu} \cos\left(\frac{a}{2}\right) \Gamma(\mu) \mathbf{J}_{\mu-1/2}\left(\frac{a}{2}\right),$$

$$\frac{1}{C_2} = \frac{i}{2} a^{-(1+\nu)} \left[\gamma(1+\nu, -ia) \exp\left\{ \frac{(\nu\pi - 2a)i}{2} \right\} - \gamma(1+\nu, ia) \exp\left\{ -\frac{(\nu\pi - 2a)i}{2} \right\} \right]$$

and

$$\frac{1}{C_3} = \sqrt{\frac{2\pi}{a}} C\left(\sqrt{a}\right),$$

respectively. The modes of (41) are the solutions of the equation:

$$a \tan(ax) = \frac{\mu - 1}{1 - x} - \frac{\nu - 1}{x}.$$

Using equation (3.768.12) in Gradshteyn and Ryzhik (2000), the nth moment associated with (41) can be derived as

$$E\left(X^n\right) = \frac{B(\mu, n+\nu) \left\{ {}_1F_1(n+\nu; n+\nu+\mu; ia) + {}_1F_1(n+\nu; n+\nu+\mu; -ia) \right\}}{B(\mu, \nu) \left\{ {}_1F_1(\nu; \nu+\mu; ia) + {}_1F_1(\nu; \nu+\mu; -ia) \right\}}$$

for $n \geq 1$.

The complementary sine pdf associated with (41) is:

$$f(x) \quad = \quad C x^{\nu-1} (1-x)^{\mu-1} \sin(ax)$$

for $0 < x < 1$, $\nu > -1$ ($\nu \neq 0$), $\mu > 0$ and $0 < a < \pi$, where the constant C is given by

$$\frac{1}{C} \; = \; -\frac{i}{2} B(\nu, \mu) \left\{ {}_1F_1 (\nu; \nu+\mu; ia) - {}_1F_1 (\nu; \nu+\mu; -ia) \right\}.$$

The modes of this pdf are the solutions of the equation:

$$a \cot(ax) \quad = \quad \frac{\mu-1}{1-x} - \frac{\nu-1}{x}$$

while its nth moment turns out to be

$$\mathrm{E}\left(X^n\right) \; = \; \frac{B(\mu, n+\nu) \left\{ {}_1F_1 (n+\nu; n+\nu+\mu; ia) - {}_1F_1 (n+\nu; n+\nu+\mu; -ia) \right\}}{B(\mu, \nu) \left\{ {}_1F_1 (\nu; \nu+\mu; ia) - {}_1F_1 (\nu; \nu+\mu; -ia) \right\}}$$

for $n \geq 1$. The latter result follows by using equation (3.768.11) in Gradshteyn and Ryzhik (2000).

The second generalization is given by the pdf

$$f(x) \quad = \quad C x^{2\nu-1} \left(1-x^2\right)^{\mu-1} \cos(ax) \tag{42}$$

for $0 < x < 1$, $\nu > 0$, $\mu > 0$ and $0 \leq a < \pi/2$, where the constant C is given by

$$\frac{1}{C} \; = \; \frac{1}{2} B(\nu, \mu) \, {}_1F_2 \left(\nu; \frac{1}{2}, \nu+\mu; -\frac{a^2}{4} \right).$$

Setting $a = 0$ into (42), one obtains a simple transformation of the standard beta pdf (1). Some other particular cases of (42) are:

$$f(x) \quad = \quad C_1 \left(1-x^2\right)^{\nu-1/2} \cos(ax)$$

and

$$f(x) \quad = \quad C_2 x \left(1-x^2\right)^{\nu-1/2} \cos(ax)$$

for $0 < x < 1$, $\nu > -1/2$ and $0 \leq a < \pi/2$, where the constants C_1 and C_2 are given by

$$\frac{1}{C_1} = \frac{\sqrt{\pi}}{2} \left(\frac{2}{a}\right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) \mathbf{J}_{\nu}(a)$$

and

$$\frac{1}{C_2} = \frac{1}{1+2\nu} - \frac{\sqrt{\pi}}{2} \left(\frac{2}{a}\right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) \mathbf{H}_{\nu+1}(a),$$

respectively. The modes of (42) are the solutions of the equation:

$$a \tan(ax) = \frac{2(\mu-1)x}{1-x^2} - \frac{2\nu-1}{x}.$$

By equation (3.771.4) in Gradshteyn and Ryzhik (2000), the nth moment becomes

$$\mathrm{E}\left(X^n\right) = \frac{B\left(\nu + \frac{n}{2}, \mu\right) \ {}_1F_2\left(\nu + \frac{n}{2}; \frac{1}{2}, \nu + \mu + \frac{n}{2}; -\frac{a^2}{4}\right)}{B\left(\nu, \mu\right) \ {}_1F_2\left(\nu; \frac{1}{2}, \nu + \mu; -\frac{a^2}{4}\right)}$$

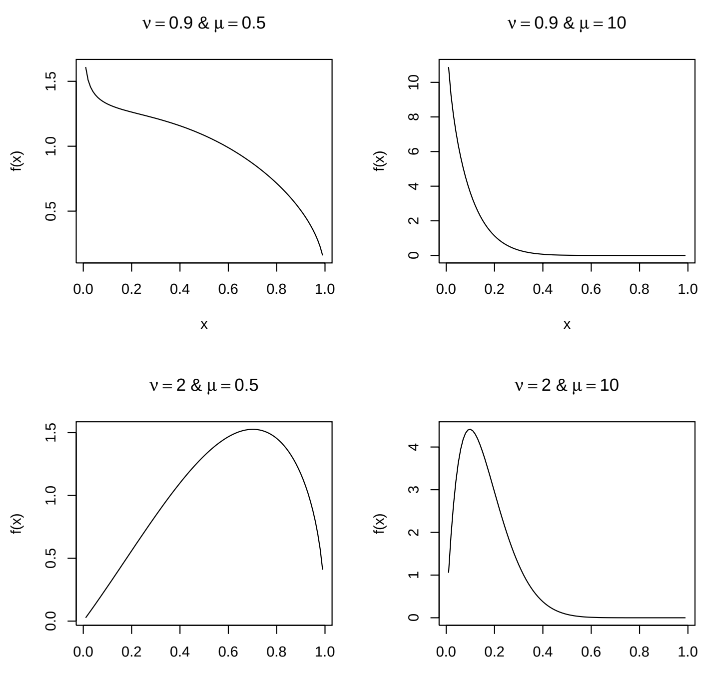

Figure 7: Plots of the pdf (41) with $a = \pi/2$.

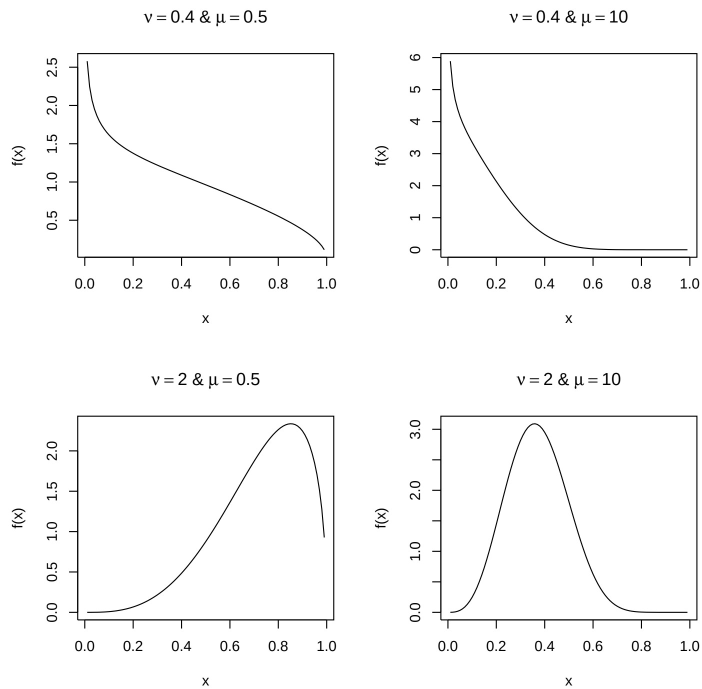

Figure 8: Plots of the pdf (42) with $a = \pi/2$.

for $n \geq 1$.

The complementary sine pdf associated with (42) is:

$$f(x) = Cx^{2\nu-1}\left(1-x^2\right)^{\mu-1}\sin(ax)$$

for $0 < x < 1$, $\nu > -1/2$, $\mu > 0$ and $0 < a < \pi$, where the constant C is given by

$$\frac{1}{C} = \frac{a}{2}B\left(\nu+\frac{1}{2},\mu\right)\ {}_1F_2\left(\nu+\frac{1}{2};\frac{3}{2},\nu+\mu+\frac{1}{2};-\frac{a^2}{4}\right).$$

The modes of this pdf are the solutions of the equation:

$$a\cot(ax) = \frac{2(\mu-1)x}{1-x^2}-\frac{2\nu-1}{x}$$

while the nth moment takes the form

$$\mathrm{E}\left(X^n\right) = \frac{B\left(\nu+\frac{n+1}{2},\mu\right)\ {}_1F_2\left(\nu+\frac{n+1}{2};\frac{3}{2},\mu+\frac{n+1}{2};-\frac{a^2}{4}\right)}{B\left(\nu+\frac{1}{2},\mu\right)\ {}_1F_2\left(\nu+\frac{1}{2};\frac{3}{2},\nu+\mu+\frac{1}{2};-\frac{a^2}{4}\right)}$$

for $n \geq 1$. The latter result follows by application of equation (3.771.3) in Gradshteyn and Ryzhik (2000).

Figures 7 and 8 show some of the possible shapes of the pdfs (41) and (42). The parameter a is fixed at $a = \pi/2$. In Figures 7 and 8, the pdf appears monotonically decreasing with x when $\nu < 1$. When $\nu > 1$ the pdf appears uni-modal with its tails exhibiting a left skew ($\mu < 1$) or right skew ($\mu > 1$).

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